Computer Assisted Mechanics and Engineering Sciences, 14: 167–181, 2007.


# Interval and fuzzy eigenfrequency analysis of uncertain substructured FE models

Hilde De Gersem, David Moens, Wim Desmet, Dirk Vandepitte
*Department of Mechanical Engineering, PMA*
*Katholieke Universiteit Leuven, Belgium*

(Received in the final form April 6, 2007)

This paper uses an interval and fuzzy finite element approach for the eigenfrequency analysis of a mechanical structure with uncertain parameters. The component mode synthesis method is applied for the numerical reduction of the structure, in order to reduce the calculation time of the interval and fuzzy analyses. Special attention is paid to the effect of uncertainties on the description of the substructuring technique and the consequences on the calculation time. All concepts are illustrated through a benchmark structure example.

## 1. Introduction

Nowadays, the finite element method (FEM) has become an indispensable tool for the numerical optimisation and validation of structural designs. This numerical method is capable of predicting the static and dynamic behaviour of a structure based on its geometry and material characteristics, the applied loads and constraints. The finite element method allows to adopt a *Virtual Prototyping* concept, as a reliable FE analysis reduces the need for expensive physical prototype production and testing.

However, it is often very difficult to define a reliable finite element model, especially when a number of physical properties are uncertain. For instance material and damping characteristics may be subject to variability, and manufacturing tolerances inevitably cause geometric uncertainties in the physical product. Furthermore, it is often very difficult to model all relevant physical phenomena correctly. For instance, connections and boundary conditions in FE models are often an idealisation of the physical reality. In the early design stage, some property values even may not yet be fixed. Due to the ever growing demands imposed on new products, it is of utmost importance for design engineers to investigate the influence of all present uncertainties on the static and/or dynamic behaviour of structures.

The probabilistic concept is most commonly used for the description of uncertainties in finite element models. This has led to a number of probabilistic finite element procedures [9], of which the Monte Carlo Simulation method is the most complete one. This method is particularly suitable when information on both the range and the probability density function (PDF) of the uncertain parameters in the FE model is available. This condition is met in advanced stages of the design process. However, the use of the Monte Carlo Simulation method can lead to subjective results when not all statistical data are available and assumptions on the PDF have to be made [7].

The concept of fuzzy sets as introduced by Zadeh [17] in 1965 provides an alternative for the description of uncertainties in finite element models. It describes linguistic and therefore incomplete information in a non-probabilistic manner. During the last decade, the use of the fuzzy concept has led to the development of the fuzzy finite element method [15]. Its aim is to calculate the membership function of an output quantity based on the membership functions of the uncertain input parameters. Moens [13, 14] has developed a fuzzy finite element method (FFEM) for the

calculation of fuzzy eigenfrequencies and frequency response functions (FRFs) for structures with uncertain parameters. Using the $\alpha$-level strategy, a hybrid interval finite element method (IFEM) for envelope FRF calculation forms the core of the fuzzy FRF procedure. Various numerical cases prove that the IFEM and the FFEM can be useful tools to perform early design validation and optimisation [4].

For industrially sized models with a large number of uncertainties, the computation time of the IFEM and the FFEM can be considerable. A reduction in calculation time can be achieved by the substructuring of large structures into substructures, which are then independently processed and reduced [1]. Subsequently all reduced substructures are recombined to form the reduced model for which the desired output quantities are calculated. However, the introduction of uncertainties in a substructuring method requires a concept for the description of uncertainty on the substructure level. In this paper, the interval FE methodology is combined with the Craig–Bampton Component Mode Synthesis substructuring technique, in which the static and dynamic behaviour of each substructure are represented by component modes.

Section 2 of this paper describes successively the basic concepts of fuzzy sets and fuzzy arithmetics, the interval and fuzzy eigenfrequency analysis, and the component mode synthesis method. In Section 3, these methods are combined into an interval procedure for the eigenfrequency analysis of a substructured model. Different approaches to handle the influence of uncertainties on substructures are proposed. In Section 4, interval and fuzzy eigenfrequency analyses are performed on two substructured finite element models of the Garteur benchmark aircraft. The accuracy and numerical efficiency of the proposed approaches for handling uncertainties in substructured models are compared.

## 2. Methodology

### 2.1. Fuzzy sets and fuzzy arithmetics

The concept of fuzzy sets, introduced by Zadeh [17] in 1965, has gained an increasing popularity during the last two decades. Its most important property is that it is capable of describing linguistic and therefore incomplete information in a non-probabilistic manner. Whereas a classical set clearly distinguishes between members and non-members, a fuzzy set introduces a degree of membership, represented by the *membership function*. For a fuzzy set $\tilde{x}$, the membership function $\mu_{\tilde{x}}(x)$ describes the grade of membership to the fuzzy set for each element $x$ in the domain $X$,

$$\tilde{x} = \{(x, \mu_{\tilde{x}}(x)) \mid (x \in X)\,(\mu_{\tilde{x}}(x) \in [0,1])\}\,. \tag{1}$$

If $\mu_{\tilde{x}}(x) = 1$, $x$ is definitely a member of the set $\tilde{x}$, whereas if $\mu_{\tilde{x}}(x) = 0$, $x$ is definitely not a member of the set $\tilde{x}$. For all $x$ with $0 < \mu_{\tilde{x}}(x) < 1$, the membership is not certain. The most frequently applied membership function shapes are the triangular and Gaussian shape.

The description of an uncertain parameter using a membership function can practically be implemented using the *$\alpha$-level strategy*. This approach subdivides the membership function range into a number of $\alpha$-levels. The intersection with the membership function of the input uncertainties at each $\alpha$-level results in an interval $x_{i,\alpha}^{I} = [\underline{x}_i\,,\,\overline{x}_i]_\alpha$. With these input intervals of the $\alpha$-sublevel, an interval finite element (IFE) analysis is performed, resulting in an interval for the analysis result at the considered $\alpha$-level. Finally, the fuzzy solution is assembled from the resulting intervals at each sublevel. Figure 1 clarifies this procedure for a function of two triangular fuzzy parameters.

