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Evolutionary shape optimization in fracture problems

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The aim to the paper is to optimize 2-dimensional elastic structures subjected to cyclic load. The loading can result in crack forming, so the aim of the optimization is to reduce the possibility of crack growth. The number of loading cycles necessary to crack growth is maximized. To solve the optimization task the evolutionary algorithm is used. The boundary element method is applied to solve the crack problem. In order to reduce the number of design variables the parametrical NURSB curves are used to model the geometry of parts of the structural element boundary.

Keywords: boundary element method, optimization, evolutionary algorithm, crack, parametric curves

1. Formulation of the task

Cracks can occur in many real structures, so fracture mechanics problems seem to be very important from the technical point of view. Crack may occur in the structure due to the technological processes or can arise during the structure operation. The ability of the crack identification during the exploitation of the structure is essential. There are different methods of non-destructive crack identification, mainly using responses of the structure (e.g. [3]).

The reduction of the crack disadvantageous influence on the structure can be achieved using shape optimization methods. The publications devoted to the shape optimization of the cracked structures can be divided in two general parts:

- minimization of the stress intensity factors (e.g. [14, 16, 17]);
- maximization of the fatigue life time of the structure (e.g. [7, 18]).

The problem considered in present paper belongs to the second part. The modification of the structure shape to reduce the possibility of crack arising is considered. If the cyclic load is applied to the structure it can be treated as the increasing of the loading cycle number necessary to extend the crack.

The aim of the paper is to develop a method of the shape optimization of mechanical structures under cyclic load in order to reduce the influence of arising cracks and as a consequence to increase the lifetime of these structures. To solve this task evolutionary algorithm is used as the optimization method and the boundary element method is employed to solve the crack problem. The external boundary of the optimized structure is modelled using parametric curves to reduce the number of design variables.

2-dimensional structural elements in plane stress conditions, made of an isotropic, linear material are optimized. The optimization task is defined as the maximization of an objective function representing number of loading cycles necessary to extend the crack.

2. Evolutionary algorithms

In many practical optimization problems the application of the "traditional", especially gradient methods is limited. It is connected with difficulties that occur when the objective function gradient is calculated. Gradient methods are very accurate and fast, but the objective function must be continuous and the probability of convergence to a local optimum is very large.

In a few recent decades alternating optimization methods, especially based on the "evolution" of the possible solutions towards better result, have arisen [9]. Methods like genetic algorithms, evolutionary strategies, genetic programming and evolutionary programming are nowadays called evolutionary algorithms (EA) or the evolutionary programs.

EAs can be treated as modified and generalised genetic algorithms with modified operators (e.g. simple, arithmetical, heuristic crossovers; uniform, boundary and non-uniform mutations and others). The probability of operators can be variable. Apart from the classical selection (roulette wheel selection) other methods of the selection, like the ranking or tournament selections can be used. The floating point chromosome coding is usually used [1].

EAs are especially convenient when the objective function gradient value is difficult of impossible to obtain (e.g. discrete optimization problems) or when the objective function is multi-modal. EA maps the process of the individuals' adaptation to the environment. The individuals are called chromosomes and the objective (fitness) function plays role of the environment.

An EA starts from the initial population P_0 of chromosomes, being the feasible solutions of the optimization task. Working on the population of the possible solutions highly increases the probability of reaching the global optimum. Each chromosome has its value calculated by means of fitness function. The best part of the population is selected to the temporary population T_0 (some chromosomes more than once). Genetic operators like mutation and crossover modify some chromosomes in the temporary population which results in the offspring population O_0. Then, the new population P_1 is created using O_0 and P_0 populations. This procedure is repeated until the termination condition is satisfied (2).

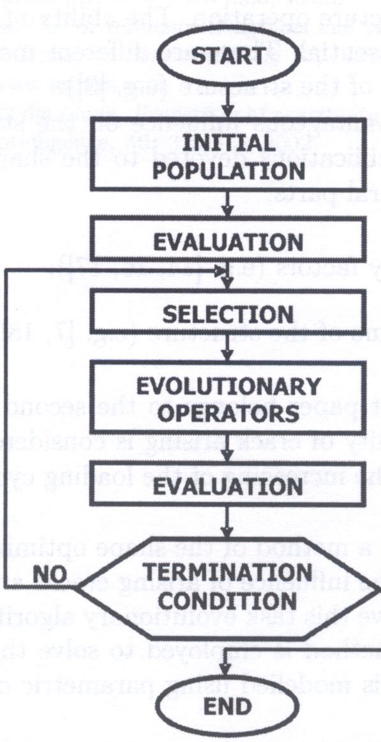

Fig. 1. The block diagram of an evolutionary algorithm

Due to the relatively long computation time and difficulties with finding the precise optimal value, EAs are often treated as the "last chance optimization algorithms". In many practical engineering optimization tasks the most time-consuming part of calculations is the evaluation of the fitness function because the boundary-value problem has to be solved.

