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The application of the differential quadrature method based on a piecewise polynomial to the vibration analysis of geometrically nonlinear beams

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The paper deals with the application of the differential quadrature method based on a piecewise polynomial to the nonlinear vibration analysis of beams. The initial-boundary-value problem is solved to study the computational stability of the method. The results are compared with those, obtained by the conventional differential quadrature. The effects of the spline degree, the number of nodes and the distribution of sampling points on the convergence and stability is also presented. The nonlinear free vibration analysis is carried out to verify the accuracy of the method.

1. Introduction

In recent years, a significant development of methods based on the differential quadrature (DQ) has taken place. It is a result of high efficiency and high accuracy provided by these methods and their simple formulations. A lot of papers, concerning some applications of the DQ methods to various mechanical problems have appeared [3]. Some modifications of the DQ methods are still introduced in order to improve their accuracy and computational stability.

The DQ methods are some special types of the finite difference method. They belong to so-called global numerical techniques. In one-dimensional case, they are equivalent to the finite difference method of the highest order of accuracy. Using high order approximation, they take into account not only the nearest neighborhood of a sampling point to approximate the solution at this point but all computational domain. It makes these methods very efficient, particularly for nonlinear problems, where the interaction of the function values far from each other in entire domain can be much stronger then for linear problems.

Higher efficiency of the DQ methods then the finite element and the finite difference method was shown in [1, 2, 4, 8, 15]. A great advantage of the DQ methods in comparison with other global techniques (i.e. Rayleigh–Ritz, Galerkin) is that the trial functions do not need to satisfy boundary conditions for the problems considered and the DQ methods require much less formulation effort.

Very important feature that conditions the use of a method in engineering problems is its computational stability. In most problems of computational mechanics, the only way to estimate the accuracy of results is to carry out the computation again using larger number of nodes. The conventional DQ method is very sensitive to the number of sampling points. When this number is too large the method shows instability. It is because the wanted solution of a differential equation is approximated by the interpolation polynomial. It is well known that the degree of this polynomial depends on the number of discrete points and the interpolation polynomial of a high degree is subjected to unfavorable oscillations near the boundaries. The use of the special type of the mesh with points concentrated near the boundaries is not a panacea. Under these circumstances the established approximation leads to great inaccuracy. It significantly limits possibilities of the application of the

DQ method, especially, when the only way to verify the accuracy of the solution is to repeat the computation with larger number of nodes.

To overcome this drawback, Zhong [26] and Guo and Zhong [11] used B-splines as the trial functions in the DQ and successfully solved the problem of bending and buckling of Kirchhoff plates and the problem of the vibration analysis of geometrically nonlinear beams. The results indicate that the proposed approximation improves the stability of the method but the rate of convergence is less comparing to the method based on the interpolation polynomial. Another approach that improves the stability is to divide the domain into small subdomains and use the DQ rules in each of them [25].

In [12], a piecewise polynomial of any degree was used to approximate the wanted solution of a governing equation. The application of symbolic computing allowed easily to determine the weighting coefficients for this type of approximation. The method was examined on the example of the vibration analysis of rectangular plates. The results obtained show that by increasing the number of nodes the accuracy improves without concern for loosing computational stability. The use of the polynomial of suitably high degree makes the rate of convergence higher. The approach presented in [12] gives also accurate results using the uniform distribution of discrete points but the rate of convergence is weaker then. The method is effective even in the problems where the conventional DQ fails [13].

The features mentioned above, particularly the computational stability and the accuracy comparable with the polynomial-based DQ, allow to suppose that the method based on spline functions considered as a piecewise polynomial will be effective in solving problems of nonlinear mechanics. To examine it, in the present paper, the spline-based differential quadrature method aided by symbolic computing (symbolic spline-based differential quadrature — SSDQ) is applied to the vibration analysis of geometrically nonlinear beams. The initial–boundary-value problem is solved, where the special attention is focused on the stability of the system of ordinary differential equations, resulting from the DQ discretization. To verify the accuracy of the method, the fundamental frequencies of the beams are determined.

The drawbacks of the method based on the interpolation polynomial, mentioned above and confirmed in this paper, force to look for new ideas based on the DQ. One of them is the conception presented in [12] and used here for the nonlinear analysis.

2. Spline-based differential quadrature method aided by symbolic computing

The idea of the DQ method was proposed by Bellman and his associates [1] in the early seventies as an analogy to the integral quadrature. In the DQ method, spatial derivatives from a governing equation are approximated by a linear weighed sum of all function values from the domain, what can be put in the form

$$\left.\frac{\mathrm{d}^r f(x)}{\mathrm{d}x^r}\right|_{x=x_i} = \sum_{j=1}^{N} a_j^{(r)}(x_i)\, f(x_j) = \sum_{j=1}^{N} a_{ij}^{(r)} f_j\,, \qquad i = 1,\ldots,N, \tag{1}$$

where N denotes the number of grid points and $a_{ij}^{(r)}$ are the weighting coefficients of the r-th order derivative.