### 2.2. Interval and fuzzy eigenfrequency analysis

Using the $\alpha$-level strategy, it is clear that the interval finite element method forms the core of the fuzzy method. The interval eigenfrequency analysis aims at the calculation of the minimum and maximum values of the eigenfrequencies of all considered modes, given the input uncertainty

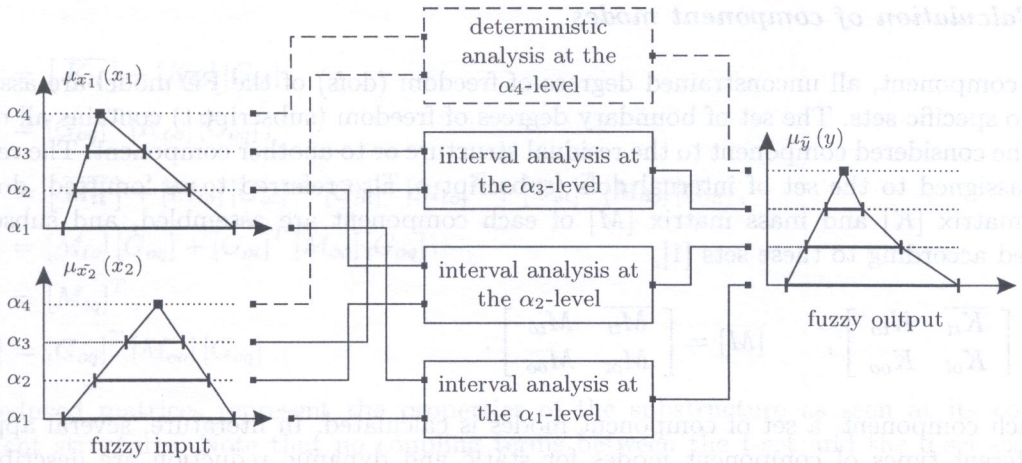



**Fig. 1.** $\alpha$-level strategy with 4 $\alpha$-levels, for a function of two triangular fuzzy parameters

intervals. It can easily be shown that a vertex method, which takes all combinations of the boundaries of the input intervals into account, correctly predicts the intervals of output quantities if the problem is monotonic. However, in the application of structural dynamics, it is almost impossible to prove the property of monotonicity in a general way. Experience shows that many problems do exhibit indeed a monotonic behaviour, especially when global model parameters are uncertain. For parameters that affect only part of the structure, it is much harder to predict the effect of a parameter change. In these cases, a vertex analysis can provide a first impression of the scatter on the eigenfrequency results, but the detection of the worst-case scenarios is not guaranteed. For those cases, it is necessary to apply a global optimisation procedure in search of the minimum and maximum eigenfrequency values. With $\Lambda_i^x$ an explicit eigenvalue function of the input parameters ($\lambda_i = \Lambda_i^x$), the global optimisation problem for mode $i$ can be written as

$$\langle \Lambda_i^x \rangle_{\{\mathbf{x}\}} = \left[ \min_{\{x\} \in \{\mathbf{x}\}} \left( \Lambda_i^x \left( \{x\} \right) \right) , \max_{\{x\} \in \{\mathbf{x}\}} \left( \Lambda_i^x \left( \{x\} \right) \right) \right] . \tag{2}$$

For each mode considered in the interval eigenfrequency analysis, a global minimisation and maximisation is required. As for each goal function evaluation a finite element eigenvalue analysis has to be performed, the interval method becomes very time-consuming for large numerical models. The use of a reduced model can substantially decrease the model evaluation time and hence increase the efficiency of the interval and fuzzy finite element methods drastically. Therefore, this paper aims at the introduction of a substructuring in the algorithm for interval eigenfrequency analysis.

### 2.3. Component mode synthesis method

The aim of the well-known Component Mode Synthesis (CMS) method is to reduce the computational cost of large structures and to enable a solution strategy in which individual components can be optimised without the need of the recalculation of the total structure. The CMS technique consists of three global steps. First the structure is divided into a residual structure and a number of components, also referred to as substructures or superelements. For each component, a set of component modes is calculated, which represent its static and dynamic behaviour. In the second step, these component modes are used to reduce the component model. In the third step, all reduced component models are combined with the non-reduced part of the structure to form the global reduced system, that is further used for global FE analyses.

A short overview of the basic principles and steps of the method is given in this section. A complete mathematical description of the CMS technique can be found in [1, 3, 12].

### *2.3.1. Calculation of component modes*

For each component, all unconstrained degrees of freedom (dofs) of the FE model are assigned to one of two specific sets. The set of boundary degrees of freedom (subscript t) contains all dofs that connect the considered component to the residual structure or to another component. The remaining dofs are assigned to the set of internal dofs (subscript o, also referred to as 'omitted' dofs). The stiffness matrix $[K]$ and mass matrix $[M]$ of each component are assembled, and subsequently partitioned according to these sets [1],

$$[K] = \begin{bmatrix} K_{tt} & K_{to} \\ K_{ot} & K_{oo} \end{bmatrix}, \qquad [M] = \begin{bmatrix} M_{tt} & M_{to} \\ M_{ot} & M_{oo} \end{bmatrix}. \tag{3}$$

For each component, a set of component modes is calculated. In literature, several approaches using different types of component modes for static and dynamic reduction are described. The Craig–Bampton method [1] has been widely used and implemented in many finite element codes because the procedure for formulating the component modes is very straightforward, and because the method produces highly accurate results. The basic and most commonly applied version of the Craig–Bampton method uses constraint modes and fixed-interface normal modes to describe the static, respectively the dynamic behaviour of a component.

Each *constraint mode* describes the static deformation of the component when a unit displacement (translation or rotation) is applied to one boundary degree of freedom, while the other dofs of the t-set are restrained. The static transformation matrix $[G_{ot}]$ with all constraint modes $\{\phi_i^C\}$ as columns, has the following mathematical description [1],

$$[G_{ot}] = [\, \{\phi_1^C\} \; \{\phi_2^C\} \; \cdots \; \{\phi_t^C\} \,] = -[K_{oo}]^{-1}[K_{ot}]\,. \tag{4}$$

The *fixed-interface normal modes*, which form the columns of the dynamic transformation matrix $[G_{oq}]$, are calculated from the eigenvalue analysis of the component with all boundary degrees of freedom fixed,

$$[K_{oo}]\{\phi_i\} = \lambda_i [M_{oo}]\{\phi_i\} \tag{5}$$

$$[G_{oq}] = [\, \{\phi_1\} \; \{\phi_2\} \; \cdots \; \{\phi_q\} \,]\,. \tag{6}$$

Each of the modes is assigned to a generalized degree of freedom (q-set). The accuracy of the dynamic reduction step is determined by the number of retained normal modes $q$. As the number of generalized degrees of freedom $q$ needed for an accurate description of the dynamic behaviour of the component is usually several orders smaller than the number of internal dofs, the numerical dimension of the component is drastically reduced.

