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Solution of the nonlinear equation for isothermal gas flows in porous medium by Trefftz method

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This paper presents numerical solution to a problem of the transient flow of gas within a two-dimensional porous medium. A method of fundamental solution for space variables and finite difference method for time variable are employed to obtain a solution of the non-linear partial differential equation describing the flow of gas. The inhomogeneous term is expressed by radial basis functions at each time steps. Picard iteration is used for treating nonlinearity.

Keywords: isothermal gas flow, porous medium, Trefftz method, fundamental solution

Nomenclature

φ – porosity
μ – viscosity [Pa s]
k – permeability [darcys, m2], related to hydrodynamic conductivity coefficient K[m/day] by $k = \frac{\mu}{\rho g} K$
ρ – mass density of the fluid [kg/m^3]
g – gravity acceleration [m/s^2]
p – pressure [Pa]
$\mathbf{q}$ – superficial fluid velocity [m/s]
T – temperature [K]
x, y – geometry variables [m]
a, b, c – geometry parameters [m]
t – time
R – individual gas constant
τ – dimensionless time parameter
X, Y – dimensionless geometry variables
D, E – dimensionless geometry parameters

1. Introduction

Trefftz method can be understood as a method in which a differential equation is satisfied exactly whereas the boundary condition are satisfied in an approximate way. There are two basic ways of choosing sets of functions that are satisfying the differential equation. The first one uses T-Herrera function and the second one uses the fundamental solution of the differential equation. The version of the Trefftz method with the second choice is known as the method of fundamental solution. For the case of non-linear equations the set of functions that fulfils exactly this equation is usually unknown. Then the Trefftz method and, in frame of it, method of fundamental solution cannot be used in a straight way for non-linear Boundary Value Problems (BVPs).

However, it does not mean that the Trefftz method cannot be used in any way for non-linear BVPs. The first case when it can be used is BVPs with a linear equation but with non-linear boundary conditions. Examples of such applications of this method are given in papers [5, 10]. The second case known in the literature is BVP with non-linear Poisson equation in the form [1, 2, 4, 7, 8]

$$\nabla^2 u = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \tag{1}$$

where u is an unknown function, and f is a known function in which the same arguments are unknown.

In the paper [4] the non-linear thermal explosions problem was solved by the method of fundamental solutions. The radial basis functions were used for interpolation of the right hand side and Picard iteration method was used to tread non-linearity. In the paper [2] a method called "particular solution Trefftz method" was used. Really it is an extension and improvement of ideas proposed in the paper [4]. Another version of Trefftz method for solution of non-linear Poisson equation was presented in the paper [1]. For the non-linear thermal conductivity problem by Kirchoff transformation the nonlinearity exists only in boundary conditions. The non-linear algebraic equation was solved by the stabilized continuation method. Kita *et al.* [7] considered steady state heat conduction problems for functionally gradient materials. For overcoming the difficulty with non-linear Poisson equation they presented the combination scheme of the Trefftz method with the computing point analysis method. Also steady state heat conduction problem with temperature dependent conductivity was considered in the paper [8]. Combination of the method of fundamental solutions with Picard iteration was used for non-linear Poisson equation. Evolutionary algorithm was applied for optimal determination of method parameters. More complicated application of Trefftz method was presented in the paper [3] where the method of operator splitting with the method of fundamental solution was used for transient non-linear Poisson problems. These problems are widely encountered in modelling of many physical phenomena and the governing differential equation has the form

$$\frac{\partial u}{\partial t} = \nabla^2 u + f(u) \tag{2}$$

where t is time.

The purpose of the present paper is application some kind of Trefftz method to a problem of the transient flow of gas within a two-dimensional porous medium. Unsteady gas flow through semi-infinite porous medium was considered in the paper [6]. In such a case the problem is described by an ordinary differential equation. In the case of finite porous region the governing equation for pressure of gas as unknown has a form similar to Eq. (2). In our proposition the method of fundamental solution for space variables and the finite difference method for time variable are employed to obtain a solution of the non-linear partial differential equation describing the flow of gas. The inhomogeneous term is expressed by radial basis functions at each time step. Picard iteration is used for treating nonlinearity.