Applying the governing equation with the derivatives described by Equation (1) at each interior point of the domain and implementing boundary conditions, one obtains a set of algebraic or ordinary differential equations, depending on the case considered. A key stage of the method is to determine the weighting coefficients. These coefficients depend on the way the wanted solution is approximated. Therefore, they influence the convergence, accuracy and stability of the method. Often, the solution is approximated by the interpolation polynomial (polynomial based differential quadrature — PDQ) and the use of the Lagrange base functions allowed to derive the explicit formulas for the weighting coefficients for the first [17] and higher order derivatives [23]. The PDQ method ensures very high accuracy, when the appropriate grid point distribution is imposed. Unfortunately, although the grid with points strongly concentrated near the boundaries is used, the method may fail. Particularly nonlinear problems are very sensitive to the way of the DQ discretisation by their nature.

In the present paper, the DQ method based on a piecewise polynomial of any degree is applied to the nonlinear analysis of beams. The problem is undertaken to examine the efficiency of the method (rate of convergence, accuracy), computational stability and possibilities of using various grid distributions in the application to nonlinear mechanics.

According to the SSDQ method, the solution is searched in the following form,

$$f(x) \approx \{s_i(x),\ x \in [z_i\,,\ z_{i+1}],\ i = 1, \dots, N-1\}\,, \tag{2}$$

where N is the number of nodes and the i-th spline section $s_i(x)$ can be written as

$$s_i(x) = \sum_{j=0}^{n} c_{ij} x^j\,. \tag{3}$$

When the degree n of the polynomial is odd, coordinates z_i in Eq. (2) agree with the nodes of imposed mesh x_i but when n is even they are defined as follows,

$$z_1 = x_1\,, \qquad z_{i+1} = \frac{1}{2}\,[x_i + x_{i+1}], \quad i = 1, \dots, N{-}1, \qquad z_{N+1} = x_N\,. \tag{4}$$

The introduction of the auxiliary points (spline knots) defined by Eq. (4) enables to meet conditions, required for the determination of the interpolation function given by Eq. (2), in the case of an even spline degree.

In order to determine the interpolation function, the coefficients c_{ij} have to be calculated. To this end, $(n{+}1)\times(N{-}1)$ equations are required, when the spline degree is odd and $(n{+}1)\times N$ equations for an even spline degree. The detailed description of these equations (interpolation conditions, continuity conditions of the derivatives, natural end conditions) is presented in [12].

The calculated coefficients c_{ij} can be expressed as a function of grid points and unknown values of the solution at these points $f_1 \dots, f_N$,

$$c_{ij} = \sum_{k=1}^{N} C_{ijk}(x_1, \dots, x_N)\, f_k\,, \qquad i = 1, \dots, \bar{N}, \quad j = 0, \dots, n, \tag{5}$$

where $\bar{N} = N - 1$ when n is odd and $\bar{N} = N$ when n is even.

In order to determine the weighting coefficients for the DQ one should calculate a suitable order derivative for the interpolation function (2) at each node, what can be expressed as

$$f^{(r)}(x_i) \approx \begin{cases} s_i^{(r)}(x_i), & i = 1, \dots, \bar{N}, \\ s_{i-1}^{(r)}(x_i), & i = N \quad \text{when } n \text{ is odd.} \end{cases} \tag{6}$$

In Eq. (6), the derivatives $s_i^{(r)}(x_i)$ and $s_{N-1}^{(r)}(x_N)$ can be written as

$$\begin{aligned} s_i^{(r)}(x_i) \quad &= \sum_{j=r}^{n}\left[\left(\sum_{k=1}^{N} C_{ijk} f_k\right) x_i^{j-r} \prod_{l=j-r+1}^{j} l\right], \qquad i = 1, \dots, \bar{N}, \\ s_{N-1}^{(r)}(x_N) &= \sum_{j=r}^{n}\left[\left(\sum_{k=1}^{N} C_{N-1\,jk} f_k\right) x_N^{j-r} \prod_{l=j-r+1}^{j} l\right], \end{aligned} \tag{7}$$

where the advantage has been taken of Eq. (5).

Changing the summation indicators, Eq. (7) can be put in the following form,

$$
\begin{aligned}
s_i^{(r)}(x_i) &= \sum_{k=1}^{N}\left[\sum_{j=r}^{n}\left(C_{ijk}x_i^{j-r}\prod_{l=j-r+1}^{j} l\right)\right] f_k\,, \qquad i=1,\ldots,N,\\
s_{N-1}^{(r)}(x_N) &= \sum_{k=1}^{N}\left[\sum_{j=r}^{n}\left(C_{N-1\,jk}x_N^{j-r}\prod_{l=j-r+1}^{j} l\right)\right] f_k\,.
\end{aligned}
\tag{8}
$$

Comparing with Eq. (1), it is easy to notice that the expressions in the square brackets are the weighting coefficients for the r-th order derivative ($r < n-1$) in the DQ method based on the spline interpolation.