### *2.3.2. Component reduction*

In the second step, the set of component modes is used for the transformation $[\Gamma]$ of the FE model from the set of physical dofs into the set of reduced dofs,

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = [\Gamma] \begin{Bmatrix} x_t \\ q \end{Bmatrix} = \begin{bmatrix} [I] & [0] \\ [G_{ot}] & [G_{oq}] \end{bmatrix} \begin{Bmatrix} x_t \\ q \end{Bmatrix}. \tag{7}$$

Using this transformation, the component stiffness and mass matrix are reduced to form the reduced superelement matrices [1, 12],

$$[K]_{\text{reduced}} = [\Gamma]^T[K][\Gamma] = \begin{bmatrix} K_{tt} & 0 \\ 0 & K_{qq} \end{bmatrix}, \qquad [M]_{\text{reduced}} = [\Gamma]^T[M][\Gamma] = \begin{bmatrix} M_{tt} & M_{tq} \\ M_{qt} & M_{qq} \end{bmatrix}, \tag{8}$$

with

$$[K_{tt}] = [\overline{K_{tt}}] + [K_{to}]\,[G_{ot}]\,, \tag{9}$$

$$[K_{qq}] = [G_{oq}]^T\,[K_{oo}]\,[G_{oq}]\,, \tag{10}$$

$$[M_{tt}] = [\overline{M_{tt}}] + [M_{to}]\,[G_{ot}] + [G_{ot}]^T\,[M_{to}]^T + [G_{ot}]^T\,[M_{oo}]\,[G_{ot}]\,, \tag{11}$$

$$[M_{tq}] = [M_{to}]\,[G_{oq}] + [G_{ot}]^T\,[M_{oo}]\,[G_{oq}]\,, \tag{12}$$

$$[M_{qt}] = [M_{tq}]^T\,, \tag{13}$$

$$[M_{qq}] = [G_{oq}]^T\,[M_{oo}]\,[G_{oq}]\,. \tag{14}$$

These reduced matrices represent the properties of the substructure as seen at its connections to adjacent structures. Note that no coupling terms between the t-set and the q-set exist in the reduced stiffness matrix, due to the definition of the static transformation matrix. In case of mass normalised normal modes, the matrices $[K_{qq}]$ and $[M_{qq}]$ simplify to respectively a diagonal matrix with the eigenvalues, and a unity matrix,

$$[K_{qq}] = [\Lambda_{qq}]\,, \qquad [M_{qq}] = [I_{qq}]\,. \tag{15}$$

### *2.3.3. Assembly of the reduced structural system*

In a last step, the reduced stiffness and mass matrices of all components are assembled with the non-reduced residual structure, to form the reduced stiffness and mass matrix of the complete structural system. These can then be used to perform finite element analyses (e.g. an eigenfrequency or frequency response function analysis) on the global structure. Data recovery for each substructure is performed by expanding the solution at the attachment points, using the same transformation matrices that were used to perform the original reduction on the substructure.

## 3. Uncertainty in a substructured model

For the calculation of the eigenfrequency intervals of an uncertain structure using the global optimisation strategy, multiple eigenvalue problems have to be solved (cf. Section 2.2). Hence, the interval finite element method becomes very time-consuming for large systems. In this paper, the IFEM is combined with the component mode synthesis method, in order to reduce the calculation time. The uncertain parameters affecting the structural behaviour of a substructured finite element model can be located either in the non-reduced part of the structure, or in one or more components. Sections 3.1 and 3.2 describe the combination of the interval FE analysis with the substructuring technique for the two uncertainty location cases.

### 3.1. Uncertainties in the residual structure

In case that all uncertain parameters are included in the residual structure, all components can be reduced deterministically, as described in Eqs. (9)–(14). Hence, only the residual structure needs to be recalculated during an interval or fuzzy analysis of the global structure, while the reduced component matrices remain unchanged. In this case, the use of a substructuring technique does not change the implementation of the interval or fuzzy analysis. However, this approach limits the numerical reduction and hence the gain in calculation time that can be achieved, e.g. when large parts of the structure are affected by global uncertainties. In [5] this method was successfully applied for interval eigenfrequency and frequency response function analysis; a reduction of 20% in computation time was achieved.

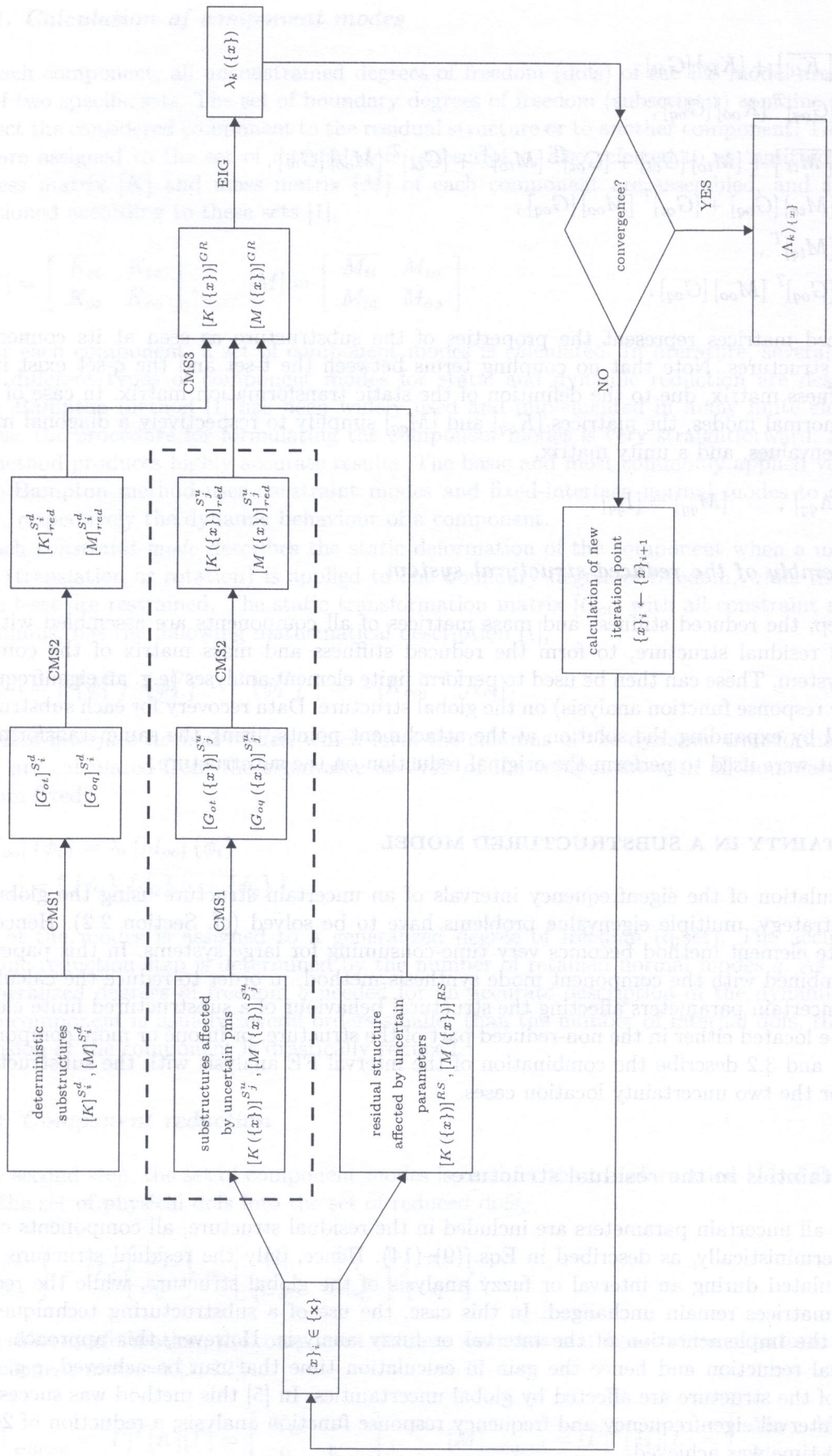



**Fig. 2.** Flow chart for the global optimisation procedure for interval eigenfrequency analysis, in case of substructure uncertainties