2. Problem description

The considered region of porous medium with flowing fluid is presented in Fig. 1. The porous medium is filled with gas under uniform pressure. The edges of considered reservoir are insulated, except one piece of edge which is opened. Pressure outside the reservoir is lower than the pressure in porous medium.

The reservoir is a rectangle with edges of lengths $2a$ and b. The open edge has length equal to $2c$. The geometry of the considered region is presented in Fig. 2.

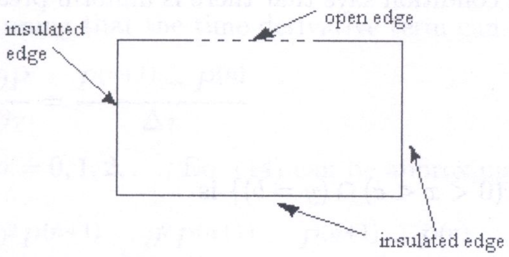

Fig. 1. The porous region with flowing fluid

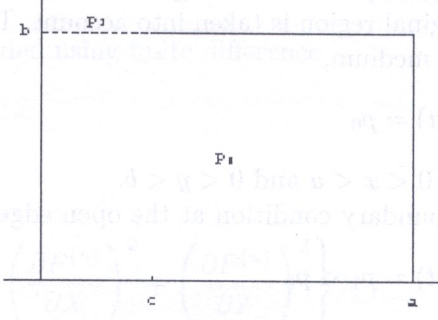

Fig. 2. The geometry of the considered region

To investigate gas flow in porous medium we introduce the following assumptions:

- The flow of gas follows the Darcy law;
- The only phase flowing is a gas of constant composition and viscosity;
- The gas is perfect and the gas flow is isothermal;
- Permeability of the porous medium is constant and uniform;
- Gravitational forces are neglected.

3. Motion equations

Darcy law is the filtration equation for fluid flow in porous media and in 2-D case it has the form

$$q_x = -\frac{k}{\mu}\frac{\partial p}{\partial x}, \tag{3}$$

$$q_y = -\frac{k}{\mu}\frac{\partial p}{\partial y}. \tag{4}$$

The continuity equation for porous media is

$$\frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) = -\frac{\partial}{\partial t}(\varphi\rho). \tag{5}$$

The gas equation for isothermal phenomena is

$$\rho = \frac{p}{RT} \tag{6}$$

and the temperature T is constant.

Applying Eq. (4) and Eq. (5) to Eq. (6) gives

$$\frac{\partial}{\partial x}\left(\frac{p}{RT}\frac{k}{\mu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{p}{RT}\frac{k}{\mu}\frac{\partial p}{\partial y}\right) = \frac{\mu}{k}\frac{\partial}{\partial t}\left(\varphi\frac{p}{RT}\right). \tag{7}$$

Rearranging Eq. (7) yields the equation

$$\frac{\partial}{\partial x}\left(p\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial p}{\partial y}\right) = \frac{\mu}{k}\frac{\partial}{\partial t}(\varphi p) \tag{8}$$

The region presented in Fig. 2 is considered. Because of the symmetry in the geometry, the half of the original region is taken into account. The initial condition says that there is uniform pressure in porous medium,

$$p(x,y,t) = p_0 \tag{9}$$

for $t = 0$, $0 < x < a$ and $0 < y < b$.

The boundary condition at the open edge $\{(x,y) \mid (0 < x < c) \cap (y = b)\}$ is

$$p(x,y,t) = p_1 < p_0 \tag{10}$$

for $0 < t < \infty$.

For the other reservoir edges

$$\{(x,y) \mid ((0 < x < a) \cap (y = 0)) \cup ((c < x < a) \cap (y = b)) \cup ((x = a) \cap (0 < y < b))\}$$

the insulated boundary condition is

$$\frac{\partial p}{\partial n} = 0 \tag{11}$$

and for $\{(x,y) \mid (x = 0) \cap (0 < y < b)\}$ the symmetry condition is applied

$$\frac{\partial p}{\partial n} = 0 \tag{12}$$

for $0 < t < \infty$.