The lack of the explicit formulas for the weighting coefficients is a drawback of the method. These coefficients can be determined with the use of symbolic–numeric computing, where the unknown function values are noted as symbols. Then, the weighting coefficients are obtained by separating the numbers from symbols f_k in Eq. (8). It is well known that the efficiency of symbolic–numeric computing is significantly weaker then pour numeric. However, the main advantage of the DQ method is that the accuracy is very high using relatively few sampling points. Under these conditions, less efficiency of symbolic computing is not a serious problem.

3. Initial–boundary-value problem of a geometrically nonlinear beam

The nonlinear vibrations of beams have been analyzed using various procedures, including analytical, perturbation, finite element and finite deference methods. An extensive literature survey for the problem was given by Sathyamoorthy [19, 20] The DQ method has been also applied to this problem [8, 11]. Both these papers use the DQ discretisation to determine the fundamental frequency of the system.

In the present section, the initial–boundary-value problem is considered. The DQ method based on the spline function, described in the previous section, is applied to discretize the governing equation for the vibration of a geometrically nonlinear beam with immovable ends. In the system, the nonlinearity is due to stretching of the neutral axis caused by large-amplitude oscillations. It makes the strain–displacement relationship become nonlinear in mathematical model of the beam. According to [24], the governing equation for a such vibrating beam has the following form,

$$
m\,\frac{\partial^2 w}{\partial t^2} + EI\,\frac{\partial^4 w}{\partial x^4} - N\,\frac{\partial^2 w}{\partial x^2} = 0,
\tag{9}
$$

where w and m represent the transverse displacement and the mass density per unit length, respectively, E denotes Young's modulus and I is the moment of inertia.

Upon neglect of the axial inertia force, the dynamic axial force N can be expressed in the form

$$
N(x,t) = EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right],
\tag{10}
$$

where u is the axial displacement and A is the cross-section area.

Taking into account that the ends are axially immovable, i.e. $u(0,t) = u(L,t) = 0$, the axial force depends only on time and Eq. (10) is simplified to

$$
N(t) = \frac{EA}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x.
\tag{11}
$$

In the paper, the dimensionless form of Eq. (9) is analyzed,

$$\frac{\partial^2 v}{\partial \tau^2} + \frac{\partial^4 v}{\partial \xi^4} - \kappa \left(\int_0^1 \left(\frac{\partial v}{\partial \xi} \right)^2 \mathrm{d}\xi \right) \frac{\partial^2 v}{\partial \xi^2} = 0, \tag{12}$$

where

$$v = \frac{w}{L}, \qquad \xi = \frac{x}{L}, \qquad \tau = \sqrt{\frac{EI}{m}} \frac{t}{L^2}, \qquad \kappa = \frac{AL^2}{2I}. \tag{13}$$

The boundary conditions are as follows,

$$v(0) = \frac{\partial^p v}{\partial \xi^p}(0) = v(1) = \frac{\partial^q v}{\partial \xi^q}(1) = 0. \tag{14}$$

For $p = 2$ and $q = 2$ both ends are simply supported, when $p = 1$ and $q = 1$ the ends are clamped and for $p = 2$, $q = 1$ one end is simply supported and the other end is clamped.

Applying the DQ rules to discretize the spatial derivatives in Eq. (12), one obtains

$$\frac{\mathrm{d}^2 v_i}{\mathrm{d}\tau^2} + \sum_{j=2}^{N-1} a_{ij}^{(4)} v_j - \kappa \left[\sum_{k=1}^{N} C_k \left(\sum_{r=2}^{N-1} a_{kr}^{(1)} v_r \right) \left(\sum_{s=2}^{N-1} a_{ks}^{(1)} v_s \right) \right] \sum_{j=2}^{N-1} a_{ij}^{(2)} v_j = 0, \qquad i = 3, \dots, N-2, \tag{15}$$

where $a_{ij}^{(r)}$ are the weighting coefficients for the r-th order derivative, obtained using the spline function. In the above equation, C_k are the coefficients derived by the Newton–Cotes integration formulas.

Equation (15) is applied at $N - 4$ interior points due to the direct substitution of the boundary conditions (14) into the above equation. The procedure is described in detail in [22]. The boundary conditions are also written using the DQ rules,

$$v_1 = 0, \qquad\qquad v_N = 0, \tag{16a}$$

$$\sum_{j=2}^{N-1} a_{1j}^{(p)} v_j = 0, \qquad \sum_{j=2}^{N-1} a_{Nj}^{(q)} v_j = 0. \tag{16b}$$

The first two conditions (16a) are imposed by appropriate setting of summation limits in Eqs. (15) and (16b). The remaining conditions (16b) are used to determine the function values at the points adjacent to the boundaries in terms of the interior points (v_i, $i = 3, \dots, N-2$). To this end, the set of Eqs. (16b) is solved with respect to the values v_2 and v_{N-1}, which gives

$$v_2 = \frac{1}{D} \sum_{j=3}^{N-2} E_j v_j, \qquad v_{N-1} = \frac{1}{D} \sum_{j=3}^{N-2} F_j v_j, \tag{17}$$

where

$$D = a_{N2}^{(q)} a_{1N-1}^{(p)} - a_{12}^{(p)} a_{NN-1}^{(q)},$$

$$E_j = a_{1j}^{(p)} a_{NN-1}^{(q)} - a_{1N-1}^{(p)} a_{Nj}^{(q)},$$

$$F_j = a_{12}^{(p)} a_{Nj}^{(q)} - a_{1j}^{(p)} a_{N2}^{(q)}.$$

Introducing the values (17) into Eq. (15), $N - 4$ function values v_i at the interior points ($i = 3, \dots, N-2$) have to be determined.