To reduce the computation time distributed version of EA (DEA) is used [5]. DEA can use many processors to calculate fitness function. Each population of chromosomes is divided into two or more subpopulations evolving almost independently, interchanging some chromosomes during the migration phase. The block diagram of DEA for one subpopulation is presented in Fig. 2.

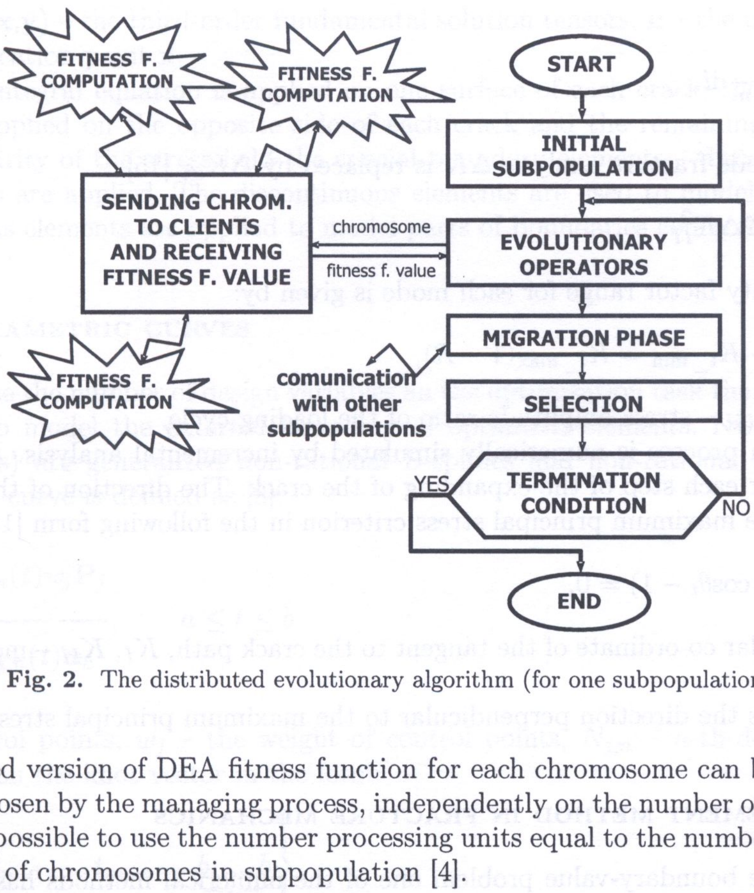

Fig. 2. The distributed evolutionary algorithm (for one subpopulation)

In the presented version of DEA fitness function for each chromosome can be calculated on the processing unit chosen by the managing process, independently on the number of subpopulations. So it is theoretically possible to use the number processing units equal to the number of subpopulations times the number of chromosomes in subpopulation [4].

3. Fatigue crack growth

Arising of the crack may significantly reduce the lifetime of structures. The most common fracture case seems to be a fatigue crack growth. It is also very dangerous for the structure, since the crack grows from a very small size to a critical one with no visible effect of this. As the effect a damage of the structure occurs. The possibility of the element lifetime prediction is crucial. In general, the velocity of the crack growth (determining the lifetime of the element) can be presented as follows [10]:

$$\frac{\mathrm{d}l}{\mathrm{d}N} = f(\sigma, l, C, Y, R, \chi), \tag{1}$$

where: N – the number of loading cycles, l – current crack length, σ – stress expressed by stress amplitude, C – material constants, Y – geometrical parameters of the element or crack, $R = \sigma_{\max}/\sigma_{\min}$ – cycle ratio, χ – function (functional) representing loading history.

There are many formulas for $f(\cdot)$ function describing velocity of the crack growth. One of the most frequently used is the Paris equation in the form:

$$\frac{dl}{dN} = c(\Delta K)^m, \tag{2}$$

where: c, m – experimentally determined material constants, $\Delta K = K_{\max} - K_{\min}$, K – stress intensity factor for single-mode fracture analysis.