### 3.2. Uncertainties on the substructure level

Uncertainties in a substructure of a reduced model affect each step of the reduction procedure: the uncertainties generally affect the constraint modes, the normal modes and corresponding eigenfrequencies, the static and dynamic transformation matrices, and the reduced stiffness and mass matrices. The flow chart of Fig. 2 gives an overview of the global optimisation procedure for the calculation of the $k$-th eigenfrequency interval of the reduced model. In this figure, the dashed rectangle denotes the reduction of the substructures that are affected by the uncertain parameters. This paper introduces three approaches for the implementation of this reduction step: repeated component reduction, approximate component reduction based on component eigenvalue ranges, and first order Taylor series expansion of the transformation matrices. A legend with the notations used in Fig. 2 is given in Table 1.

**Table 1.** Notations used in Fig. 2

| | | |
|---|---|---|
| $\{x\} \in \{\mathbf{x}\}$ | : | vector with values of the uncertain parameters in the design point, inside the uncertainty domain $\{\mathbf{x}\}$ |
| $S_i^d$ | : | $i$-th deterministic substructure |
| $S_j^u$ | : | $j$-th uncertain substructure |
| $RS$ | : | residual structure i.e. non-reduced part of the structure |
| $GR$ | : | global reduced structure |
| $[G_{ot}]$ | : | static transformation matrix |
| $[G_{oq}]$ | : | dynamic transformation matrix |
| $[K]_{red}, [M]_{red}$ | : | reduced stiffness, respectively mass matrix of a substructure |
| CMS1 | : | calculation of component modes and transformation matrices |
| CMS2 | : | calculation of reduced substructure matrices |
| CMS3 | : | assembly of the global reduced structure |
| EIG | : | eigenvalue analysis of the global reduced structure |
| $\lambda_k$ | : | $k$-th eigenvalue of the global reduced structure |

*Repeated component reduction*

A first possible approach to handle uncertainties on the substructure level during an interval eigenfrequency analysis, is to recalculate the component modes, static and dynamic transformation matrices, and reduced matrices of each substructure affected by uncertain parameters, during each goal function evaluation of the global optimisation procedure. As in each iteration step all substructures are correctly reduced, this approach gives the correct eigenfrequency intervals and can hence be used as reference for approximate methods. Although the implementation of this approach is straightforward, it limits the efficiency of the substructuring technique as for each eigenvalue analysis of the global structure, one or more component reduction steps have to be repeated.

*Approximate component reduction based on component eigenvalue ranges*

Another approach to handle uncertainties on the substructure level, is to approximate the description of an uncertain substructure, such that it can represent the effect of the uncertain parameters on the static and dynamic behaviour adequately, and no component recalculation has to be performed during a global interval or fuzzy analysis of the total reduced structural system.

In literature, the approximation of uncertain substructures is most often carried out by presuming that only the eigenvalues of the substructures are affected by the uncertain parameters, hereby assuming that the component modes remain unaffected [10, 11, 16]. For the Craig–Bampton method with mass normalized modes, this means that only the reduced matrix $[K_{qq}]$ is affected, while all other reduced matrices remain constant, cf. Eqs. (9)–(15). Therefore, in the reduction step of the

algorithm, a separate interval eigenfrequency analysis is performed on the substructure in order to calculate the substructure eigenvalue ranges,

$$\langle [\Lambda_{qq}] \rangle_{\{\mathbf{x}\}} = \left[ \min_{\{x\} \in \{\mathbf{x}\}} [\Lambda_{qq}(\{x\})] \, , \max_{\{x\} \in \{\mathbf{x}\}} [\Lambda_{qq}(\{x\})] \right] . \tag{16}$$

After this preliminary interval eigenvalue analysis on the uncertain substructure, the reduced component stiffness matrix $[K_{qq}(\{x\})]$, containing the uncertainties through the calculated eigenvalue ranges, is included in the global reduced FE model. Hence, the uncertainty originally defined on the substructure is translated towards uncertainty affecting directly the reduced structure, and no further component recalculations are required during all global interval FE analyses performed on the total reduced structure, resulting in a decrease of calculation time for these global uncertainty analyses.

*Taylor series expansion of the transformation matrices*

The assumption that component modes such as constraint and normal modes are (almost) not affected by uncertain parameters, is in general cases not valid, e.g. when local uncertainties such as local geometric changes affect the substructure. To include the component modes into an approximate reduction of an uncertain substructure, a concept for the description of uncertain vectors is required.

In this paper, a method is presented to approximate the influence of uncertain parameters on component modes of a substructure using a first order Taylor series expansion of the transformation matrices, as written here for one uncertain parameter $s$,

$$[G_{ot}]_{s_x} = [G_{ot}]_{s_d} + (s_x - s_d) \left. \frac{\partial [G_{ot}]}{\partial s} \right|_{s_d} , \tag{17}$$

$$[G_{oq}]_{s_x} = [G_{oq}]_{s_d} + (s_x - s_d) \left. \frac{\partial [G_{oq}]}{\partial s} \right|_{s_d} , \tag{18}$$

with $s_x$ and $s_d$ the value of the uncertain parameter in the design point, respectively the deterministic case. In these equations, the factor $(s_x - s_d)$ explicitly describes the influence of the uncertain parameter on the transformation matrices, while the deterministic transformation matrices and their respective derivatives preserve the correlation between the degrees of freedom.