The obtained set of nonlinear ordinary differential equations (15) in time domain can be solved by the use of various numerical techniques. Although there are attempts to employ the DQ method to the temporal discretisation [9, 10], in the present paper, the set of equations is solved by the

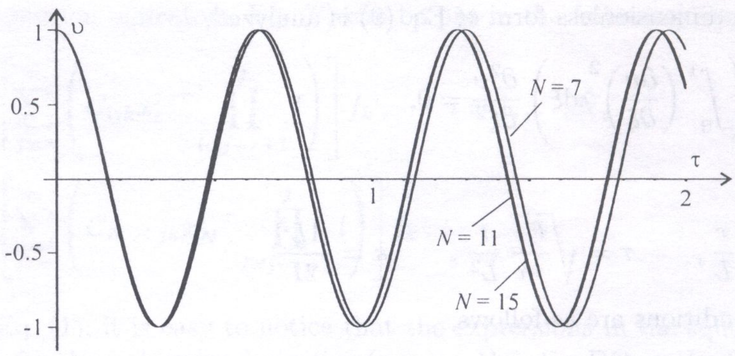

Fig. 1. Vibration of the middle point of the beam by the SSDQ (uniform grid)

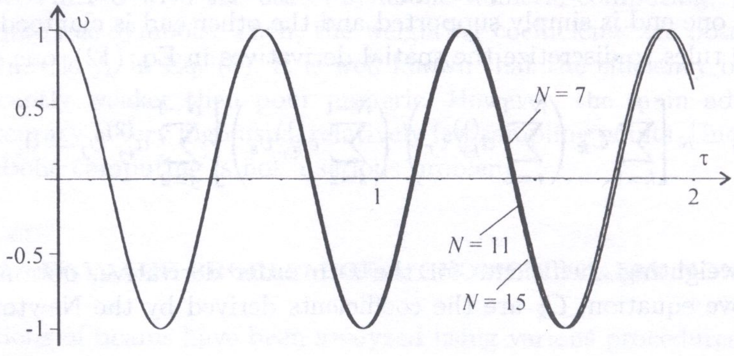

Fig. 2. Vibration of the middle point of the beam by the SSDQ (Chebyshev–Gauss–Lobatto grid)

fourth order Runge–Kutta scheme with controlled step of integration [7]. It ensures a bounded range of integration error.

Figures 1 and 2 present the time history of the oscillation of the middle point of the simply supported beam as a dependence of the number of nodes applied to the spatial discretization. The curves from Fig. 1 are effects of using the uniform grid distribution to discretize the spatial variable and the curves from Fig. 2 — the effects of using the Chebyshev–Gauss–Lobatto grid distribution

$$\xi_i = \frac{1}{2}\left[1 - \cos\left(\frac{i-1}{N-1}\cdot\pi\right)\right], \qquad i = 1, \ldots, N. \tag{18}$$

In both cases, the first form of the linear vibrations is taken as a initial displacement of the beam.

When the numbers of nodes are large enough, the curves in Fig. 1 as well as Fig. 2 are in very well agreement. It allows to be sure that the numbers of nodes applied guarantee acceptable accuracy. It is easy to notice that the rate of convergence of the method is higher when the grid with the points concentrated near the boundaries (18) is used. Figure 3 presents similar time history but as a dependence on the spline degree n used in the determination of the weighting coefficients.

In Fig. 3 one can notice that the spline degree has a significant influence on the rate of convergence.

The numerical simulations are also carried out using the weighting coefficients obtained on the basis of the interpolation polynomial. The detailed description and the formulas for the conventional DQ can be found in [21]. The oscillations of the middle point of the simply supported beam, resulting from this approximation, are shown in Fig. 4. In this case, $N = 9$ and $N = 11$ nodes are used according to the Chebyshev–Gauss–Lobatto pattern to discretize spatial variable. It is known that the imposition of the uniform grid distribution makes the results very inaccurate in the PDQ.

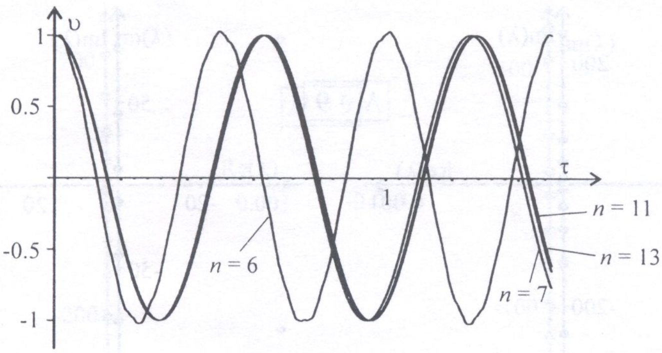

Fig. 3. Vibration of the middle point of the beam by the SSDQ (uniform grid)