The Paris law is suitable for the velocity of the crack propagation between 10^{-9} and 10^{-6} m/cycle. By integrating of Eq. (2) one can calculate the number of cycles N necessary to extend the crack from l_1 to l_2:

$$N = \int_{l_1}^{l_2} \frac{1}{C(\Delta K)^m} dl. \tag{3}$$

For the mixed-mode fracture analysis ΔK is replaced by ΔK_{eff} [15]

$$\Delta K_{\mathrm{eff}}^2 = \Delta K_I^2 + 2\Delta K_{II}^2. \tag{4}$$

The stress intensity factor range for each mode is given by:

$$\Delta K_i = K_{i\_\max} - K_{i\_\min} = K_{i\_\max}(1 - R), \tag{5}$$

where: $R = \sigma_{\min}/\sigma_{\max}$ – stress amplitude ratio of the loading cycle.

The crack-growth process is numerically simulated by incremental analysis. A boundary-value problem is solved for each step of the expanding of the crack. The direction of the crack growth is determined using the maximum principal stress criterion in the following form [13]:

$$K_I \sin\theta_t + K_{II}(3\cos\theta_t - 1) = 0, \tag{6}$$

where: θ_t – the angular co-ordinate of the tangent to the crack path, K_I, K_{II} – mode I and II stress intensity factors.

Angle θ_t indicates the direction perpendicular to the maximum principal stress direction.

4. Boundary element method in fracture mechanics

In order to solve the boundary-value problem one of the numerical methods has to be used. The most popular and wide applied one is the finite element method (FEM), but in the present case the boundary element method (BEM) is more convenient. The main reason is that the crack is a part of the boundary, so that (assuming the lack of body forces) there is no necessity to discretize the inside of the body and the dimension of the boundary-value problem is reduced. The BEM is also capable of accurate modelling the high stress gradients near the crack tip [2].

The displacement of an arbitrary point $\mathbf{x}$ can be represented by the boundary displacement integral equation:

$$\mathbf{c}(\mathbf{x})\mathbf{u}(\mathbf{x}) = \int_\Gamma \mathbf{U}(\mathbf{x},\mathbf{y})\mathbf{p}(\mathbf{y})d\Gamma(\mathbf{y}) - \int_\Gamma \mathbf{P}(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{y})d\Gamma(\mathbf{y}), \qquad x \in \Gamma, \tag{7}$$

where: $\mathbf{U}(\mathbf{x},\mathbf{y})$, $\mathbf{P}(\mathbf{x},\mathbf{y})$ – fundamental solutions of elastostatics; $\mathbf{c}(\mathbf{x})$ – a constant depending on the collocation point $(\mathbf{x})$ position; $\mathbf{y}$ – the boundary point.

If Eq. (7) is used on both crack surfaces, then two identical equations are formed. As the result the set of the algebraic equations obtained after the discretization of the body becomes singular. There are a few techniques allowing overcoming this problem [6]. In the Green's function method specific Green's functions, which include the exact solution for a traction-free crack, are used. This method is restricted to two-dimensional problems with linear cracks. In the multiregion method the body is divided into parts in the place of the crack (cracks). This method gives the bigger systems of equations (requires higher discretization) and is especially inconvenient in the case of the crack propagation.

The dual BEM technique seems to be the most general and is employed in the present paper. In this technique the additional tractions integral equation is introduced in the form [12]:

$$\frac{1}{2}\,\mathbf{p}(\mathbf{x}) = \mathbf{n}\left[\int_{\Gamma} \mathbf{D}(\mathbf{x},\mathbf{y})\mathbf{p}(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) - \int_{\Gamma} \mathbf{S}(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y})\right], \qquad x \in \Gamma \tag{8}$$

where: $\mathbf{D}(\mathbf{x},\mathbf{y})$, $\mathbf{S}(\mathbf{x},\mathbf{y})$ – the third-order fundamental solution tensors, $\mathbf{n}$ – the unit outward normal vector at the collocation point $\mathbf{x}$.

The tractions integral equation is applied on one surface of each crack; the displacement integral equation is applied on the opposite side of each crack and the remaining boundary as well. Due to the singularity of the stress field the special boundary elements – discontinuous and semi-discontinuous ones are applied. The discontinuous elements are used to model the cracks and the semi- discontinuous elements are applied to model parts of boundaries contacting the edge cracks.