As the static transformation matrix has an explicit prescription (Eq. 4), its first derivative is calculated as follows,

$$\frac{\partial [G_{ot}]}{\partial s} = - [K_{oo}]^{-1} \left( \frac{\partial [K_{ot}]}{\partial s} + \frac{\partial [K_{oo}]}{\partial s} [G_{ot}] \right) . \tag{19}$$

The normal modes are calculated with the implicit equation 5. For each normal mode, the first order derivative can be calculated as proposed by Fox and Kapoor [8],

$$\begin{bmatrix} [K_{oo}] - \lambda_i [M_{oo}] & -[M_{oo}] \{\phi_i\} \\ -\{\phi_i\}^T [M_{oo}] & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \{\phi_i\}}{\partial s} \\ \dfrac{\partial \lambda_i}{\partial s} \end{Bmatrix} = \begin{Bmatrix} -\left( \dfrac{\partial [K_{oo}]}{\partial s} - \lambda_i \dfrac{\partial [M_{oo}]}{\partial s} \right) \{\phi_i\} \\ \dfrac{1}{2} \{\phi_i\}^T \dfrac{\partial [M_{oo}]}{\partial s} \{\phi_i\} \end{Bmatrix} . \tag{20}$$

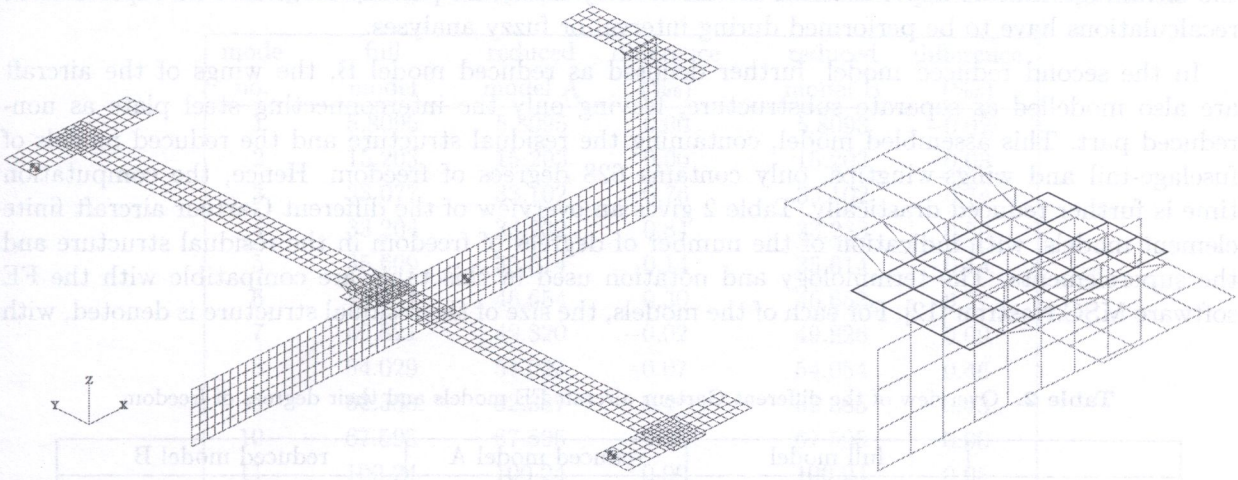
For the calculation of the first order derivatives of the transformation matrices, the first derivatives of the substructure stiffness and mass matrices are required, as can be seen in Eqs. (19) and (20). These can be calculated analytically, based on the finite element description of the substructure, or numerically, e.g. with the finite differences approach.

By using an approximation of the transformation matrices of uncertain substructures, no substructure recalculation has to be performed during a global uncertainty analysis of the total reduced system. However, as only a first order approximation of the component modes is used, the method does not guarantee to reach the exact interval or fuzzy eigenfrequency results. Higher order derivatives of the component modes could lead to more accurate results for largely non-monotonic uncertain parameters and/or large uncertainty intervals.

## 4. Case study: the Garteur benchmark problem

### 4.1. Problem description

The Garteur benchmark problem [6] consists of a small-scale, simplified aluminium aircraft model with a length of 1.5 m, a wing span of 2 m and a mass of 44 $kg$. The fuselage of the aircraft consists of a rectangular plate with a thickness of 50 mm. The tail with a thickness of 10 mm is connected rigidly to the fuselage. The wings are connected to the fuselage through an intermediate steel plate. Wingtips are connected rigidly at both ends of the wings. Both the wings and wingtips are rectangular plates with a thickness of 10 mm. The FE model, as illustrated in Fig. 3(a), contains almost 20000 degrees of freedom.




(a) Global finite element model

(b) Detail of the connection between fuselage and wings

**Fig. 3.** Finite element model of the Garteur benchmark aircraft

The Garteur aircraft model contains some inherent uncertainties, due to a lack of knowledge on the physical model as well as due to uncertainty on the modelling level. Three uncertainties are considered during the dynamic analyses described in this paper:

1. A first source of uncertainty is the thickness of the visco-elastic layer, glued onto a part of the wings. The uncertainty on this thickness ranges between 0.1 and 1.6 mm, with a nominal value of 1.1 mm.
2. A second source of uncertainty is the stiffness of a part of the connection between wings and fuselage – an inherent modelling uncertainty. In the assembled model, this connection is modelled with an interconnecting plate parallel to the wings. In the FE model, the fuselage is connected rigidly to this plate, cf. Fig. 3(b). The translational degrees of freedom of the interconnecting plate are connected rigidly to the wings, except for the dofs on the edge of the plate. These dofs are flexibly connected to the corresponding dofs on the wings using linear springs. This modelling concept results in mode shapes that are very similar to the mode shapes observed during ground vibration tests on the physical structure [6]. The dimension and the stiffness of the connection between the wings and the fuselage can then be varied in a continuous way by changing the stiffness of these springs. In the performed analyses this stiffness ranges between 10 N/m and $10^{15}$ N/m with a nominal value of $10^{8}$ N/m.
3. A third uncertainty is introduced on the Young's modulus of the wing material, with a range of 67.5 GPa up to 68.5 GPa, with nominal value 68.0 GPa.

Two reduced models of the Garteur aircraft have been constructed. In the first model (reduced model A), four parts of the structure are modelled as deterministic substructures: the tail, fuselage and both wingtips. Hence, all uncertain parameters affect only the non-reduced residual structure. The reduced model contains 7380 degrees of freedom (versus 19898 dofs in the full model), which gives a reduction of computation time of 20%. This reduction affects every goal function evaluation in the global optimisation of the aircraft eigenfrequencies; therefore, the same proportional time gain is achieved for the total IFEM and FFEM procedure. By substructuring the Garteur aircraft such that all model parts affected by uncertain parameters are located in the residual structure only, the number of degrees of freedom of this global reduced system remains quite large. In this case, the strength of the component mode synthesis method is not fully exploited, as a model reduction normally leads to a model size of 500 à 1000 degrees of freedom. However, this reduced model has the advantage that no superelements are affected by uncertain parameters, hence no superelement recalculations have to be performed during interval or fuzzy analyses.