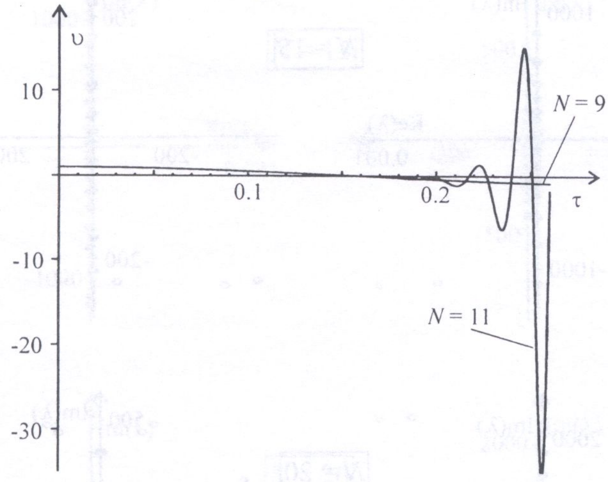

Fig. 4. Vibration of the middle point of the beam by the PDQ

Although the integration in time domain is carried out by the numerical scheme with controlled step of integration to ensure a bounded error of the process, the solution diverges. It indicates that the conventional DQ discretisation is the reason of the instability. Too many sampling points make much oscillation of the solution. This fact limits possibilities of the application of the PDQ method, especially, when the only way to verify the results is to thicken the mesh.

More precise analysis of the stability of the solution can be done by observing the eigenvalues of a linear set of equations which corresponds with the nonlinear system given by Eq. (15). This linear set of equations, written in the state space, has the following matrix form,

$$\dot{\mathbf{q}} = \mathbf{B}\,\mathbf{q}, \tag{19}$$

where the state vector $\mathbf{q}$ and matrix $\mathbf{B}$ are as follows,

$$\mathbf{q} = \begin{bmatrix} \boldsymbol{v} \\ \dot{\boldsymbol{v}} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{A} & \mathbf{0} \end{bmatrix}. \tag{20}$$

In the above formula, $\boldsymbol{v}$ denotes the vector that contains the function values v_i at the interior points ($i = 3, \ldots, N-2$) and $\dot{\boldsymbol{v}}$ is its derivative with respect to time. $\mathbf{A}$ is the matrix of the N−4-th order that contains the weighting coefficients for the fourth order derivative and is modified by the direct substitution of the boundary conditions into the governing equation. $\mathbf{I}$ and $\mathbf{0}$ are the identity and zero matrices of the $N-4$ order.

