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Sensitivity analysis of frames with unspecified dynamic load and joint parameters accounting for damping

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The paper is concerned with a class of generalized structural optimization problems, in which geometrical nonlinearities play an important role in a response of dynamically loaded structure. Forced, steady-state periodic vibrations of linear elastic frame and beam structures are considered. Both, viscous and complex modulus damping models are used. Using the adjoint variable method, sensitivity operators with respect to variation of stiffness, damping and mass parameters, as well as loading and support conditions are derived. The loading corresponds to an excitation induced by a rotational machine founded on vibro-isolation. The forms of response functionals expressed in displacements are discussed. Numerical examples of frame structures illustrate the theory and demonstrate the accuracy of the derived sensitivity operators.

Keywords: sensitivity analysis, optimal design, second order geometric effects, structural dynamics, vibrations

1. Introduction

Structural sensitivity analysis has focused much attention in the last two decades. Sensitivity gradients find applications in gradient optimization algorithms and directly in engineering practice. They are useful in the practice because they provide information on the sensitivity of structural response due to design and constructional tolerances. Most studies reported in the literature take up the theoretical issues of sensitivity analysis. Therefore it seems worthwhile to consider an engineering problem, when the machine of rotational type is founded on a base structure by means of elastic connectors with damping. Hence, the extended problem of optimization appears, where the damping mass and visco-elastic joint parameters can be considered as control variables. Thus, displacements of the machine and all displacements of the base structure can be minimized.

The problem of optimal design of supports in structures was first formulated in [11]. The study of minimum stiffness of supports maximizing the eigenfrequencies of vibrating beams was presented in [1]. Optimal location of supports in vibrating structures with complex damping was discussed in [9]. Sensitivity with respect to dynamic load parameters allowing for viscous and complex damping model was taken up in [4]. Extension to dynamic problems with excitation induced by rotational machine and viscous damping models was presented in [6]. The problem of optimal design of support conditions was generalized by extension to optimization of joints within the structure. Sensitivity of frames to variation of hinges in dynamic and stability problem was discussed in [5]. In [3] the effect of joint flexibility on the behavior of frames loaded statically was presented. The formulation of optimization problems due to joint position and stiffness distribution in frames with damping constraints was presented in [12]. Considerations of the sensitivity of dynamic systems to variations of parameters describing visco-elastic joint allowing the deflection and slope line discontinuity was taken up in [7]. Mróz and Haftka [10] presented an adjoint structure approach to the calculation of vibration frequency and buckling loads variations for geometrically nonlinear plates. This formulation was extended in [8]. The approach was developed and used for beam and frame structures in [2].

The present paper is concerned with the sensitivity analysis and optimal design of frame structures allowing for damping. The influence of rotational machine location, the manner of load transmission from the machine to the structure and specifying passive dampers and joints in the structure are analyzed in detail. The analysis is carried out involving second order geometric effects on the behavior of the frame structure.

2. Problem formulation

Consider a linear elastic frame structure subjected to dynamic loading by a rotational machine. A practical engineering problem is studied, when the machine is founded on a frame structure by means of visco-elastic connectors playing the role of dampers. The structure is illustrated schematically in Fig. 1. Viscous or complex damping in the structure and the connectors are allowed for. The case of small, forced and periodic vibrations is analyzed. The second order geometric effects are also taken into account.

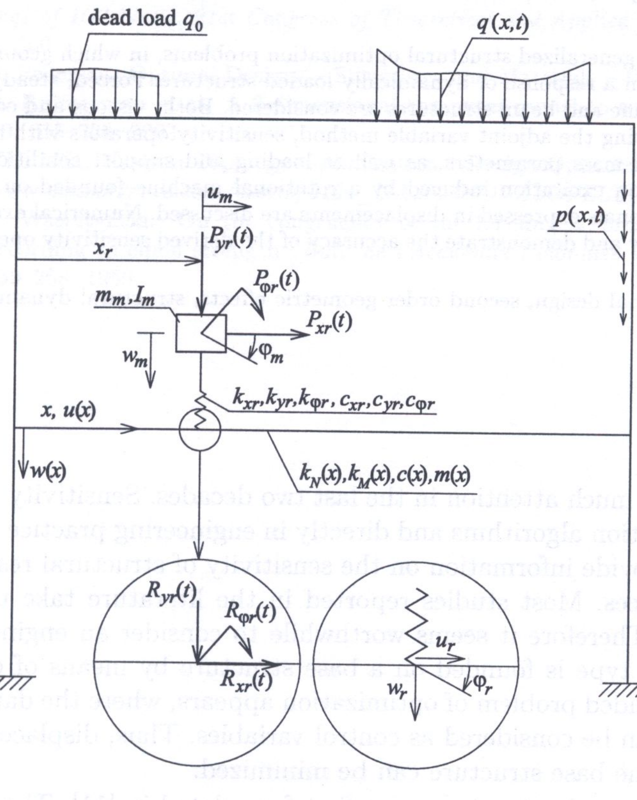

Fig. 1. Concept model – visco-elastic frame structure

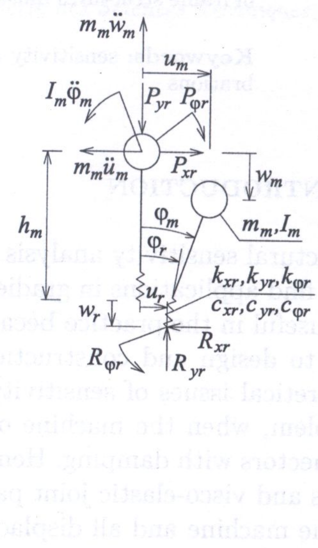

Fig. 2. Model of a machine

The task is to find optimal parameters of connectors and damping mass as well as optimal position of the machine modelled as a rigid body with the center of gravity at height h_m. A model of the machine is presented in Fig. 2.