5. NURBS parametric curves

In order to decrease the number of design variables an the optimization task the parametric NURBS curves are used to model the external boundary of optimized elements. NURBS (Non-Uniform Rational B-Splines) are generalized non-rational B-splines and non-rational and rational Bezier curves. A NURBS curve is defined as [8]:

$$\mathbf{C}(t) = \frac{\sum_{j=0}^{r} N_{j,n}(t) w_j \mathbf{P}_j}{\sum_{k=0}^{r} N_{k,n}(t) w_k}, \qquad a \le t \le b \tag{9}$$

where: $\mathbf{P}_j$ – control points, w_j – the weight of control points, $N_{j,n}$ – n-th-degree B-spline basis functions defined on the knot vector in the form:

$$T = \Big\{\underbrace{a,...,a}_{n+1}, t_{n+1},...,t_{m-n-1}, \underbrace{b,...,b}_{n+1}\Big\}. \tag{10}$$

An example of a NURBS curve is presented in Fig. 3.

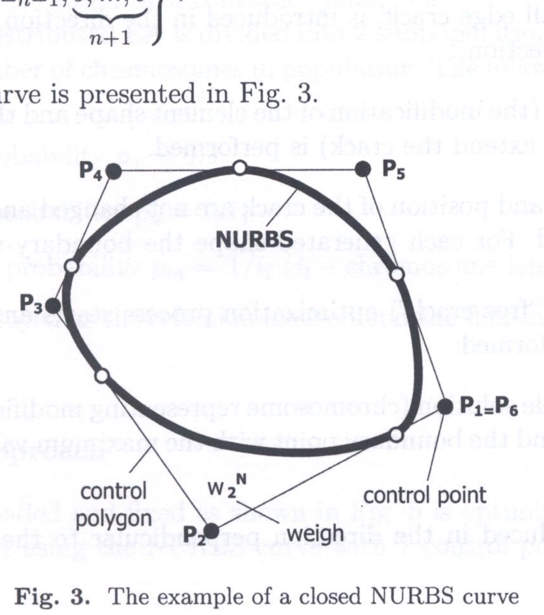

Fig. 3. The example of a closed NURBS curve

Using NUSBS curves it is possible to design a large variety of shapes in a flexible way. Changing discontinuous position of control points and the weight of control points, it is possible to control the shape of the curve precisely. If control point $\mathbf{P}_j$ is moved and/or the weight w_j is changed, only a part of the curve on the interval $t \in [t_i,\ t_{i+p+1}]$ is modified, which can be significant from the practical point of view (local approximation).

There are fast, numerically stable and accurate algorithms allowable for NURBS curves [11].

6. Optimization procedure

The aim of the optimization is to minimize the possibility of the crack propagation. The optimization task can be formulated as the maximization of an objective function J_0 representing the number of cycles N necessary to extend the crack:

$$\max_{\mathbf{a}}(J_0) = \max_{\mathbf{a}}(N). \tag{11}$$

The external shape is expressed by a vector of design variables $\mathbf{a} = (a_r)$, $r = 1,...,2R$, $R - 2$ times number of control points.

Design variables represent co-ordinates of the control points of NURBS curves. Chromosome, consisting of genes representing design variables, has the form:

$$ch = [a_1, a_2, ...a_R] \tag{12}$$

with constraints:

$$a_r^{\min} \leq a_r \leq a_r^{\max}, \qquad r = 1..R. \tag{13}$$

Each pair of genes, starting from a_1 and a_2, presents the x and y co-ordinates of one control point.

Limitations for the maximum von Mises stresses on the boundary and limitations for the elements area and are imposed. The number of cycles N necessary to extend the crack is calculated using Eq. (3).

Two approaches to solve the problem are considered (Fig. 4). In the first one, ("fixed crack") the analysis of the element with primary shape is performed:

- the position of the boundary point with the maximum value of the von Mises stresses is found;
- an initial, relatively small edge crack, is introduced in the direction perpendicular to the maximum principal stress direction;
- the optimization process (the modification of the element shape and the calculation of the number of cycles N necessary to extend the crack) is performed.

In this approach the size and position of the crack are not changed and the part of boundary with initial crack is not modified. For each generated shape the boundary-value problem is calculated only once.