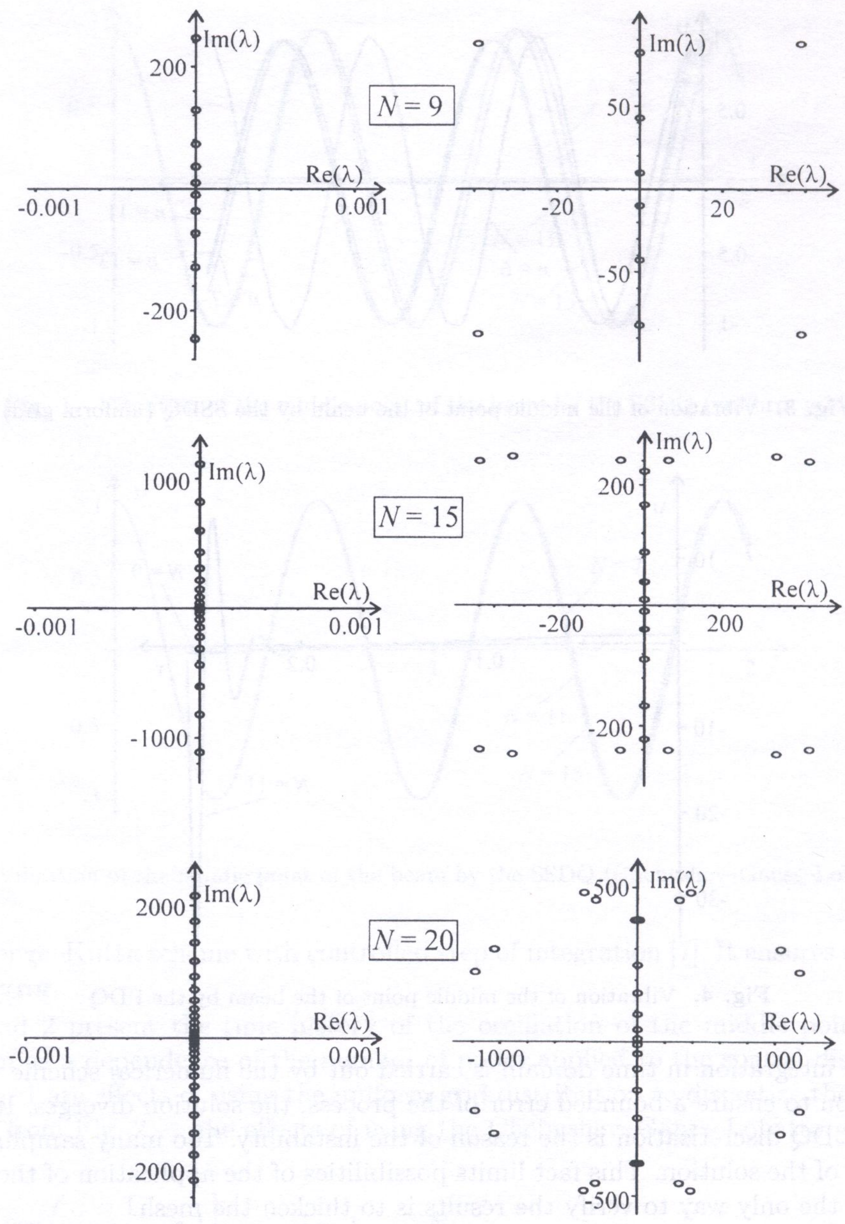

Fig. 5. Eigenvalue distribution (uniform grid): left column – SSDQ, right column – PDQ

Since the solution of Eq. (19) can be expressed as

$$\mathbf{q} = \sum_{i=1}^{2(N-4)} d_i \mathbf{u}^{(i)} e^{\lambda_i \tau} \tag{21}$$

where $\mathbf{u}^{(i)}$ are the eigenvectors, λ_i are the eigenvalues of matrix $\mathbf{B}$ and d_i denote constant values defined by the initial state of the system, the real parts of λ_i determine the stability of the system (15).

The matrix $\mathbf{B}$ in Eq. (19) possesses $N-4$ pairs of complex conjugate eigenvalues in general. If one at least eigenvalue has a positive real part then the solution is unstable. The calculations are done using the weighting coefficients obtained on the basis of the spline function of the eleventh degree and the interpolation polynomial. Two patterns of grid distribution are introduced (uniform and Chebyshev–Gauss–Lobbato). The dependence of obtained results on the number of nodes is presented on the complex plane (Figs. 5, 6).

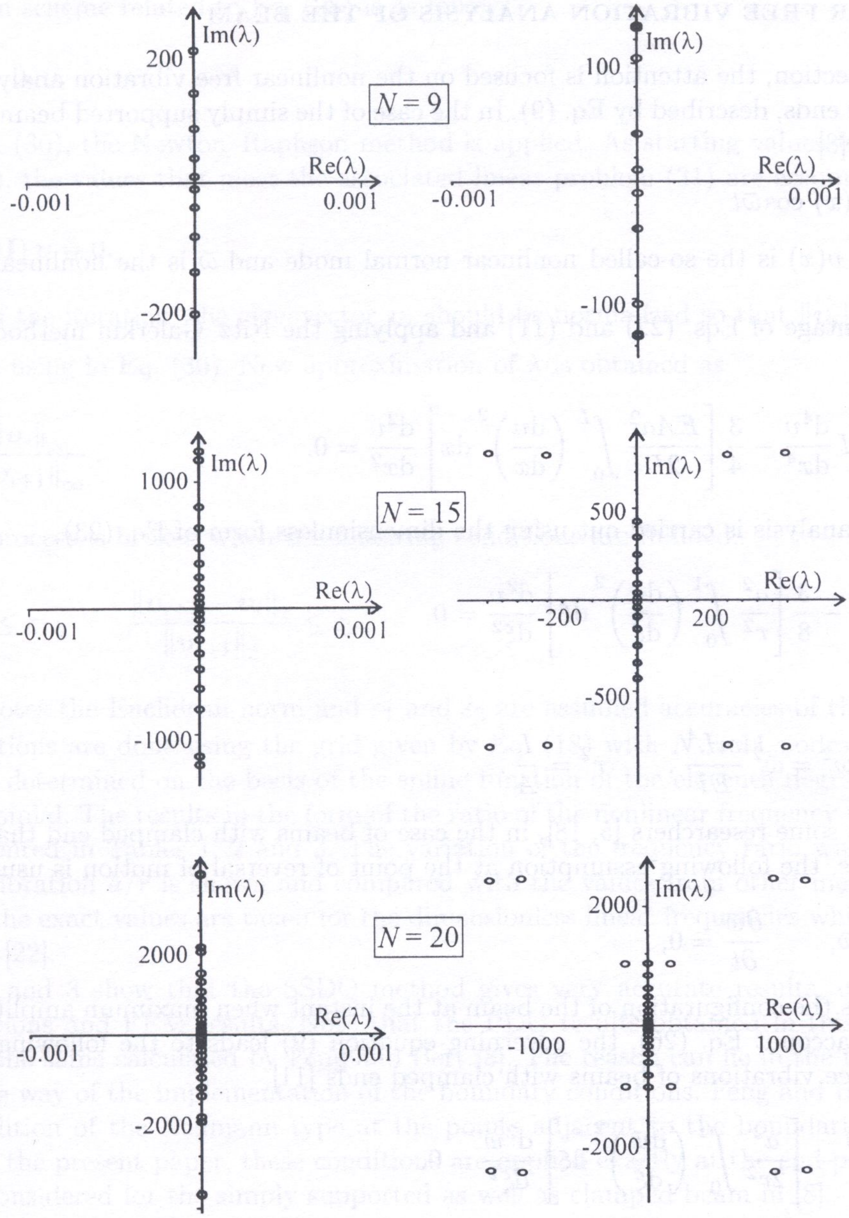

Fig. 6. Eigenvalue distribution (Chebyshev–Gauss–Lobatto grid): left column – SSDQ, right column – PDQ