The dimensionless variables are introduced

$$X = \frac{x}{a}, \quad Y = \frac{y}{a}, \quad E = \frac{b}{a}, \quad D = \frac{c}{a}, \quad P = \frac{p}{p_0}, \quad P_1 = \frac{p_1}{p_0}, \quad \tau = \frac{kp_0}{\varphi \mu a^2} t. \tag{13}$$

Therefore Eq. (8) has the dimensionless form

$$\frac{\partial}{\partial x}\left(P\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(P\frac{\partial P}{\partial y}\right) = \frac{\partial P}{\partial \tau} \tag{14}$$

and the initial condition is

$$P(X,Y,\tau) = 1 \tag{15}$$

for $\tau = 0$, $0 < X < 1$, $0 < Y < E$.

The boundary conditions are:

- the boundary condition for the open edge

$$P(X,Y,\tau) = P_1 < 1 \tag{16}$$

for $\tau = 0$, $0 < X < D$, $y = E$,

- and insulating and symmetry condition is

$$\frac{\partial P}{\partial n} = 0 \tag{17}$$

for the boundary

$$\{(X,Y) \mid ((0 < X < 1) \cap (Y = 0)) \cup ((D < X < 1) \cap (Y = E)) \cup ((X = 1) \cap (0 < Y < E))\}.$$

4. ALGORITHM FOR SOLVING INITIAL-BOUNDARY PROBLEM

Assuming that the time derivative term can be expanded using finite difference

$$\frac{\partial P}{\partial \tau} = \frac{P^{(n+1)} - P^{(n)}}{\Delta \tau} \tag{18}$$

for $n = 0, 1, 2, \ldots$, Eq. (14) can be approximated as

$$\frac{\partial^2 P^{(n+1)}}{\partial X^2} + \frac{\partial^2 P^{(n+1)}}{\partial Y^2} - \frac{P^{(n+1)} - P^{(n)}}{P^{(n)}\Delta \tau} = -\frac{1}{P^{(n)}} \left\{ \left(\frac{\partial P^{(n)}}{\partial X}\right)^2 + \left(\frac{\partial P^{(n)}}{\partial Y}\right)^2 \right\} \tag{19}$$

with the initial condition

$$P^{(0)}(X, Y, \tau) = 1 \tag{20}$$

for $\tau = 0$, $0 < X < 1$, $0 < Y < E$, and boundary conditions

$$P^{(n+1)} = P_1^{(n+1)} < 1 \tag{21}$$

for $0 < X < D$, $Y = E$, and

$$\frac{\partial P^{(n+1)}}{\partial n} = 0 \tag{22}$$

for the boundary

$$\{(X,Y) \mid ((0 < X < 1) \cap (Y = 0)) \cup ((D < X < 1) \cap (Y = E)) \cup ((X = 0) \cap (0 < Y < E)) \cup ((X = 1) \cap (0 < Y < E))\},$$

where $P^{(n)}$ is dimensionless pressure at n-th time step, $P^{(n+1)}$ is this pressure in the next time step.

For the first time step the dimension-less pressure P is uniform. Therefore Eq. (19) may be treated as the Helmholtz equation,

$$\frac{\partial^2 P}{\partial X^2} + \frac{\partial^2 P}{\partial Y^2} - k^2 P = f(X, Y) \tag{23}$$

where $P = P^{(n+1)}$, $k^2 = 1/(P^{(0)}\Delta\tau)$, $f(X, Y) = -1/(\Delta\tau)$.