In the second approach ("free crack") optimization process starts and in each optimization step the following stages are performed:

- for each generated possible solution (chromosome representing modified geometry) the boundary-value problem is solved and the boundary point with the maximum value of the von Mises stresses is found;
- an initial crack is introduced in the direction perpendicular to the maximum principal stress direction;

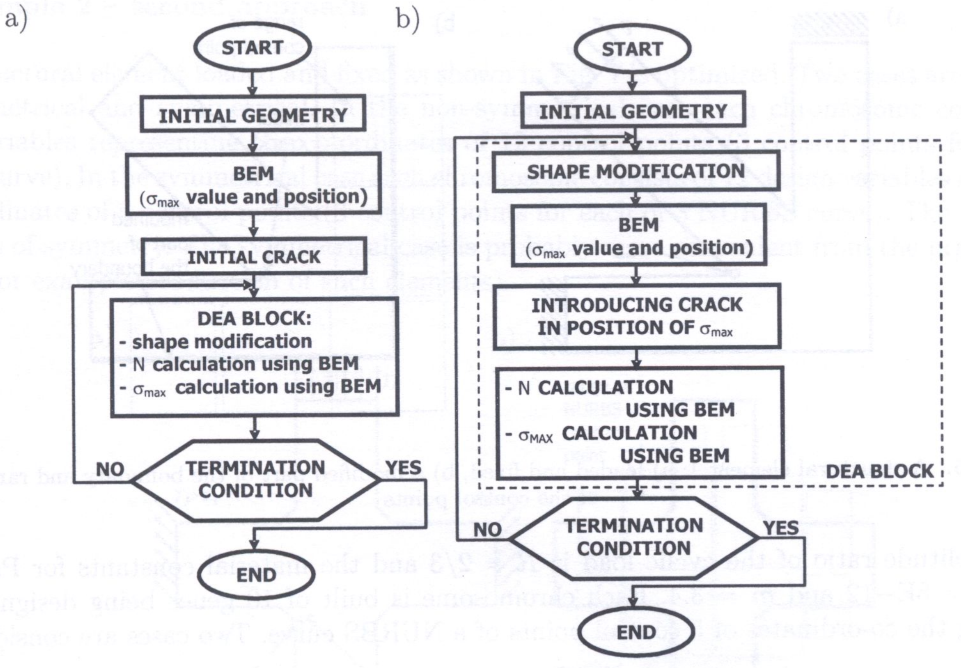

Fig. 4. Two approaches to the optimization: a) "fixed crack", b) "free crack"

- the boundary-value problem is solved once again and the number of cycles N necessary to extend the crack is calculated.

In this approach the boundary-value problem has to be calculated twice, but the crack can also occur on the modified part of the boundary. As a result the position and direction of the crack can be different for various generated shapes of the structural element.

7. Numerical examples

The shape optimization of structural elements being in the plane stress state and subjected to the cyclic load is performed.

The material elastic constants of the elements are: $E = 2\text{e}5$ MPa, $\nu = 0.3$. It is assumed that the fatigue cracking process is caused by a constant cycling load.

Each population of the distributed EA is divided into 2 subpopulations, regardless the number of used processors and the number of chromosomes in population. The following evolutionary operators are used:

- simple crossover with probability $p_s = 0.9$;
- uniform mutation with probability $p_u = 0.1$;
- Gaussian mutation with probability $p_m = 1/i_l$ (i_l – chromosome length).

Ranking selection is employed as the selection method and the floating-point chromosome coding is applied.

7.1. Example 1 – first approach

A 2-D structural element loaded and fixed as shown in Fig. 5 is optimized. The modified free part of the boundary is modelled using the NURBS curve with 7 control points. The first and the last control points are fixed.

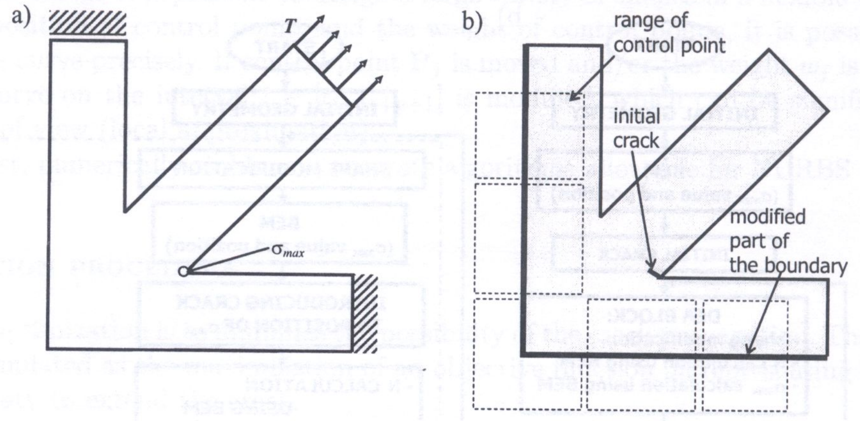

Fig. 5. A structural element 1: a) loaded and fixed, b) a modified part of the boundary and ranges of the control points