Figures 5 and 6 show that the PDQ method is very sensitive to the number of sampling points and the imposition of the special grid does not improve significantly the stability of the method. Therefore, the application of the PDQ method to the nonlinear (15) or associated linear (19) problem is risky undertaking even though the method is very accurate if the number of nodes is small enough. The SSDQ method acts differently. Regardless of the way the nodes are distributed and their number, the real parts of the eigenvalues are not positive. It means that the method can be successfully applied to the linear problem (19). According to the stability theory of the first approximation, a further analysis is needed to verify the stability of the nonlinear system (15), when the real parts of the eigenvalues are zeros. The numerical simulations presented in Figs. 1, 2 and 3 confirm that the solution does not diverge in this case.

The similar computations are done for other types of the boundary conditions mentioned in Eq. (14) and the same conclusion can be drawn.

4. Nonlinear free vibration analysis of the beam

In the present section, the attention is focused on the nonlinear free vibration analysis of the beam with immovable ends, described by Eq. (9). In the case of the simply supported beam it is reasonable to assume that [8]

$$w(x,t) = av(x)\cos\bar{\omega}t \tag{22}$$

where quantity $v(x)$ is the so-called nonlinear normal mode and $\bar{\omega}$ is the nonlinear free vibration frequency.

Taking advantage of Eqs. (22) and (11) and applying the Ritz–Galerkin method, Eq. (9) takes the form [24]

$$-\bar{\omega}^2 mv + EI\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} - \frac{3}{4}\left[\frac{EAa^2}{2L}\int_0^L \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 \mathrm{d}x\right]\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = 0. \tag{23}$$

The further analysis is carried out using the dimensionless form of Eq. (23),

$$-\omega^2 v + \frac{\mathrm{d}^4 v}{\mathrm{d}\xi^4} - \frac{3}{8}\left[\frac{a^2}{r^2}\int_0^1 \left(\frac{\mathrm{d}v}{\mathrm{d}\xi}\right)^2 \mathrm{d}\xi\right]\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} = 0, \tag{24}$$

where

$$\xi = \frac{x}{L}, \qquad \omega^2 = \bar{\omega}^2\frac{mL^4}{EI}, \qquad r^2 = \frac{I}{A}. \tag{25}$$

According to some researchers [5, 18], in the case of beams with clamped end that oscillate with large amplitudes, the following assumption at the point of reversal of motion is usually done,

$$\frac{\partial^2 \bar{w}}{\partial t^2} = -\bar{\omega}^2\bar{w}, \qquad \frac{\partial \bar{w}}{\partial t} = 0, \tag{26}$$

where $\bar{w}$ denotes the configuration of the beam at the instant when maximum amplitude is reached.

Taking into account Eq. (26), the governing equation (9) leads to the following expression for the nonlinear free vibrations of beams with clamped ends [11],

$$-\omega^2\bar{w} + \frac{\mathrm{d}^4\bar{w}}{\mathrm{d}\xi^4} - \left[\frac{a^2}{2r^2}\int_0^1 \left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\xi}\right)^2 \mathrm{d}\xi\right]\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\xi^2} = 0, \tag{27}$$

where ξ, ω^2 and r^2 are given in Eq. (25).

Equations (24), (27) and the associated boundary conditions (14) are discretized with the aid of the SSDQ method resulting in nonlinear algebraic set of equations, which can be presented in the following matrix form,

$$\mathbf{F}(\boldsymbol{v},a) = \lambda\boldsymbol{v}, \tag{28}$$

where $\lambda = \omega^2$, $\boldsymbol{v}$ is the nodal values vector (with elements v_i or $\bar{w}_i$) and $\mathbf{F}(\boldsymbol{v},a)$ is the vector whose elements are nonlinear functions of the elements of $\boldsymbol{v}$ and the parameter a (the amplitude of vibration at the point p – the point of the maximum deflection of the beam).

In order to determine the fundamental frequency of the beam, Eq. (28) has to be solved. To this end, the vector iteration method, described in detail in [14], is used. It is assumed that following conditions are fulfilled,

$$\mathbf{F}(0,a) = 0, \qquad \|\boldsymbol{v}\|_\infty = \max_j v_j = v_p = 1, \qquad \mathbf{G}(\boldsymbol{v},a) = \frac{\partial\mathbf{F}}{\partial\boldsymbol{v}}, \tag{29}$$

and the $\mathbf{G}(\boldsymbol{v},a)$ matrix is positive definite.