The boundary conditions for Eq. (23) are

$$P = P_1 < 1 \tag{24}$$

for $0 < X < D$, $Y = E$, and

$$\frac{\partial P}{\partial n} = 0 \tag{25}$$

for the boundary

$$\{(X,Y) \mid ((0 < X < 1) \cap (Y = 0)) \cup ((D < X < 1) \cap (Y = E)) \cup ((X = 0) \cap (0 < Y < E)) \cup ((X = 1) \cap (0 < Y < E))\}.$$

The calculation of pressure in the next time steps is based also on Eq. (19). However, the pressure distribution is not uniform any more (as the result from the first time step), the equation is transformed into Poisson equation,

$$\frac{\partial^2 P^{(n+1)}}{\partial X^2} + \frac{\partial^2 P^{(n+1)}}{\partial Y^2} = \frac{1}{\Delta\tau} - \frac{P^{(n)}}{P^{(n+1)}\Delta\tau} - \frac{1}{P^{(n+1)}} \left\{ \left(\frac{\partial P^{(n+1)}}{\partial X}\right)^2 + \left(\frac{\partial P^{(n+1)}}{\partial Y}\right)^2 \right\} \tag{26}$$

with boundary conditions (24)–(25). The equation is strongly non-linear with respect to $P^{(n+1)}$, therefore, it is solved in an iterative fashion,

$$\frac{\partial^2 P^{(n+1,i+1)}}{\partial X^2} + \frac{\partial^2 P^{(n+1,i+1)}}{\partial Y^2} = \frac{1}{\Delta\tau} - \frac{P^{(n)}}{P^{(n+1,i)}\Delta\tau} - \frac{1}{P^{(n+1,i)}}\left\{\left(\frac{\partial P^{(n+1,i)}}{\partial X}\right)^2 + \left(\frac{\partial P^{(n+1,i)}}{\partial Y}\right)^2\right\} \tag{27}$$

with the boundary conditions (24), (25), where $P^{(n+1,i)}$ is the i-th iteration result at $(n+1)$-th time step. We introduce an initial condition for iterative procedure, e.g. trial equation in the Laplace form, which is the modified version of Eq. (27),

$$\frac{\partial^2 P^{(n+1,i+1)}}{\partial X^2} + \frac{\partial^2 P^{(n+1,i+1)}}{\partial Y^2} = 0 \tag{28}$$

with trial boundary conditions

$$P^{(n+1,1)} = P_1 < 1 \tag{29}$$

for $0 < X < D$, $Y = E$ and

$$\frac{\partial P^{(n+1,1)}}{\partial n} = 0 \tag{30}$$

for the boundary

$$\{(X,Y) \mid ((0 < X < 1) \cap (Y = 0)) \cup ((D < X < 1) \cap (Y = E)) \cup ((X = 0) \cap (0 < Y < E)) \cup ((X = 1) \cap (0 < Y < E))\}.$$

One extra boundary condition is added,

$$P^{(n+1,1)} = P^{(n)} \tag{31}$$

for $\{(X,Y) \mid (0 < X < 1) \cap (Y = 0)\}$, to combine the previous time step pressure distribution with the solution at the next time step.

Equation (28) is solved by the fundamental solution method, including the appropriate boundary conditions into calculation.

Solutions at the second and next iteration steps are found by the Trefftz method, based on Eq. (27) with its boundary conditions. Therefore, in one time step we obtain the sequence of solutions: $P^{(n+1,1)}$, $P^{(n+1,2)}$, ...

The iterative process is terminated when the difference between solutions of two successive iteration steps is quite small, less than a chosen small parameter. We introduce m, which points the iteration step number, at which the solution is taken as the solution at n-th time step, denoted as $P^{(n+1,m)} = P^{(n+1)}$.

5. Trefftz method to solve the boundary problem

The partial differential inhomogeneous equation

$$Lu = f(x,y) \tag{32}$$

is considered in the region Ω. The operator L is a partial differential operator, which includes Laplace operator.

The boundary condition has the general form

$$Bu = f(x,y) \tag{33}$$

where B is an operator imposed as boundary conditions, such as Dirichlet, Neumann, and Robin.

Let us denote by $\{P_i = (x_i, y_i)\}_{i=1}^N$ N collocations points in $\Omega \cup \partial\Omega$ of which $\{(x_i, y_i)\}_{i=1}^{N_l}$ are interior points; $\{(x_i, y_i)\}_{i=N_l+1}^{N}$ are boundary points.