Assume the response functional as a function of horizontal, vertical and angular displacements of the machine u_m, w_m, φ_m, respectively, and integrated displacements of the base structure $u(x)$, $w(x)$,

$$G(\mathbf{s}) = \int_{t_0}^{t_1} \left\{ f_1[u_m(t)] + f_2[w_m(t)] + f_3[\varphi_m(t)] + \int_L f_4[u(x,t), w(x,t)]\,\mathrm{d}x \right\} \mathrm{d}t\,. \tag{1}$$

Functions f_1, f_2, f_3 and f_4 are arbitrary, differentiable. The integration $\int_L \cdots \mathrm{d}x$ is carried out over the length of all beams and columns of the frame with the extraction of the singular point $x = x_r$

of machine position. Effects of action of the machine will be considered separately. The vector of design parameters consists of: position of the machine, mass and moment of mass inertia of the machine and six parameters describing the connection, bending stiffness, axial stiffness, damping and mass of the frame structure. Hence, the control vector is represented by

$$\mathbf{s} = \{x_r, m_m, I_m, k_{xr}, k_{yr}, k_{\varphi r}, c_{xr}, c_{yr}, c_{\varphi r}, k_M(x), k_N(x), c(x), m(x)\} \ . \tag{2}$$

Subscript r denotes that the symbol refers to the machine parameters, subscript m refers to the point mass. Our aim is to find the variation of functional (1) with respect to small variations of design parameters (the components of the control vector $\mathbf{s}$)

$$\delta G = \int_{t_0}^{t_1} \left\{ f_{1,\mathbf{s}} + f_{2,\mathbf{s}} + f_{3,\mathbf{s}} + \int_L f_{4,\mathbf{s}}\, \mathrm{d}x \right\} \delta\mathbf{s}\, dt\,. \tag{3}$$

3. Sensitivity operators

Variations of machine and the base structure displacements are implicit functions of design parameters. In order to transform Eq. (3) to an explicit form we use adjoint variable method. The primary structure is described by the following geometrical equations,

$$e = \varepsilon + \gamma = u_{,x} + \frac{1}{2} w_{,x} w_{,x}\,, \tag{4}$$

$$\kappa = -w_{,xx}\,, \tag{5}$$

where comma denotes partial differentiation and u and w denote axial and transversal displacements, respectively. Generalized strains e are nonlinear with respect to the derivative of w. The constitutive, linear equations for the primary structure have the form

$$N = k_N e\,, \tag{6}$$

$$M = k_M \kappa\,, \tag{7}$$

where e, κ, N, M, k_N and k_M denote elongation, curvature, normal force, bending moment, axial stiffness and bending stiffness.

The adjoint structure is described by similar equations, but now geometrical equations are linear and the derivative of w of the primary structure plays the role of a multiplier. The equations (8)–(11) may be thought of as the "tangent" structure to the primary one. For the adjoint structure, the second term in Eq. (8) can be considered as an initial strain.

$$e^a = \varepsilon^a + \gamma^a = u^a_{,x} + \frac{1}{2}(w_{,x} w^a_{,x} + w^a_{,x} w_{,x})\,, \tag{8}$$

$$\kappa^a = -w^a_{,xx}\,, \tag{9}$$

$$N^a = k_N e^a\,, \tag{10}$$

$$M^a = k_M \kappa^a\,. \tag{11}$$

There are two differences more if we compare primary and adjoint structure, namely loadings and damping coefficients. The adjoint structure should be loaded by the following forces,

$$P^a_{xr} = \frac{\partial f_1}{\partial u_m}\,, \quad P^a_{yr} = \frac{\partial f_2}{\partial w_m}\,, \quad P^a_{\varphi r} = \frac{\partial f_3}{\partial \varphi_m}\,, \quad p^a = \frac{\partial f_4}{\partial u}\,, \quad q^a = \frac{\partial f_4}{\partial w}\,. \tag{12}$$

The damping coefficients for the adjoint structure have the form

$$c^a_{xr} = -c_{xr}\,, \qquad c^a_{yr} = -c_{yr}\,, \qquad c^a_{\varphi r} = -c_{\varphi r}\,, \qquad c^a = -c. \tag{13}$$

An alternative approach to assuming negative damping is an integration in of the adjoint problem in time domain, in the inverse direction of time.

Using the concepts of primary and adjoint structure, the virtual work equation for adjoint forces and variation of primary kinematic fields can be written as follows

$$\begin{aligned}
&\int_{t_0}^{t_1} \left\{ \int_L [N^a(\varepsilon^* - \varepsilon) + M^a(\kappa^* - \kappa) + N^a(\gamma^* - \gamma) + N w^a_{,x}(w^*_{,x} - w_{,x})]\, dx \right\} dt \\
&= \int_{t_0}^{t_1} \left\{ \int_L [(p^a - m\ddot{u}^a)(u^* - u) + (q^a - m\ddot{w}^a - c^a\dot{w}^a)(w^* - w)]\, dx \right. \\
&\quad \left. + R^a_{xr}(u^*_r - u_r) + R^a_{yr}(w^*_r - w_r) + R^a_{\varphi r}(\varphi^*_r - \varphi_r) \right\} dt\,. \qquad (14)
\end{aligned}$$

Conversely, using variation of primary forces and adjoint kinematics, we obtain

$$\begin{aligned}
&\int_{t_0}^{t_1} \left\{ \int_L [(N^* - N)\varepsilon^a + (M^* - M)\kappa^a + (N^* w^*_{,x} - N w_{,x}) w^a_{,x}]\, dx \right\} dt \\
&= \int_{t_0}^{t_1} \left\{ \int_L [((p^* - m^*\ddot{u}^*) - (p - m\ddot{u}))u^a + ((q^* - m^*\ddot{w}^* - c^*\dot{w}^*) - (q - m\ddot{w} - c\dot{w}))w^a]\, dx \right. \\
&\quad \left. + R^*_{xr*}u^a_{r*} - R_{xr}u^a_r + R^*_{yr*}w^a_{r*} - R_{yr}w^a_r + R^*_{\varphi r*}\varphi^a_{r*} - R_{\varphi r}\varphi^a_r \right\} dt\,. \qquad (15)
\end{aligned}$$

All quantities denoted by star refer to the structure with perturbed components of design vector $\mathbf{s}$. In particular, in the symbol w^*_{r*} the superscript star denotes perturbed displacement field w, whereas the subscript r^* denotes that this displacement is measured in perturbed position of the machine $x^*_r = x_r + \delta x_r$. The forces R specify reactions transferred from the machine to the base structure. Note that the rotation φ_r is equal to $w_{r,x}$ in the case of Bernoulli beam.