The amplitude ratio of the cyclic load is $R = 2/3$ and the material constants for Paris equation are: $c = 5\text{E}{-}12$ and $m = 3.4$. Each chromosome is built of 10 genes being design variables representing the co-ordinates of 5 control points of a NURBS curve. Two cases are considered:

- the final area of the element is not bigger than the area of the primary element;
- the final area of the element can be increased by 10%.

Maximum von Mises reduced stresses are limited to $\sigma_p = 220$ MPa. The population consists of 40 chromosomes divided into 2 subpopulations.

The EA is stopped after 50 generations. The optimization results are presented in Fig. 6.

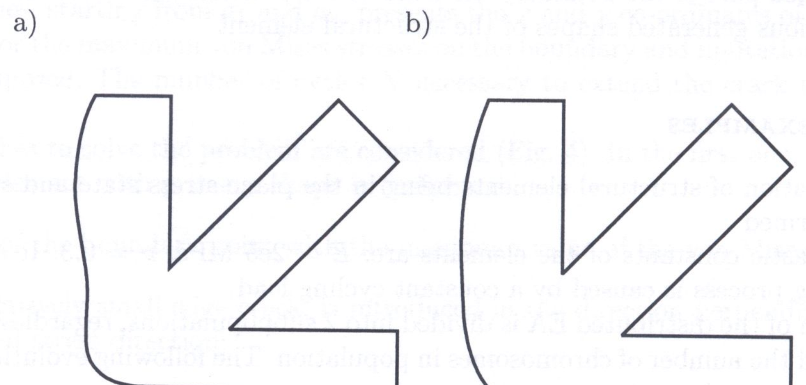

Fig. 6. A structural element 1 – optimal shapes: a) fixed area, b) increased area

The initial and final values of cycle numbers, the maximum stresses and the areas of the element are collected in Table 1.

Table 1. A structural element 1 – results

	Number of cycles	Allow. σ [MPa]	$\sigma_{\max}$ [MPa]	Allow. area [m^2]	Area [m^2]
Primary shape with initial crack	$2.9343 \cdot 10^7$	–	197.92	–	0.1084
Final shape with fixed area	$4.6659 \cdot 10^7$	220.0	219.99	0.1084	0.1083
Final shape with increase area	$5.3538 \cdot 10^8$		219.99	0.1192	0.1141

7.2. Example 2 – second approach

A 2-D structural element loaded and fixed as shown in Fig. 7 is optimized. Two cases are considered: non-symmetrical and symmetrical. In the non-symmetrical case each chromosome consists of 24 design variables representing the co-ordinates of 12 control points (3 control points for each of 4 NURBS curve). In the symmetrical case each chromosome consists of 12 design variables representing the co-ordinates of 6 control points (3 control points for each of 3 NURBS curve). The vertical axis is the axis of symmetry. The symmetrical case is probably more convenient from the practical point of view (for example production of such elements).

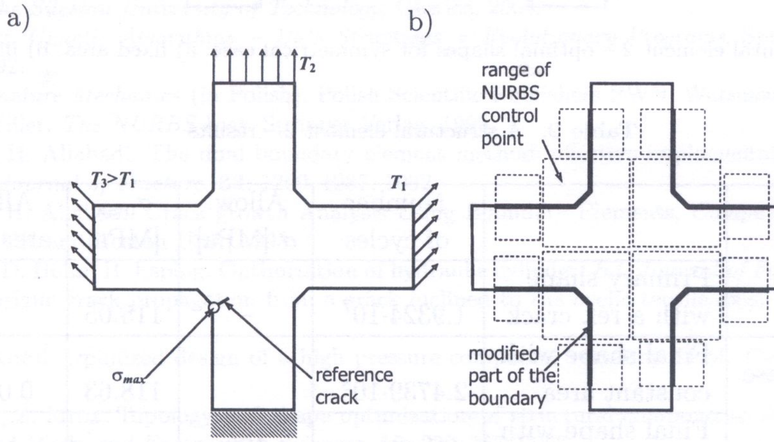

Fig. 7. A structural element 2: a) loaded and fixed, b) modified parts of the boundary and ranges of the control points