In the second reduced model, further denoted as reduced model B, the wings of the aircraft are also modelled as separate substructure, leaving only the interconnecting steel plate as non-reduced part. This assembled model, containing the residual structure and the reduced models of fuselage-tail and wings-wingtips, only contains 628 degrees of freedom. Hence, the computation time is further reduced drastically. Table 2 gives an overview of the different Garteur aircraft finite element models, with indication of the number of degrees of freedom in the residual structure and the superelements. The terminology and notation used in this table are compatible with the FE software MSC.Nastran [12]. For each of the models, the size of the residual structure is denoted, with

**Table 2.** Overview of the different Garteur aircraft FE models and their degrees of freedom

| | full model | reduced model A | reduced model B |
|---|---|---|---|
| number of dofs in the residual structure | 19896 dofs<br>- constrained: 3542<br>- free: 16354 | 7391 dofs<br>- constrained: 1295<br>- free: 6096 | 628 dofs<br>- constrained: 86<br>- free: 542 |
| uncertain superelements | / | / | superelement WINGS<br>- before reduction: 6948<br>boundary: 150<br>internal: 6798<br>- after reduction: 185<br>t-set: 150<br>q-set: 35 |
| deterministic superelements | / | superelement TAIL<br>- before reduction: 2544<br>boundary: 54<br>internal: 2490<br>- after reduction: 78<br>t-set: 54<br>q-set: 24<br>superelement LEFT WINGTIP<br>- before reduction: 1446<br>boundary: 48<br>internal: 1398<br>- after reduction: 60<br>t-set: 48<br>q-set: 12<br>superelement RIGHT WINGTIP<br>- before reduction: 1446<br>boundary: 48<br>internal: 1398<br>- after reduction: 60<br>t-set: 48<br>q-set: 12<br>superelement FUSELAGE<br>- before reduction: 7344<br>boundary: 42<br>internal: 7302<br>- after reduction: 77<br>t-set: 42<br>q-set: 35 | |

the degrees of freedom subdivided into degrees of freedom that are constrained during the analyses, and degrees of freedom that are free to displace/rotate. For both reduced Garteur FE models, the size of the deterministic and uncertain superelements is indicated, before reduction as well as after reduction. Before reduction, the degrees of freedom are subdivided between boundary and internal degrees of freedom. After reduction, the dofs are subdivided according to the t-set (boundary dofs, connected to the constraint modes) and the q-set (generalized dofs, connected to the component normal modes).

Table 3 gives a comparison of the first 14 eigenfrequencies of the full (i.e. non-reduced) model versus the two substructured models, as well as the relative differences per thousand. For both reduced models, the deterministic eigenfrequencies are accurately predicted.

**Table 3.** Nominal eigenfrequency values [Hz] for the non-reduced model and both substructured models

| mode no. | full model | reduced model A | difference (‰) | reduced model B | difference (‰) |
|---|---|---|---|---|---|
| 1 | 5.8095 | 5.8095 | 0.00 | 5.8093 | −0.02 |
| 2 | 15.203 | 15.203 | 0.00 | 15.203 | −0.03 |
| 3 | 33.077 | 33.102 | 0.76 | 33.102 | 0.74 |
| 4 | 33.204 | 33.232 | 0.84 | 33.233 | 0.89 |
| 5 | 35.609 | 35.614 | 0.14 | 35.614 | 0.14 |
| 6 | 46.641 | 46.654 | 0.30 | 46.653 | 0.27 |
| 7 | 49.821 | 49.820 | −0.02 | 49.826 | 0.09 |
| 8 | 54.029 | 54.025 | −0.07 | 54.054 | 0.46 |
| 9 | 62.358 | 62.387 | 0.47 | 62.385 | 0.43 |
| 10 | 67.595 | 67.595 | 0.00 | 67.595 | 0.00 |
| 11 | 100.24 | 100.24 | 0.00 | 100.24 | 0.05 |
| 12 | 128.81 | 128.89 | 0.62 | 128.89 | 0.59 |
| 13 | 137.44 | 137.77 | 2.40 | 137.77 | 2.34 |
| 14 | 150.85 | 151.04 | 1.26 | 151.04 | 1.24 |

### 4.2. Interval eigenfrequency analysis with uncertain residual structure

An interval eigenfrequency analysis with the three above mentioned uncertain parameters has been performed on the reduced model A, i.e. with all uncertain parameters in the residual structure. Table 4 gives the resulting eigenfrequency intervals of the first 14 modes for the reduced model, compared to the results of the full model obtained using the global optimisation strategy. The results prove that the use of a substructuring technique with all uncertain parameters in the residual structure, enhances the efficiency of the IFEM without compromising the accuracy of the interval results. During the analyses, it was noticed that the uncertainty on the thickness of the visco-elastic layer has a non-monotonic influence on the eigenfrequencies. Consequently, the global optimisation strategy was required to find the correct extrema on the eigenfrequencies.

### 4.3. Interval and fuzzy eigenfrequency analysis with uncertain substructures

An interval and fuzzy eigenfrequency analysis have been performed on the reduced model B, with two uncertain parameters affecting the wing substructure. An uncertainty interval [0.6–1.6] mm is taken on the thickness of the visco-elastic layer, and an uncertainty interval [67.5–68.5] GPa on the Young's modulus of the wing material. For the interval eigenfrequency analysis, the three approaches introduced in Section 3.2 have been applied. Following the first approach, the uncertain substructure is recalculated for each goal function evaluation of each optimisation procedure. This approach gives

**Table 4.** Eigenfrequency intervals [Hz] and percentage width of the intervals (%) of the first 14 modes of the Garteur aircraft model, for the full model and the reduced model A

| | full model | | reduced model A | |
|---|---|---|---|---|
| mode no. | frequency interval | interval width (%) | frequency interval | interval width (%) |
| 1 | 5.4886–5.9137 | 7.75 | 5.4885–5.9137 | 7.75 |
| 2 | 14.727–15.365 | 4.33 | 14.727–15.365 | 4.33 |
| 3 | 32.417–33.379 | 2.97 | 32.442–33.404 | 2.97 |
| 4 | 32.485–33.532 | 3.22 | 32.512–33.560 | 3.22 |
| 5 | 35.263–35.766 | 1.43 | 35.267–35.772 | 1.43 |
| 6 | 43.926–47.804 | 8.83 | 43.937–47.818 | 8.83 |
| 7 | 49.269–50.007 | 1.50 | 49.269–50.008 | 1.50 |
| 8 | 53.482–54.347 | 1.62 | 53.476–54.345 | 1.63 |
| 9 | 59.361–63.617 | 7.17 | 59.382–63.647 | 7.18 |
| 10 | 67.583–67.603 | 0.03 | 67.583–67.603 | 0.03 |
| 11 | 100.10–100.33 | 0.23 | 100.10–100.33 | 0.23 |
| 12 | 126.26–129.58 | 2.63 | 126.37–129.64 | 2.59 |
| 13 | 129.50–141.12 | 8.97 | 129.77–141.47 | 9.02 |
| 14 | 143.89–153.91 | 6.97 | 143.99–154.14 | 7.05 |