The right-hand side function f is approximated with Radial Basis Functions (RBFs) as

$$f_N(x, y) = \sum_{j=1}^{N} a_j\, \varphi(r_j) + \sum_{k=1}^{l} b_k\, p_k(x, y) \tag{34}$$

where $r_j = \sqrt{(x - x_j)^2 + (y - y_j)^2}$ and $\varphi(r_j) : R^d \to R^+$ is a RBF, $\{p_k\}_{k=1}^l$ is the complete basis for d-variate polynomials of degree $\leq m - 1$, and C_{m+d-1}^d is the dimension of P_{m-1} . The coefficients $\{a_j\}$, $\{b_k\}$ can be found by solving the system

$$\sum_{j=1}^{N} a_j \varphi(r_{ji}) + \sum_{k=1}^{l} b_k p_k(x_i, y_i) = f(x_i, y_i), \qquad \text{for } 1 \leq i \leq N, \tag{35}$$

$$\sum_{k=1}^{l} a_j p_k(x_j, y_j) = 0 \qquad \text{for } 1 \leq k \leq l, \tag{36}$$

where $r_{ji} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$, $\{(x_i, y_i)\}_{i=1}^l$ are the collocation points on $\Omega \cup \partial\Omega$.

The approximate particular solution u_p can be obtained using the coefficients $\{a_j\}$ and $\{b_j\}$ by

$$u_p = \sum_{j=1}^{N} a_j\, \phi(r_j, r_j) + \sum_{k=1}^{l} b_k\, \psi_k(x, y) \tag{37}$$

where

$$L\phi = \varphi\,, \tag{38}$$

$$L\psi_k = p_k\,. \tag{39}$$

The solution of differential equation (32) can be given now as

$$u = u_p + v \tag{40}$$

where v is the solution of boundary value problem in the form

$$Lv = 0 \qquad \text{in } \Omega, \tag{41}$$

$$Bv = g\,(x, y) - Bu_p \qquad \text{on } \partial\Omega. \tag{42}$$

The method of fundamental solution is used to solve the problem presented above, which means that

$$v(x, y) = \sum_{j=1}^{N} c_j\, f_S(r_j) \tag{43}$$

where $f_S\,(r_j)$ is the fundamental solution function. Putting Eq. (43) into the boundary condition (42),

$$\sum_{j=1}^{N} c_j\, B\, f_S(r_{ji}) = g(x_i, y_i) - Bu_p(x_i, y_i) \qquad \text{for } 1 \leq i \leq N, \tag{44}$$

coefficients c_j are obtained. The solution of the boundary problem (32) and (33) is calculated by Eq. (40).

5.1. Numerical implementation

For Helmholtz equation, the differential operator is

$$L = \nabla^2 - k^2 \tag{45}$$

where ∇ is the Laplace operator and k is constant real number.

The inhomogeneous boundary problem (32) with the operator (45) is solved by numerical implementation of the solution given by Eqs. (37), (43) and (40). The function f is approximated as is presented in Eq. (34). The radial basis functions are [9]:

Case I

$$\varphi(r) = \begin{cases} 0 & \text{for } r = 0 \\ r^2 \ln r & \text{for } r \neq 0 \end{cases} \tag{46}$$

Case II

$$\varphi(r) = \begin{cases} 0 & \text{for } r = 0 \\ r^4 \ln r & \text{for } r \neq 0 \end{cases} \tag{47}$$

Case III

$$\varphi(r) = \begin{cases} 0 & \text{for } r = 0 \\ r^6 \ln r & \text{for } r \neq 0 \end{cases} \tag{48}$$

Case IV

$$\varphi(r) = \begin{cases} 0 & \text{for } r = 0 \\ r^8 \ln r & \text{for } r \neq 0 \end{cases} \tag{49}$$

Case V

$$\varphi(r) = \begin{cases} 0 & \text{for } r = 0 \\ r^{10} \ln r & \text{for } r \neq 0 \end{cases} \tag{50}$$

The solutions of the problem (39) with the right-hand side function given by Eqs. (46), (47), (48), (49) and (50) are, respectively,