Developing the perturbations of displacements in Taylor series and retaining only linear terms we arrive at

$$u^*_{r*} - u_r = \delta u_r \ (\text{def.})\,, \qquad u^*_r - u_r = \delta u_r - u^-_{r,x}\delta x_r\,, \qquad u^a_{r*} - u^a_r = u^{a+}_{r,x}\delta x_r\,, \qquad (16)$$

$$w^*_{r*} - w_r = \delta w_r \ (\text{def.})\,, \qquad w^*_r - w_r = \delta w_r - w^-_{r,x}\delta x_r\,, \qquad w^a_{r*} - w^a_r = w^{a+}_{r,x}\delta x_r\,, \qquad (17)$$

$$\varphi^*_{r*} - \varphi_r = \delta \varphi_r \ (\text{def.})\,, \qquad \varphi^*_r - \varphi_r = \delta \varphi_r - w^-_{r,xx}\delta x_r\,, \qquad \varphi^a_{r*} - \varphi^a_r = w^{a+}_{r,xx}\delta x_r\,. \qquad (18)$$

Please note that we distinguished between left-side $(\cdot)^-$ and right-side $(\cdot)^+$ derivatives in Eqs. (16)–(18). In fact, only the second derivative $w_{,xx}$ is not continuous.

Subtracting Eq. (14) from Eq. (15), introducing Eqs. (16)–(18) and (4)–(11), assuming limits of integration in time $t_1 = 0$, $t_2 = T$, where T is a period of vibrations, and integrating partially the terms with $\delta\ddot{u}$, $\delta\ddot{w}$, $\delta\dot{w}$, we arrive at

$$\begin{aligned}
&\int_0^T \left\{ R^a_{xr}\delta u_r + R^a_{yr}\delta w_r + R^a_{\varphi r}\delta\varphi_r + \int_L (p^a\delta u + q^a\delta w)\, dx \right\} dt \\
&= \int_0^T \left\{ (R^a_{xr}u^-_{r,x} + R^a_{yr}w^-_{r,x} + R^a_{\varphi r}w^-_{r,xx} + u^{a+}_{r,x}R_{xr} + w^{a+}_{r,x}R_{yr} + w^{a+}_{r,xx}R_{\varphi r})\,\delta x_r \right. \\
&\quad + u^a_r\delta R_{xr} + w^a_r\delta R_{yr} + w^a_{r,x}\delta R_{\varphi r} \\
&\quad \left. - \int_L [\kappa^a\kappa\delta k_M + e^a e\delta k_N + u^a\ddot{u}\delta m + w^a\ddot{w}\delta m + w^a\dot{w}\delta c]\, dx \right\} dt\,. \qquad (19)
\end{aligned}$$

All terms in Eq. (19) express the work. For complex damping model, where the quantities have complex form, the work should be formulated in terms of scalar product. The scalar product $\langle f(x), g(x)\rangle$ of the functions f and g, which are generally complex and which belong to the $\overline{L^2}$ space, is defined as

$$\langle f(x), g(x)\rangle = \int_0^L \overline{f(x)}g(x)\, dx\,, \qquad (20)$$

where the symbol $\overline{\cdots}$ denotes complex conjugate. The scalar product has the following property,

$$\langle f(x), g(x)\rangle = \overline{\langle \overline{g(x)}, \overline{f(x)}\rangle}\,. \tag{21}$$

Therefore, the order of functions of Eq. (19) and henceforth in each term is very important. First place in all terms should by occupied by values from the adjoint system and the second place — by the values from the primary structure. The opposite, but consistent order, is also correct.

To transform implicit variations δR, δu_r, δw_r and $\delta\varphi_r$ to explicit form, we use balance equations and virtual work equations for the concentrated mass (compare [6]). The balance equations for the mass (Fig. 2) are

$$P_{xr} - m_m\ddot{u}_m - R_{xr} = 0\,, \tag{22}$$

$$P_{yr} - m_m\ddot{w}_m - R_{yr} = 0\,, \tag{23}$$

$$(P_{xr} - m_m\ddot{u}_m)h_m + P_{\varphi r} - I_m\ddot{\varphi}_m - R_{\varphi r} = 0\,. \tag{24}$$

The forces and the moment in visco-elastic spring can be expressed as

$$R_{xr} = k_{xr}(u_m - u_r) + c_{xr}(\dot{u}_m - \dot{u}_r)\,, \tag{25}$$

$$R_{yr} = k_{yr}(w_m - w_r) + c_{yr}(\dot{w}_m - \dot{w}_r)\,, \tag{26}$$

$$R_{\varphi r} = k_{\varphi r}(\varphi_m - \varphi_r) + c_{\varphi r}(\dot{\varphi}_m - \dot{\varphi}_r)\,. \tag{27}$$

The work equation of adjoint forces on variations of primary displacements has the form

$$\begin{aligned}
&\int_0^T \{(P^a_{xr} - m_m\ddot{u}^a_m)\delta u_m - R^a_{xr}\delta u_r + (P^a_{yr} - m_m\ddot{w}^a_m)\delta w_m - R^a_{yr}\delta w_r \\
&\qquad + (P^a_{\varphi r} - I_m\ddot{\varphi}^a_m)\delta\varphi_m - R^a_{\varphi r}\delta\varphi_r\}\,\mathrm{d}t \\
&= \int_0^T \{[k_{xr}(u^a_m - u^a_r) + c^a_{xr}(\dot{u}^a_m - \dot{u}^a_r)](\delta u_m - \delta u_r) \\
&\qquad + [k_{yr}(w^a_m - w^a_r) + c^a_{yr}(\dot{w}^a_m - \dot{w}^a_r)](\delta w_m - \delta w_r) \\
&\qquad + [k_{\varphi r}(\varphi^a_m - \varphi^a_r) + c^a_{\varphi r}(\dot{\varphi}^a_m - \dot{\varphi}^a_r)](\delta\varphi_m - \delta\varphi_r)\}\,\mathrm{d}t\,.
\end{aligned} \tag{28}$$