The parameters of Paris equation are assumed as: $c = 4.62\text{E}{-12}$ and $m = 3.3$ and the amplitude ratio of the cyclic load is $R = 2/3$. To compare the number of cycles N with the initial shape the position of the maximum von Mises stress is found and the reference crack is introduced in the proper direction. Then the boundary-value problem is solved and N is calculated.
Two cases are considered:

- the final area of the element is not bigger than the area of the primary element;
- the final area of the element can be increased by 10%.

Maximum von Mises reduced stresses are limited to $\sigma_p = 120$ MPa. The population consists of 60 chromosomes divided into 2 subpopulations. The EA is stopped after 80 generations.

Shapes after optimization are presented in Fig. 8 (the non-symmetrical case) and Fig. 9 (the symmetrical case). The initial and final values of cycle numbers, maximum stresses and areas of the element are collected in Table 2.

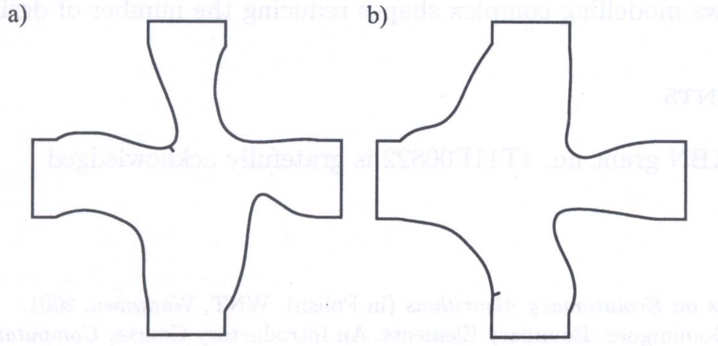

Fig. 8. A structural element 2 – optimal shapes for non-symmetrical case: a) fixed area, b) increased area

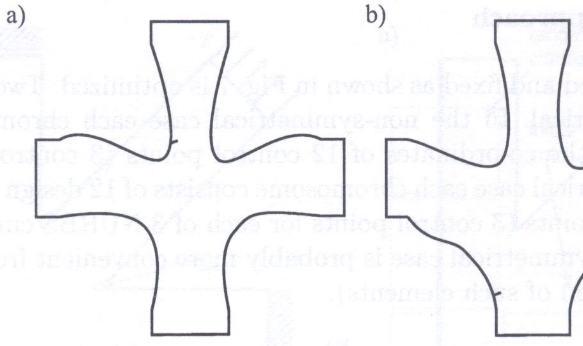

Fig. 9. A structural element 2 – optimal shapes for symmetrical case: a) fixed area, b) increased area

Table 2. A structural element 2 – results

		Number of cycles	Allow. σ [MPa]	σ_{max} [MPa]	Allow. area [m^2]	Area [m^2]
	Primary shape with a ref. crack	$1.9324 \cdot 10^7$	–	115.05	–	0.7080
non-symmetrical case	Final shape with constant area	$2.4739 \cdot 10^8$	120	118.63	0.0708	0.07078
	Final shape with increased area	$1.1097 \cdot 10^{10}$		112.57	0.0779	0.07514
symmetrical case	Final shape with constant area	$1.7073 \cdot 10^8$		114.79	0.0708	0.06289
	Final shape with increased area	$1.6019 \cdot 10^{10}$		108.98	0.0779	0.07129

8. Concluding remarks

In the present paper the shape optimization of the elements subjected to the cyclic loading has been presented. The aim of optimization is to maximize the number of loading cycles to extend the possible crack. Two approaches to the problem has been presented and the non-symmetrical and symmetrical cases has been considered.

In order to solve the problem the evolutionary algorithms, the boundary element method and parametric curves have been employed. The only information evolutionary algorithms need to work is the information about the objective function – the calculation of the fitness function gradient (often not possible or hard to perform) is unnecessary. Time-consuming evolutionary calculations can be accelerated by using the distributed version of the EA. The use of parametric curves like NURBS curves allows modelling complex shapes reducing the number of design variables.

Acknowledgments

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