Case I

$$\phi(r) = \begin{cases} -\frac{4}{k^4}\left(K_0(kr) + \ln r\right) - \frac{r^2 \ln r}{k^2} - \frac{4}{k^4} & \text{for } r > 0 \\ \frac{4}{k^4}\left(\gamma + \ln\frac{k}{2}\right) - \frac{4}{k^4} & \text{for } r = 0 \end{cases} \tag{51}$$

Case II

$$\phi(r) = \begin{cases} -\frac{64}{k^6}\left(K_0(kr) + \ln r\right) - \frac{r^2 \ln r}{k^2}\left(\frac{16}{k^2} + r^2\right) - \frac{8r^2}{k^4} - \frac{96}{k^6} & \text{for } r > 0 \\ \frac{64}{k^6}\left(\gamma + \ln\frac{k}{2}\right) - \frac{96}{k^6} & \text{for } r = 0 \end{cases} \tag{52}$$

Case III

$$\phi(r) = \begin{cases} -\frac{2304}{k^8}\left(K_0(kr) + \ln r\right) - \frac{r^2 \ln r}{k^2}\left(\frac{576}{k^4} + \frac{36r^2}{k^2} + r^4\right) \\ \qquad - \frac{12r^2}{k^4}\left(\frac{40}{k^4} + r^2\right) - \frac{4224}{k^8} & \text{for } r > 0 \\ \frac{2304}{k^8}\left(\gamma + \ln\frac{k}{2}\right) - \frac{4224}{k^8} & \text{for } r = 0 \end{cases} \tag{53}$$

Case IV

$$\phi(r)=\begin{cases} -\frac{147456}{k^{10}}\left(K_0(kr)+\ln r\right)-\frac{r^2\ln r}{k^2}\left(\frac{36864}{k^6}+\frac{2304r^2}{k^4}+\frac{64r^4}{k^2}+r^6\right) \\ \qquad -\frac{r^2}{k^4}\left(\frac{39936}{k^4}+\frac{1344r^2}{k^2}+r^4\right)-\frac{307200}{k^{10}} & \text{for } r>0 \\ \frac{147456}{k^{10}}\left(\gamma+\ln\frac{k}{2}\right)-\frac{307200}{k^{10}} & \text{for } r=0 \end{cases} \tag{54}$$

Case V

$$\phi(r)=\begin{cases} -\frac{14745600}{k^{12}}\left(K_0(kr)+\ln r\right) \\ \qquad -\frac{r^2\ln r}{k^2}\left(\frac{3686400}{k^8}+\frac{230400r^2}{k^6}+\frac{6400r^4}{k^4}+\frac{100r^6}{k^2}+r^8\right) \\ \qquad -\frac{r^2}{k^4}\left(\frac{4730880}{k^6}+\frac{180480r^2}{k^4}+\frac{2880r^4}{k^2}+20r^6\right)-\frac{33669120}{k^{10}} & \text{for } r>0 \\ \frac{14745600}{k^{12}}\left(\gamma+\ln\frac{k}{2}\right)-\frac{33669129}{k^{12}} & \text{for } r=0 \end{cases} \tag{55}$$

where $K_0(r)$ is the Bessel function of the second kind.

The fundamental solution for Helmholtz equation with the operator (45) is the Bessel function, therefore:

$$f_S(r)=K_0(kr) \tag{56}$$

The solution of the Poisson inhomogeneous equation is presented in [11]. However, main parts are included in this paper for reader's convenience.

In the Poisson equation the differential operator is the Laplace operator

$$L=\nabla^2. \tag{57}$$

The radial basis function are:

Case I

$$\varphi(r)=1+r \tag{58}$$

Case II

$$\varphi(r)=\begin{cases} 0 & \text{for } r=0 \\ r^2\ln r & \text{for } r\neq 0 \end{cases} \tag{59}$$

Case III

$$\varphi(r)=\sqrt{r^2+C^2} \tag{60}$$

Case IV

$$\varphi(r)=r^2+r^3 \tag{61}$$

Case V

$$\varphi(r)=\begin{cases} 0 & \text{for } r>a \\ \left(1-\frac{r}{a}\right)^4\left(1+\frac{4r}{a}\right) & \text{for } 0\le r\le a \end{cases} \tag{62}$$

where C is a parameter.