Conversely, using variations of primary forces and adjoint displacements we obtain

$$\begin{aligned}
&\int_0^T \{-(\delta m_m\ddot{u}_m + m_m\delta\ddot{u}_m)u^a_m - \delta R_{xr}u^a_r - (\delta m_m\ddot{w}_m + m_m\delta\ddot{w}_m)w^a_m - \delta R_{yr}w^a_r \\
&\qquad - (\delta I_m\ddot{\varphi}_m + I_m\delta\ddot{\varphi}_m)\varphi^a_m - \delta R_{\varphi r}\varphi^a_r\}\,\mathrm{d}t \\
&= \int_0^T \{[\delta k_{xr}(u_m - u_r) + k_{xr}(\delta u_m - \delta u_r) + \delta c_{xr}(\dot{u}_m - \dot{u}_r) + c_{xr}(\delta\dot{u}_m - \delta\dot{u}_r)](u^a_m - u^a_r) \\
&\qquad + [\delta k_{yr}(w_m - w_r) + k_{yr}(\delta w_m - \delta w_r) + \delta c_{yr}(\dot{w}_m - \dot{w}_r) + c_{yr}(\delta\dot{w}_m - \delta\dot{w}_r)](w^a_m - w^a_r) \\
&\qquad + [\delta k_{\varphi r}(\varphi_m - \varphi_r) + k_{\varphi r}(\delta\varphi_m - \delta\varphi_r) + \delta c_{\varphi r}(\dot{\varphi}_m - \dot{\varphi}_r) + c_{\varphi r}(\delta\dot{\varphi}_m - \delta\dot{\varphi}_r)](\varphi^a_m - \varphi^a_r)\}\,\mathrm{d}t\,.
\end{aligned} \tag{29}$$

Subtracting Eq. (29) from Eq. (28) and integrating by parts the terms with $\delta\ddot{u}$, $\delta\dot{u}$, $\delta\ddot{w}$, $\delta\dot{w}$, $\delta\ddot{\varphi}$, $\delta\dot{\varphi}$, we arrive at

$$\begin{aligned}
&\int_0^T (u^a_r\delta R_{xr} - R^a_{xr}\delta u_r + w^a_r\delta R_{yr} - R^a_{yr}\delta w_r + \varphi^a_r\delta R_{\varphi r} - R^a_{\varphi r}\delta\varphi_r)\,\mathrm{d}t \\
&= \int_0^T \{-P^a_{xr}\delta u_m - u^a_m\ddot{u}_m\delta m_m - (u^a_m - u^a_r)[\delta k_{xr}(u_m - u_r) + \delta c_{xr}(\dot{u}_m - \dot{u}_r)] \\
&\qquad - P^a_{yr}\delta w_m - w^a_m\ddot{w}_m\delta m_m - (w^a_m - w^a_r)[\delta k_{yr}(w_m - w_r) + \delta c_{yr}(\dot{w}_m - \dot{w}_r)] \\
&\qquad - P^a_{\varphi r}\delta\varphi_m - \varphi^a_m\ddot{\varphi}_m\delta I_m - (\varphi^a_m - \varphi^a_r)[\delta k_{\varphi r}(\varphi_m - \varphi_r) + \delta c_{\varphi r}(\dot{\varphi}_m - \dot{\varphi}_r)]\}\,\mathrm{d}t\,.
\end{aligned} \tag{30}$$

Introducing Eqs. (30) and (12) into Eq. (19), we finally arrive at the sensitivity operators

$$\begin{aligned}\delta G &= \int_0^T \Big\{ P_{xr}^a \delta u_m + P_{yr}^a \delta w_m + P_{\varphi r}^a \delta\varphi_m + \int_0^L (p^a \delta u + q^a \delta w)\mathrm{d}x \Big\}\, \mathrm{d}t \\ &= \int_0^T \Big\{ \big(u_{r,x}^{a+} R_{xr} + R_{xr}^a u_{r,x}^- + w_{r,x}^{a+} R_{yr} + R_{yr}^a w_{r,x}^- + w_{r,xx}^{a+} R_{\varphi r} + R_{\varphi r}^a w_{r,xx}^-\big)\, \delta x_r \\ &\quad - (u_m^a \ddot{u}_m + w_m^a \ddot{w}_m)\delta m_r - \varphi_m^a \ddot{\varphi}_m \delta I_m \\ &\quad - (u_m^a - u_r^a)\,[\delta k_{xr}(u_m - u_r) + \delta c_{xr}(\dot{u}_m - \dot{u}_r)] \\ &\quad - (w_m^a - w_r^a)\,[\delta k_{yr}(w_m - w_r) + \delta c_{yr}(\dot{w}_m - \dot{w}_r)] \\ &\quad - (\varphi_m^a - \varphi_r^a)\,[\delta k_{\varphi r}(\varphi_m - \varphi_r) + \delta c_{\varphi r}(\dot{\varphi}_m - \dot{\varphi}_r)] \\ &\quad - \int_L [\kappa^a \kappa \delta k_M + e^a e \delta k_N + u^a \ddot{u} \delta m + w^a \ddot{w} \delta m + w^a \dot{w} \delta c]\, \mathrm{d}x \Big\}\, \mathrm{d}t\,. \end{aligned} \tag{31}$$

The operator (31) expresses the variation (3) of the response functional (1) as an explicit function of variations of all design parameters (2). It has quite general form and can be used to many special cases. The arbitrary functions f_1 , f_2 and f_3 in Eq. (1) represent structural response expressed by the displacements of the machine and the function f_4 represents displacements of the base structure. In spite of formal complexity of Eq. (31), all sensitivity operators have similar structure. They consist of two values: one from the solution of adjoint problem and another one from the solution of primary problem. It can be schematically presented as

$$\delta G_1 = \int_0^T A(t)\, B(t)\, \mathrm{d}t\,. \tag{32}$$

The values A and B depend on time and in case of steady-state, harmonic vibrations they can be expressed as a product of amplitude and cosine function

$$A = \hat{A}\cos(\omega t - \phi_A), \qquad B = \hat{B}\cos(\omega t - \phi_B), \tag{33}$$

where ϕ is respective phase angle, ω is circular frequency of vibrations and $\hat{}$ denotes an amplitude. Integrating in time we arrive at the formula expressed by amplitudes

$$\delta G_1 = \hat{A}\hat{B}\cos(\phi_A - \phi_B)\,, \tag{34}$$

which can be applied to the solution (31), thus providing applicable form (35). For brevity we rewrite only the first part of Eq. (31)

$$\delta G = \Big(\hat{u}_{r,x}^{a+}\hat{R}_{xr}\cos(\phi_1 - \phi_2) + \hat{R}_{xr}^a \hat{u}_{r,x}^- \cos(\phi_3 - \phi_4) + \hat{w}_{r,x}^{a+}\hat{R}_{yr}\cos(\phi_5 - \phi_6) + \cdots\Big)\,\delta x_r + \cdots . \tag{35}$$