**Table 5.** Comparison of the interval eigenfrequency results [Hz] of the three approaches

| | repeated component reduction | | approximate component reduction, based on substr. eigenvalue ranges | | first order Taylor series expansion of transformation matrices | |
|---|---|---|---|---|---|---|
| mode no. | frequency interval | interval width (%) | frequency interval | interval width (%) | frequency interval | interval width (%) |
| 1 | 5.7783–5.8533 | 1.29 | 5.7798–5.8518 | 1.24 | 5.7869–5.8618 | 1.29 |
| 2 | 15.160–15.272 | 0.74 | 15.140–15.293 | 1.01 | 15.176–15.287 | 0.73 |
| 3 | 32.963–33.241 | 0.84 | 32.957–33.246 | 0.87 | 33.144–33.423 | 0.84 |
| 4 | 33.086–33.381 | 0.89 | 33.083–33.383 | 0.90 | 33.273–33.570 | 0.89 |
| 5 | 35.533–35.722 | 0.53 | 35.537–35.721 | 0.52 | 35.575–35.767 | 0.54 |
| 6 | 46.422–46.987 | 1.21 | 46.423–46.986 | 1.21 | 46.618–47.189 | 1.22 |
| 7 | 49.723–49.925 | 0.41 | 49.703–49.946 | 0.48 | 49.739–49.938 | 0.40 |
| 8 | 53.831–54.276 | 0.82 | 53.813–54.283 | 0.87 | 53.834–54.278 | 0.82 |
| 9 | 62.151–62.733 | 0.93 | 62.155–62.717 | 0.90 | 62.348–62.961 | 0.98 |
| 10 | 67.590–67.600 | 0.01 | 67.590–67.599 | 0.01 | 67.590–67.600 | 0.01 |
| 11 | 100.19–100.30 | 0.11 | 100.19–100.29 | 0.10 | 100.19–100.30 | 0.11 |
| 12 | 128.65–129.18 | 0.41 | 128.64–129.19 | 0.43 | 128.85–129.36 | 0.40 |
| 13 | 136.93–138.88 | 1.41 | 136.96–138.85 | 1.37 | 137.58–139.55 | 1.43 |
| 14 | 150.51–151.79 | 0.85 | 150.42–151.88 | 0.98 | 151.00–152.32 | 0.87 |

the correct eigenfrequency results for the reduced model, which will be used as reference. In the two other approaches, the physical uncertainty of thickness of the visco-elastic layer and the Young's modulus of the wing material is transferred to the representation of the uncertain substructure by its reduced matrices. In the second approach, only the range of eigenvalues corresponding to the fixed-interface normal modes of the wing substructure are taken into account. In the third approach, a first order approximation of the transformation matrices is used for the calculation of the reduced matrices of the wing substructure in each of the goal function evaluations. For the derivatives of the system matrices in Eqs. (19)–(20), numerical derivatives are used. Table 5 lists the eigenfrequency intervals as calculated with the three approaches.

For the approach using only the superelement eigenvalue ranges, a non-conservative result i.e. an underestimation of the eigenfrequency interval width is detected for modes 1, 5, 6, 9, 10, 11 and 13. For the approach using a first order Taylor series expansion of the component modes, the width of the interval eigenfrequency results is more accurately predicted. However, all eigenfrequency results are estimated slightly too high. This phenomenon is due to the fact that the proposed approach with Taylor series expansion on the component modes, is a Rayleigh–Ritz method with approximated 'assumed-modes'. For this type of analyses, it can be proven that the estimated eigenvalues are an upper bound to the corresponding correct eigenvalues [2].

The largest difference in results between the methods occurs for the second eigenfrequency, for which the approximate component reduction based on the component eigenvalue ranges gives a relatively large overestimation of the interval width. In this case, by only considering the influence of the component eigenvalue ranges, the method overestimates the influence of the uncertain parameters on the second eigenfrequency of the reduced assembly. Taking the first order Taylor approximation on the component modes into account better approximates the real influence of the uncertain parameters on the second eigenfrequency of the assembly. This difference between the approximate methods clearly shows that the uncertain parameters do have an influence on the component modes. This can be explained by the fact that the visco-elastic layer only partially covers the wings, hence the influence of the uncertainty on the thickness of this layer can be expected to be locally, thereby changing the shape of the fixed-interface normal modes of the wing substructure. As the second eigenfrequency of the aircraft corresponds to an eigenmode that is dominated by the first anti-symmetric bending of the wings, the influence of the local thickness variation of the visco-elastic layer is clearly visible.

For the reduced model B, a fuzzy eigenfrequency analysis has been performed. The membership function used for the input uncertain parameters have a triangular shape, as indicated in Fig. 4. Figure 5 gives the fuzzy results for eigenmodes 1 and 2 as calculated with the third approach. The fuzzy plots give a clear indication of the influence of the width of the input uncertainty interval on the eigenfrequency intervals of the global structure. In Fig. 6 the membership functions of the eigenfrequency of mode 11, as calculated with the three proposed methods for substructure uncertainty handling, are compared. The membership functions calculated with the repeated component