Appropriate solutions of Poisson equation with given above radial basis function are:

Case I

$$\phi(r)=\frac{r^2}{4}+\frac{r^3}{9} \tag{63}$$

Case II

$$\phi(r) = \frac{r^4 \ln r}{16} - \frac{r^4}{32} \tag{64}$$

Case III

$$\phi(r) = -\frac{C^2 \ln\left(C\sqrt{r^2+C^2}+C^2\right)}{3} + \frac{(r^2+4C^2)\sqrt{r^2+C^2}}{9} \tag{65}$$

Case IV

$$\phi(r) = \frac{r^4}{16} - \frac{r^5}{25} \tag{66}$$

Case V

$$\phi(r) = \begin{cases} \frac{529a^2}{5880} + \frac{a^2}{14}\ln\left(\frac{r}{a}\right) & \text{for } r > a \\ \frac{r^2}{4} - \frac{5r^4}{8a^2} + \frac{4r^5}{5a^3} - \frac{5r^6}{12a^4} + \frac{4r^7}{49a^5} & \text{for } 0 \le r \le a \end{cases} \tag{67}$$

The fundamental solution for Poisson equation is the function

$$fs(r) = \ln(r). \tag{68}$$

6. Numerical example

The example solved by the combination of the methods described above is presented. For first time step, $\Delta\tau = 1.3333$, the Helmholtz equation (23) with the boundary conditions (24)–(25) is solved using algorithm presented in the previous section. The right-hand side function is interpolated by RBF given by Eq. (46). For the solution of Eq. (39), the function (51) is applied. Moreover, to find the function described by Eqs. (43) and (44), the fundamental solution (56) is used.

Figure 3 shows the relative error of the right-hand side function interpolated by RBF. Error of the order $2 \cdot 10^{-5}$ points a very good approximation of the right-hand side function.

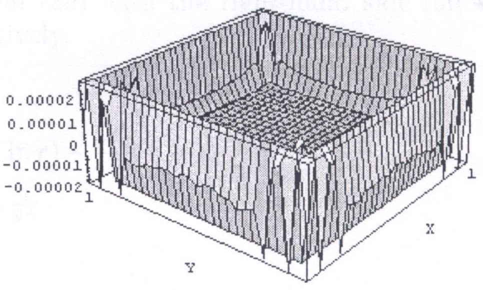

Fig. 3. Error of the approximation of the right-hand side function in the first time step

Pressure distribution in the first time step is presented in Fig. 4. The pressure is within the range (0.5,0.75) (Fig. 4a), which confirms the outflow of gas. Moreover, the isobars shown in Fig. 4b, have the exponential function shape.

The next two time steps results are shown in Fig. 5 and 6 – the gas flows out from the reservoir.

The pressure distribution at the next time step is shown in Fig. 7a. Values of pressure inside the considered region are very close to the value of pressure at the open edge. However, the isobars close to this open edge are exponential function like (see Fig. 7b), as it is expected.