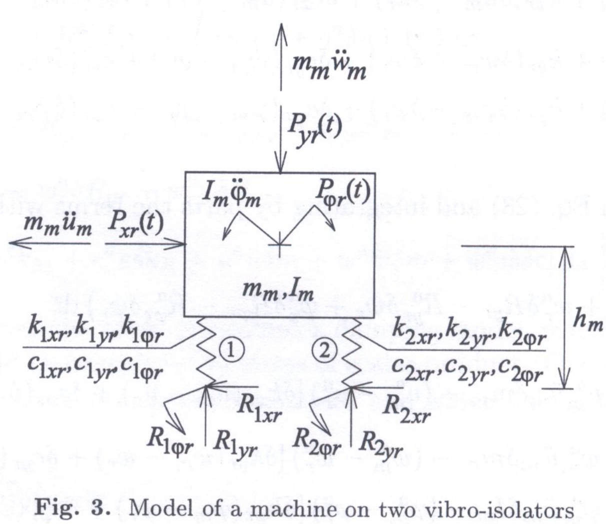

Fig. 3. Model of a machine on two vibro-isolators

This form is very convenient for numerical applications using professional FEM codes, because all quantities can be easily computed. The numerical examples demonstrating application of the derived sensitivity operators are presented in Section 4.

The derived sensitivity operators can be adjusted in a simply way to the case of machine seated on two or more vibro-isolators (Fig. 3). It requires to consider for each vibro-isolator these terms which contain reactions transferred from the machine to the base structure (variation of machine position δx_r) and the terms with variations of stiffness and damping coefficients of visco-elastic joints. In case of i connectors it is a simple summation

$$\begin{aligned}\delta G = \int_0^T \Bigg\{ & \sum_i \Big[\Big(u^{a+}_{ir,x} R_{ixr} + R^a_{ixr} u^-_{ir,x} + w^{a+}_{ir,x} R_{iyr} + R^a_{iyr} w^-_{ir,x} + w^{a+}_{ir,xx} R_{i\varphi r} + R^a_{i\varphi r} w^-_{ir,xx}\Big)\delta x_r\Big] \\ & - (u^a_m \ddot{u}_m + w^a_m \ddot{w}_m)\delta m_m - \varphi^a_m \ddot{\varphi}_m \delta I_m \\ & - \sum_i \{(u^a_{im} - u^a_{ir})[\delta k_{ixr}(u_{im} - u_{ir}) + \delta c_{ixr}(\dot{u}_{im} - \dot{u}_{ir})] \\ & \quad + (w^a_{im} - w^a_{ir})[\delta k_{iyr}(w_{im} - w_{ir}) + \delta c_{iyr}(\dot{w}_{im} - \dot{w}_{ir})] \\ & \quad + (\varphi^a_{im} - \varphi^a_{ir})[\delta k_{i\varphi r}(\varphi_{im} - \varphi_{ir}) + \delta c_{i\varphi r}(\dot{\varphi}_{im} - \dot{\varphi}_{ir})]\} \\ & - \int_L [\kappa^a \kappa \delta k_M + e^a e \delta k_N + u^a \ddot{u} \delta m + w^a \ddot{w} \delta m + w^a \dot{w} \delta c]\, dx \Bigg\}\, dt. \end{aligned} \tag{36}$$

4. Examples

4.1. Example 1

Consider a frame structure excited by vertical concentrated dynamic force $R_{yr} = 1 \cdot \cos(2\pi 5t)$ [kN] (Fig. 4). The frequency of excitation is $f = 5$ [Hz], whereas the first eigenfrequency is equal to 5.9629 [Hz]. The structure is assumed as geometrically linear structure. Technical parameters of the system are as follows: axial stiffness $k_N = 801550$ [kN], bending stiffness $k_M = 7975$ [kNm^2] and

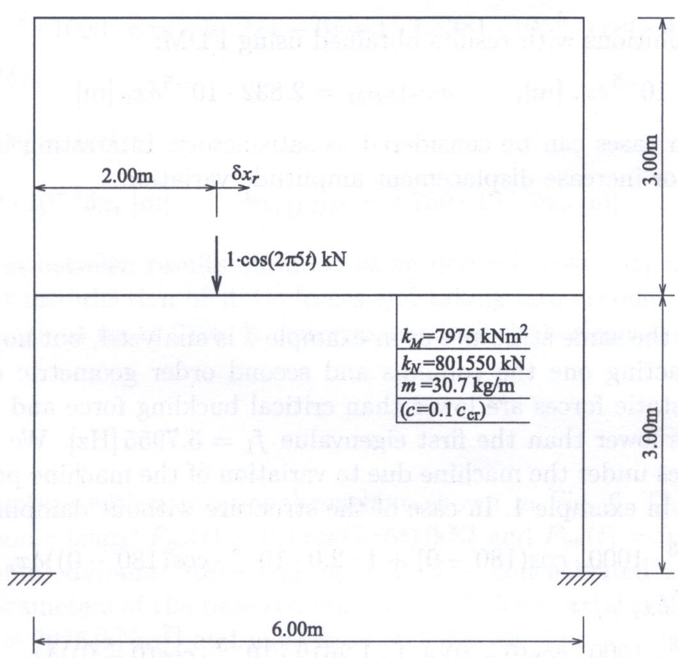

Fig. 4. Frame structure with variable position of dynamic force

mass $m = 30.7$ [kg/m]. Structures without damping and with viscous damping coefficient $c = 0.1c_{cr}$ are analyzed separately.