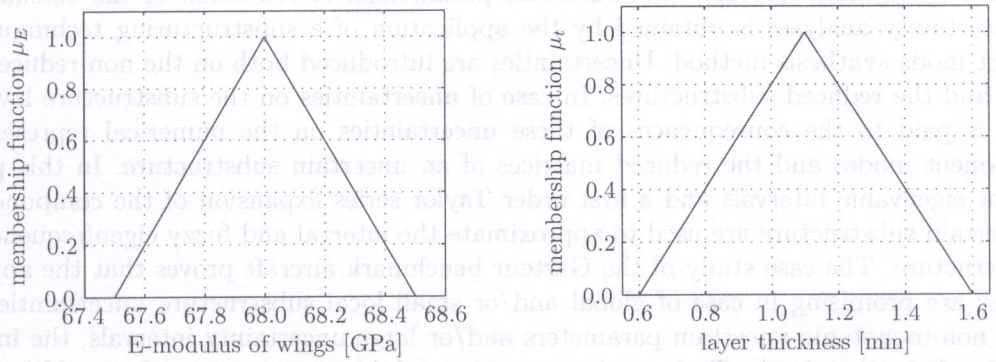



**Fig. 4.** Triangular membership functions for the input uncertain parameters

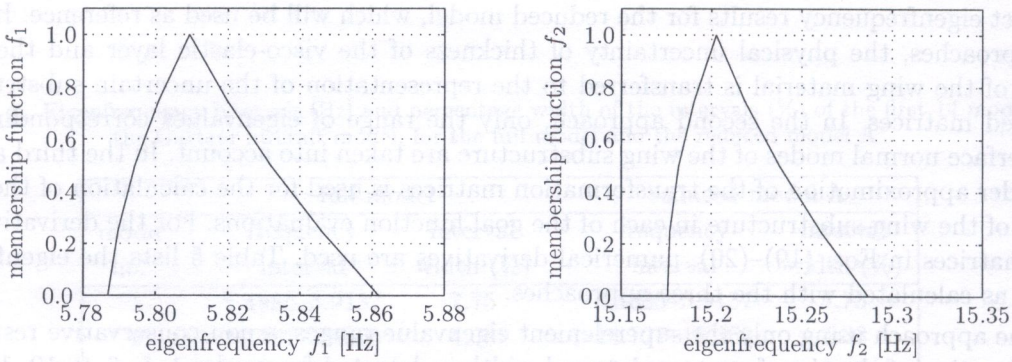



**Fig. 5.** Membership function of the 1st and 2nd eigenfrequency of the Garteur aircraft

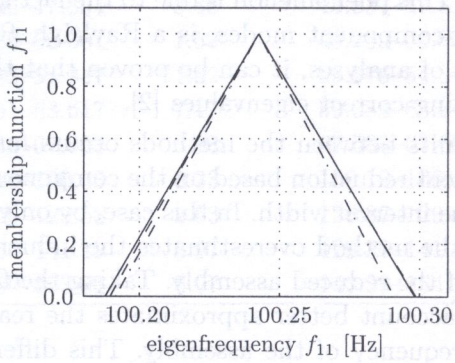



**Fig. 6.** Membership function of the 11-th eigenfrequency of the Garteur aircraft. Solid lines: repeated component reduction; dashed lines: approximate component reduction based on Taylor series expansions; dash-dotted lines: approximate component reduction based on component eigenvalue ranges.

reduction approach (solid lines) and the approximate method with Taylor series expansions of the component modes (dashed lines) almost coincide, while the membership function calculated with the approximate method that takes only the eigenvalue ranges of the uncertain wing substructure into account (dash-dotted lines) is clearly non-conservative.

## 5. Conclusions

This paper uses the interval and fuzzy finite element method for the eigenfrequency analysis of the Garteur benchmark aircraft with uncertain parameters. A reduction of the calculation time of the uncertainty analyses is obtained by the application of a substructuring technique, i.e. the component mode synthesis method. Uncertainties are introduced both on the non-reduced residual structure and the reduced substructures. In case of uncertainties on the substructure level, special attention is paid to the consequences of these uncertainties on the numerical representation of the component modes and the reduced matrices of an uncertain substructure. In this paper, the component eigenvalue intervals and a first order Taylor series expansion of the component modes of an uncertain substructure are used to approximate the interval and fuzzy eigenfrequencies of the reduced structure. The case study of the Garteur benchmark aircraft proves that the approximate approaches are promising in case of global and/or small local substructure uncertainties. In case of largely non-monotonic uncertain parameters and/or large uncertainty intervals, the inclusion of higher order derivatives in the Taylor series expansions of the component modes could lead to more accurate results.

## ACKNOWLEDGEMENTS

H. De Gersem is Research Assistant of the Research Foundation – Flanders (F.W.O.-Vlaanderen).
D. Moens is Postdoctoral Fellow of the Research Foundation – Flanders (F.W.O.-Vlaanderen).

## REFERENCES

[1] R.R. Craig, M.C.C. Bampton. Coupling of substructures for dynamic analyses. *AIAA Journal*, **6**(7): 1313–1319, 1968.

[2] R.R. Craig, Jr. *Structural Dynamics: An Introduction to Computer Methods.* John Wiley & Sons, Inc., New York, first edn., 1981.

[3] R.R. Craig, Jr. A review of time-domain and frequency-domain component mode synthesis methods. *Journal of Modal Analysis*, **2**(2): 59–72, 1987.

[4] H. De Gersem, D. Moens, W. Desmet, D. Vandepitte. A fuzzy finite element procedure for the calculation of uncertain frequency response functions of damped structures: Part 2 - numerical case studies. *Journal of Sound and Vibration*, **288**(3): 463–486, 2005.

[5] H. De Gersem, D. Moens, W. Desmet, D. Vandepitte. Interval finite element analysis of large structures with uncertain parameters. In: *Proceedings of the third M.I.T. Conference on Computational Fluid and Solid Mechanics*, 179–183. Cambridge, 2005.

[6] M. Degener, M. Hermes. Ground vibration test and finite element analysis of the Garteur SM-AG19 testbed. Tech. Rep. IB 232-96 J 08, DLR - German Aerospace Research Establishment, Institute for Aeroelasticity, 1996.

[7] I. Elishakoff. Essay on uncertainties in elastic and viscoelastic structures: from A.M. Freudenthal's criticism to modern convex modelling. *Computers & Structures*, **56**(2): 871–895, 1995.

[8] R.L. Fox, M.P. Kapoor. Rates of change of eigenvalues and eigenvectors. *AIAA Journal*, **6**(12): 2426–2429, 1968.

[9] A. Haldar, S. Mahadevan. *Reliability Assessment using Stochastic Finite Element Analysis.* John Wiley & Sons, Inc., New York, first edn., 2000.

[10] B. Lallemand, A. Cherki, T. Tison, P. Level. Fuzzy modal finite element analysis of structures with imprecise material properties. *Journal of Sound and Vibration*, **220**(2): 353–364, 1999.

[11] B.R. Mace, P.J. Shorter. A local modal/perturbational method for estimating frequency response statistics of built-up structures with uncertain properties. *Journal of Sound and Vibration*, **242**(5): 793–811, 2001.

[12] The MacNeal–Schwendler Corporation, Los Angeles. *MSC.Nastran superelement user's guide*, 2001.

[13] D. Moens, D. Vandepitte. An interval finite element approach for the calculation of envelope frequency response functions. *International Journal for Numerical Methods in Engineering*, **61**(14): 2480–2507, 2004.

[14] D. Moens, D. Vandepitte. A fuzzy finite element procedure for the calculation of uncertain frequency response functions of damped structures: Part 1 - procedure. *Journal of Sound and Vibration*, **288**(3): 431–462, 2005.

[15] S.S. Rao, P. Sawyer. Fuzzy finite element approach for the analysis of imprecisely defined systems. *AIAA Journal*, **33**(12): 2364–2370, 1995.

[16] B. Van den Nieuwenhof. *Stochastic finite elements for elastodynamics: random field and shape uncertainty modelling using direct and modal perturbation-based approaches.* Ph.D. thesis, Université Catholique de Louvain, Louvain-La-Neuve, Belgium, 2003.

[17] L. Zadeh. Fuzzy sets. *Information & Control*, **8**(3): 338–353, 1965.