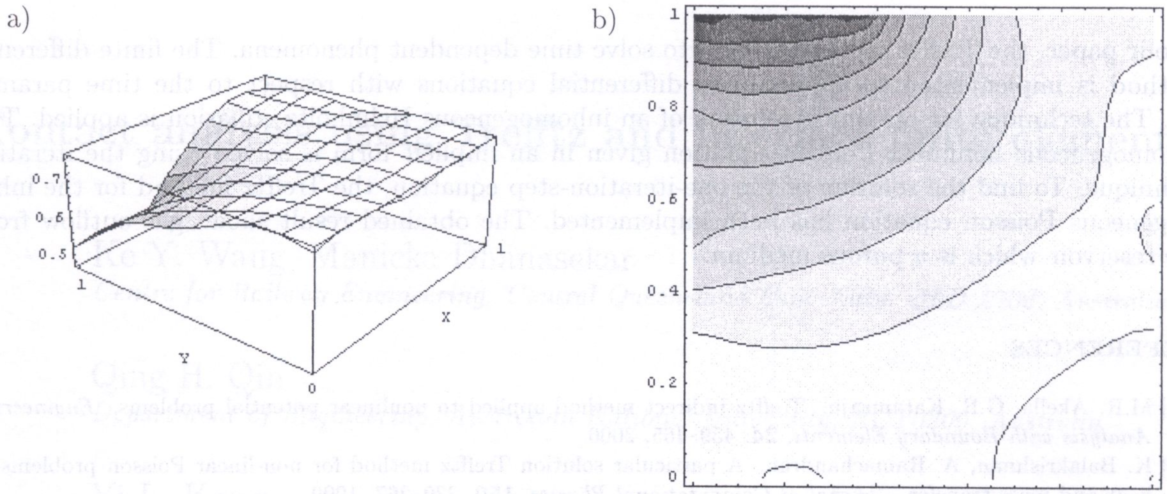

Fig. 4. Pressure field at the first time step: a) pressure, b) isobars

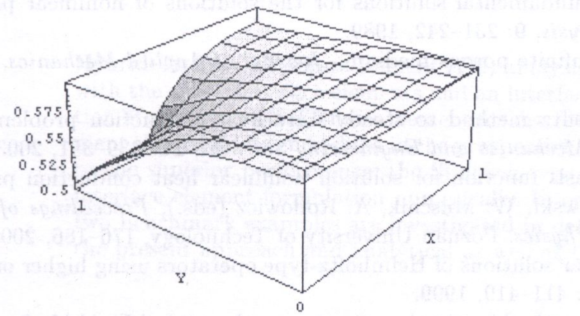

Fig. 5. Pressure distribution at the second time step

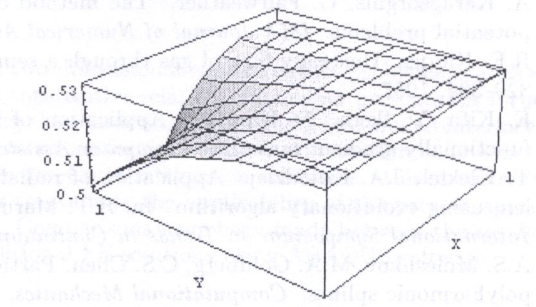

Fig. 6. Pressure distribution at the third time step

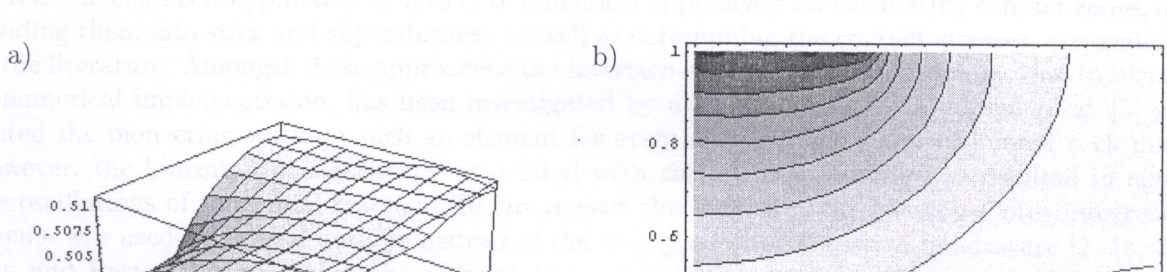

Fig. 7. The solution at the fourth time step; a) pressure distribution, b) isobars

7. Conclusions

In our paper, the Trefftz method is used to solve time dependent phenomena. The finite difference method is implemented to approximate differential equations with respect to the time parameter. The technique for obtaining solution of an inhomogeneous Helmholtz equation is applied. The inhomogeneous nonlinear Poisson equation given in an implicit form is solved using the iterative technique. To find the solution of the one-iteration-step equation, the Trefftz method for the inhomogeneous Poisson equation has been implemented. The obtained result shows gas outflow from the reservoir which is a porous medium.

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