We are interested in the variation of the horizontal and vertical displacements amplitude due to variation of the force position δx_r. The variations can be found using simplified forms of Eq. (31),

$$\delta G = \int_0^T R_{xr}^a \delta u_r \,\mathrm{d}t = \int_0^T (w_{r,x}^{a+} R_{yr} + R_{xr}^a u_{r,x}^-)\delta x_r \,\mathrm{d}t\,, \tag{37}$$

$$\delta G = \int_0^T R_{yr}^a \delta w_r \,\mathrm{d}t = \int_0^T (w_{r,x}^{a+} R_{yr} + R_{yr}^a w_{r,x}^-)\delta x_r \,\mathrm{d}t\,. \tag{38}$$

To find the variation of horizontal and vertical displacement amplitudes under the force, the adjoint structure is loaded by unitary horizontal and vertical forces. These adjoint forces start in the same phase angles as respective displacements of the primary structure. Henceforth, we denote phase angles in degrees. Application of the sensitivity operators expressed by amplitudes leads to the following calculations in case of a structure without damping,

$$\begin{aligned}\delta\hat{u}_r &= (2.6049 \cdot 10^{-8} \cdot 1000 \cdot \cos(180-0) + 1 \cdot 2.0 \cdot 10^{-8} \cdot \cos(180-0))\delta x_r\\ &= -2.607 \cdot 10^{-5}\delta x_r\,,\end{aligned} \tag{39}$$

$$\begin{aligned}\delta\hat{w}_r &= (1.3550 \cdot 10^{-8} \cdot 1000 \cdot \cos(0-0) + 1 \cdot 1.3550 \cdot 10^{-5} \cdot \cos(0-0))\delta x_r\\ &= 2.710 \cdot 10^{-5}\delta x_r\,.\end{aligned} \tag{40}$$

We compare the results with the total Finite Difference Method, for the perturbations $\delta x_r = 0.05$ [m]. FDM provided respectively,

$$\delta\hat{u}_r|_{FDM} = -2.652 \cdot 10^{-5}\delta x_r \text{ [m]}, \qquad \delta\hat{w}_r|_{FDM} = 2.672 \cdot 10^{-5}\delta x_r \text{ [m]}. \tag{41}$$

Similarly, in case of structure with viscous damping $c = 0.1c_{cr}$ we have

$$\begin{aligned}\delta\hat{u}_r &= (2.2662 \cdot 10^{-8} \cdot 1000 \cdot \cos(179.96-0) + 1 \cdot 2.0 \cdot 10^{-8} \cdot \cos(150.0+30.0))\delta x_r\\ &= -2.268 \cdot 10^{-5}\delta x_r\,,\end{aligned} \tag{42}$$

$$\begin{aligned}\delta\hat{w}_r &= (1.4664 \cdot 10^{-8} \cdot 1000 \cdot \cos(-8.075-0) + 1 \cdot 1.4662 \cdot 10^{-5} \cdot \cos(-4.282-3.793))\delta x_r\\ &= 2.904 \cdot 10^{-5}\delta x_r\,.\end{aligned} \tag{43}$$

We can compare the solutions with results obtained using FDM:

$$\delta\hat{u}_r|_{FDM} = -2.324 \cdot 10^{-5}\delta x_r \text{ [m]}, \qquad \delta\hat{w}_r|_{FDM} = 2.832 \cdot 10^{-5}\delta x_r \text{ [m]}. \tag{44}$$

The agreement in both cases can be considered as satisfactory. Interesting is that introduction of damping can decrease or increase displacement amplitude variation.

4.2. Example 2

In this example almost the same structure as in example 1 is analyzed, but now there are additional static forces 500 [kN] acting one the columns and second order geometric effects are taken into account (Fig. 5). The static forces are lower than critical buckling force and frequency of dynamic excitation $f = 5$ [Hz] is lower than the first eigenvalue $f_1 = 5.7955$ [Hz]. We can find variations of displacement amplitudes under the machine due to variation of the machine position using the same equations (37), (38) as in example 1. In case of the structure without damping we have

$$\begin{aligned}\delta\hat{u}_r &= (3.1935 \cdot 10^{-8} \cdot 1000 \cdot \cos(180-0) + 1 \cdot 2.0 \cdot 10^{-8} \cdot \cos(180-0))\delta x_r\\ &= -3.196 \cdot 10^{-5}\delta x_r\,,\end{aligned} \tag{45}$$

$$\begin{aligned}\delta\hat{w}_r &= (1.2619 \cdot 10^{-8} \cdot 1000 \cdot \cos(0-0) + 1 \cdot 1.2619 \cdot 10^{-5} \cdot \cos(0-0))\delta x_r\\ &= 2.524 \cdot 10^{-5}\delta x_r\,,\end{aligned} \tag{46}$$

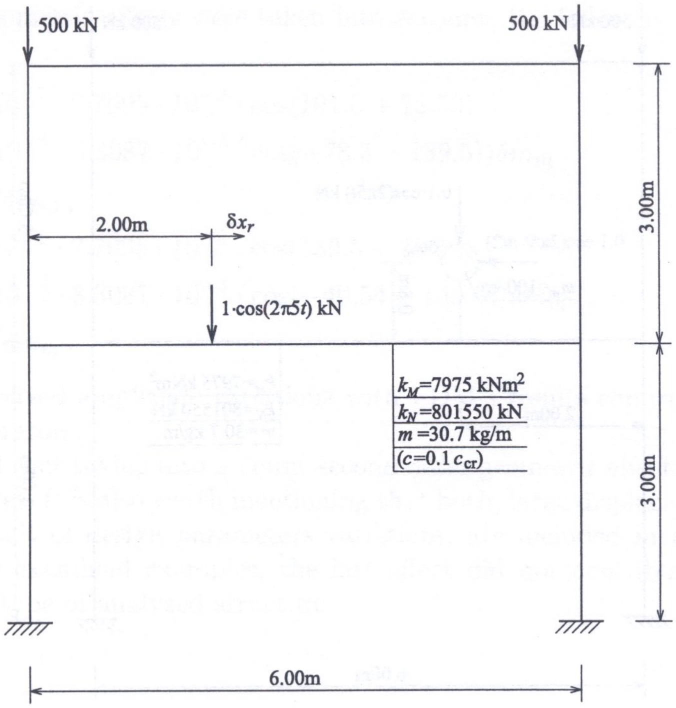

Fig. 5. Geometrically nonlinear frame structure

whereas FDM provides

$$\delta\hat{u}_r|_{FDM} = -3.374 \cdot 10^{-5}\delta x_r \text{ [m]}, \qquad \delta\hat{w}_r|_{FDM} = 2.462 \cdot 10^{-5}\delta x_r \text{ [m]}. \tag{47}$$

Analogically, for structure with damping, the following results were received:

$$\begin{aligned} \delta\hat{u}_r &= (2.6452 \cdot 10^{-8} \cdot 1000 \cdot \cos(180.01 - 0) + 1 \cdot 2.0 \cdot 10^{-8} \cdot \cos(145.5 + 34.5))\delta x_r \\ &= -2.647 \cdot 10^{-5}\delta x_r\,, \end{aligned} \tag{48}$$

$$\begin{aligned} \delta\hat{w}_r &= (1.4381 \cdot 10^{-8} \cdot 1000 \cdot \cos(-11.224 - 0) + 1 \cdot 1.4381 \cdot 10^{-5} \cdot \cos(-4.994 - 6.230))\delta x_r \\ &= 2.820 \cdot 10^{-5}\delta x_r\,. \end{aligned} \tag{49}$$

The FDM provided respectively:

$$\delta\hat{u}_r|_{FDM} = -2.712 \cdot 10^{-5}\delta x_r \text{ [m]}, \qquad \delta\hat{w}_r|_{FDM} = 2.750 \cdot 10^{-5}\delta x_r \text{ [m]}. \tag{50}$$

The relative differences between results obtained using derived sensitivity operators and FDM are about 2.5%. Note that introduction of static forces and taking into account second order geometric effects change the variations up to 27%. It concerns particularly horizontal displacements.

4.3. Example 3

Consider a frame structure with a rotational machine shown in Fig. 6. The machine is subjected to concentrated harmonic loads: $P_{yr}(t) = 0.1\cos(2\pi 5t)$ [kN] and $P_{xr}(t) = 0.1\cos(2\pi 5t - \pi/2)$ [kN]. Note that the horizontal dynamic force lags by $\pi/2$. The concentrated mass of the machine is $m_m = 100$ [kg] and parameters of the base structure are as follows: axial stiffness $k_N = 801550$ [kN], bending stiffness $k_M = 7975$ [kNm²] and mass $m = 30.7$ [kg/m]. The structure without damping is analyzed for two different cases: as a geometrically linear and nonlinear system. Additional static forces 500 [kN] are applied at the top of both columns.

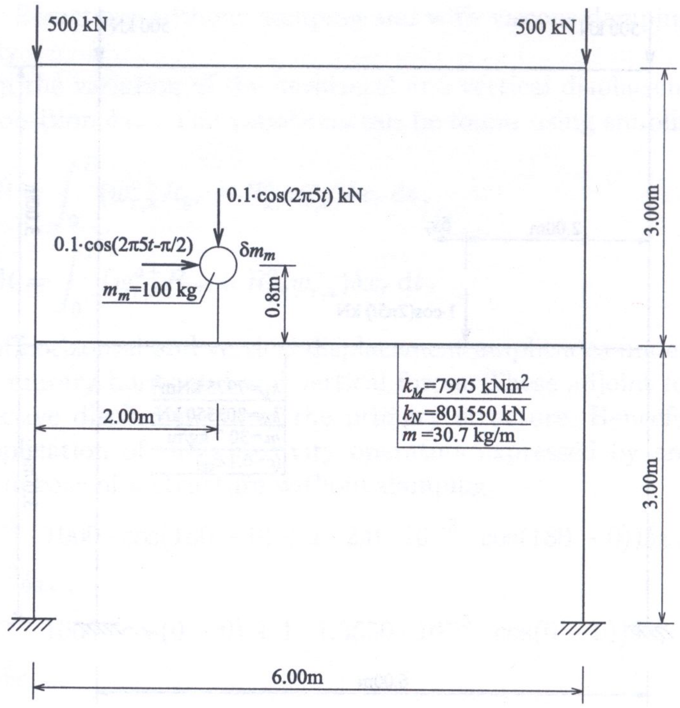

Fig. 6. The frame structure with variable mass of machine

In the example we are looking for the variation of displacement amplitude (horizontal and vertical) of the machine center due to variation of its mass. The variations can be computed using simplified forms of Eq. (31),

$$\delta G = \int_0^T P^a_{xr}\delta u_m \, dt = \int_0^T (-u^a_m \ddot{u}_m - w^a_m \ddot{w}_m)\delta m_m \, dt \,, \tag{51}$$

$$\delta G = \int_0^T P^a_{yr}\delta w_m \, dt = \int_0^T (-u^a_m \ddot{u}_m - w^a_m \ddot{w}_m)\delta m_m \, dt \,. \tag{52}$$

The adjoint structure is loaded by unitary horizontal and vertical force, respectively and simultaneously it is subjected to initial distortions specified by the displacements and strains of the primary structure. The initial phase angle of the adjoint excitation force depends on the phase angle of respective displacement in the primary system.

In case of geometrically linear structure, application of the sensitivity operators (51), (52) expressed in amplitudes gives the following results,

$$\begin{aligned}\delta\hat{u}_m &= -(2.2782\cdot 10^{-7}\cdot 2.2956\cdot 10^{-2}\cdot\cos(101.7+78.34)\\ &\quad +4.7011\cdot 10^{-8}\cdot 7.7352\cdot 10^{-3}\cdot\cos(-78.3-143.1))\delta m_m\\ &= 5.502\cdot 10^{-9}\delta m_m\,,\end{aligned} \tag{53}$$

$$\begin{aligned}\delta\hat{w}_m &= -(4.7011\cdot 10^{-8}\cdot 2.2956\cdot 10^{-2}\cdot\cos(143.1+78.34)\\ &\quad +6.2710\cdot 10^{-8}\cdot 7.7352\cdot 10^{-3}\cdot\cos(-36.86-143.14))\delta m_m\\ &= 1.294\cdot 10^{-9}\delta m_m\,.\end{aligned} \tag{54}$$

When second order geometric effects were taken into account, the following results were obtained, respectively,

$$\begin{aligned}\delta\hat{u}_m &= -(2.6902\cdot 10^{-7}\cdot 2.7095\cdot 10^{-2}\cdot\cos(101.5+78.50)\\ &\quad + 5.4721\cdot 10^{-8}\cdot 8.3087\cdot 10^{-3}\cdot\cos(-78.5-139.5))\delta m_m\\ &= 7.648\cdot 10^{-9}\delta m_m\,,\end{aligned} \tag{55}$$

$$\begin{aligned}\delta\hat{w}_m &= -(5.4721\cdot 10^{-8}\cdot 2.7095\cdot 10^{-2}\cdot\cos(139.5+78.50)\\ &\quad + 6.3974\cdot 10^{-8}\cdot 8.3087\cdot 10^{-3}\cdot\cos(-40.54-139.5))\delta m_m\\ &= 1.700\cdot 10^{-9}\delta m_m\,.\end{aligned} \tag{56}$$

The comparison of received amplitude variations with FDM's results confirmed the correctness of derived sensitivity operators.

It should be noticed that taking into account second order geometric effects dramatically changes displacement amplitudes. It is also worth mentioning that both, large displacement effects and stress redistribution as a result of design parameters variations, are included in the derived sensitivity operators (31). In the examined examples, the last effect did not contribute remarkably to find results because of the type of analyzed structure.

4.4. Example 4

The last example demonstrates complexity of the problem of optimal design of dynamically loaded structures. The system is presented in Fig. 7. Parameters of the base structure are the same as in the previous examples. The vertical excitation force with the frequency 55 [Hz] acts on the concentrated mass connected to the base structure by a spring with the stiffness k.

We are interested in minimizing of the vertical displacements of the mass and the base structure. The solution — the dependence of the displacement amplitudes on the spring stiffness is presented in Fig. 8. For a very stiff spring, displacements of the machine and the base structure are equal to each other. Decreasing the stiffness of the spring, we can reduce vibrations of the machine simultaneously increasing vibrations of the base structure and incidentally we approach resonance. In case of very

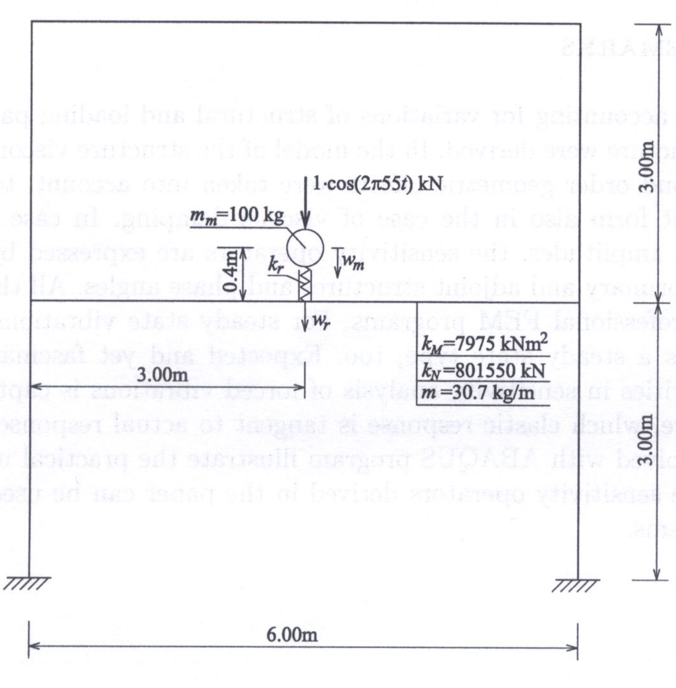

Fig. 7. Frame structure with a machine on the spring foundation

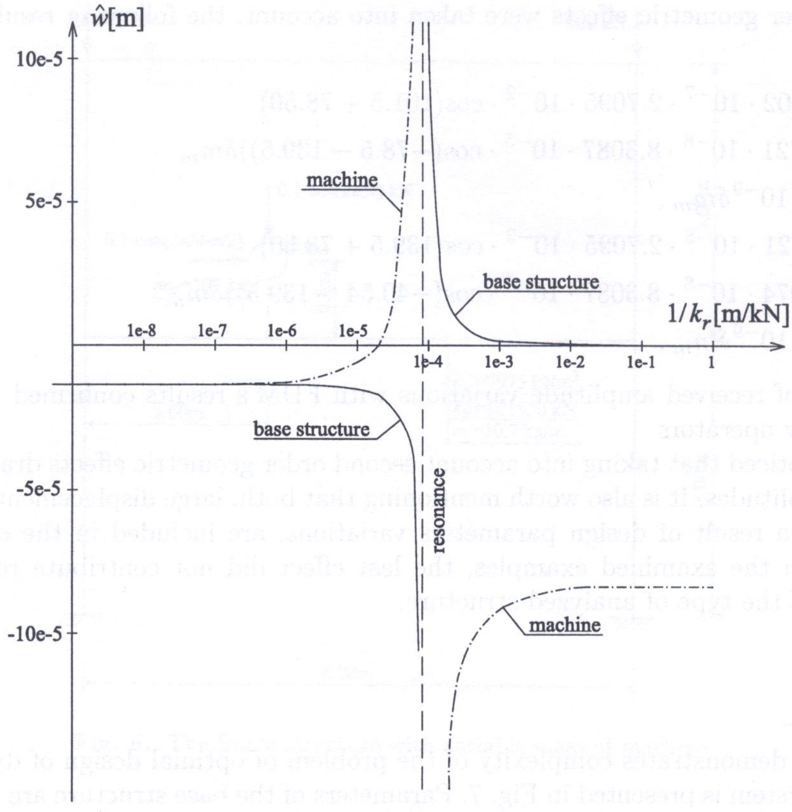

Fig. 8. Vertical displacement amplitude of the machine and the base structure versus compliance of the spring

soft spring, displacements of the base structure tend to zero, while the vibrations of the machine are relatively high. The example shows that the problems of optimal design of dynamic systems can be very interesting.

5. Concluding remarks

Sensitivity derivatives accounting for variations of structural and loading parameters of a dynamically loaded frame structure were derived. In the model of the structure viscous or complex damping were allowed for. Second order geometric effects were taken into account, too. Derived sensitivity operators have explicit form also in the case of viscous damping. In case of objective functions representing vibration amplitudes, the sensitivity operators are expressed by amplitudes of forces and displacements of primary and adjoint structures and phase angles. All these quantities are easily obtainable from professional FEM programs. For steady state vibrations of primary structure the adjoint problem is a steady state type, too. Expected and yet fascinating is that the effect of geometric nonlinearities in sensitivity analysis of forced vibrations is captured by introducing a linear adjoint structure, which elastic response is tangent to actual response of primary structure. Numerical examples solved with ABAQUS program illustrate the practical use of derived formulae and the accuracy. The sensitivity operators derived in the paper can be used in optimal design or in identification problems.

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