The iteration scheme related to Eq. (28) is as follows,

$$\mathbf{F}(\boldsymbol{v}_{i+1}, a) = \lambda_i \boldsymbol{v}_i \,. \tag{30}$$

To solve Eq. (30), the Newton–Raphson method is applied. As starting values for the nonlinear eigenpair $(\lambda, \boldsymbol{v})$, the values that meet the associated linear problem (31) are assumed

$$(\mathbf{G}(\mathbf{0}, a) - \lambda \mathbf{I})\, \boldsymbol{v} = 0. \tag{31}$$

In each cycle of the iteration, the eigenvector $\boldsymbol{v}_i$ should be normalized so that $\|\boldsymbol{v}_i\|_\infty = \max_j v_{j,i} = v_{p,i} = 1$, before using in Eq. (30). New approximation of λ is obtained as

$$\lambda_{i+1} = \lambda_i \frac{\|\boldsymbol{v}_i\|_\infty}{\|\boldsymbol{v}_{i+1}\|_\infty} \,. \tag{32}$$

The iteration process is broken when the following conditions are fulfilled,

$$\frac{|\lambda_{i+1} - \lambda_i|}{\lambda_{i+1}} \le \varepsilon_1 \,, \qquad \frac{\|\boldsymbol{v}_{i+1} - \boldsymbol{v}_i\|_2}{\|\boldsymbol{v}_{i+1}\|_2} \le \varepsilon_2 \,, \tag{33}$$

where $\|\cdot\|_2$ denotes the Euclidean norm and ε_1 and ε_2 are assumed accuracies of the calculation.

The calculations are done using the grid given by Eq. (18) with $N = 11$ nodes. The weighting coefficients are determined on the basis of the spline function of the eleventh degree and the interpolation polynomial. The results in the form of the ratio of the nonlinear frequency to the linear one ω/ω_L are presented in Tables 1, 2 and 3. The variation of the frequency ratio with dimensionless amplitude of vibration a/r is shown and compared with the values from other methods. It should be noted that the exact values are taken for the dimensionless linear frequencies which can be found for example in [22].

Tables 1, 2 and 3 show that the SSDQ method gives very accurate results, comparable with the exact solutions and FEM results. Note that the PDQ results obtained in this work are more accurate then the same calculated by Feng and Bert [8]. The reason can lie in the number of nodes applied and the way of the implementation of the boundary conditions. Feng and Bert imposed the boundary condition of the Neumann type at the points adjacent to the boundaries (see Eq. (18) in [8]) while in the present paper, these conditions are applied exactly at the end points. Moreover, Eq. (24) was considered for the simply supported as well as clamped beam in [8].

Table 1. Ratio of the nonlinear frequency to the linear frequency ω/ω_L for simply supported beam

a/r	Analytical [6]	SSDQ (present, $N = 11$)	SDQ [11]	PDQ (present, $N = 11$)	PDQ [8]	FEM [16]
0.1	1.0009	1.0009	1.0009	1.0009	1.0010	1.0009
0.2	1.0037	1.0037	1.0037	1.0037	1.0043	1.0037
0.4	1.0149	1.0148	1.0149	1.0148	1.0170	1.0148
0.6	1.0332	1.0332	1.0332	1.0332	1.0384	1.0339
0.8	1.0583	1.0582	1.0583	1.0583	1.0673	1.0578
1.0	1.0897	1.0897	1.0897	1.0897	1.1030	1.0889
1.5	1.1924	1.1923	1.1924	1.1924	1.2045	1.1902
2.0	1.3229	1.3228	1.3229	1.3228	1.3170	1.3022
3.0	1.6394	1.6393	–	1.6394	–	1.6260

Table 2. Ratio of the nonlinear frequency to the linear frequency ω/ω_L for clamped beam

a/r	FEM [18]	RQFEM [18]	GFEM [5]	SSDQ (present, $N=11$)	SDQ [11]	PDQ (present, $N=11$)	PDQ [8]
0.1	1.0003	1.0003	1.0003	1.0004	1.0003	1.0003	1.0003
0.2	1.0012	1.0012	1.0012	1.0013	1.0012	1.0012	1.0011
0.4	1.0048	1.0048	1.0048	1.0049	1.0048	1.0048	1.0044
0.6	1.0107	1.0107	1.0107	1.0108	1.0108	1.0107	1.0100
0.8	1.0190	1.0190	1.0190	1.0191	1.0190	1.0190	1.0178
1.0	1.0295	1.0295	1.0295	1.0296	1.0296	1.0295	1.0278
1.5	1.0650	1.0653	1.0650	1.0651	1.0652	1.0651	1.0628
2.0	1.1127	1.1135	1.1127	1.1128	1.1129	1.1127	1.1119
3.0	1.2377	1.2407	1.2377	1.2379	–	1.2377	–

Table 3. Ratio of the nonlinear frequency to the linear frequency ω/ω_L for the beam with one end simply supported the other end clamped

a/r	FEM [18]	RQFEM [18]	GFEM [5]	SSDQ (present, $N=11$)	SDQ [11]	PDQ (present, $N=11$)
0.1	1.0006	1.0006	1.0006	1.0006	1.0006	1.0006
0.2	1.0026	1.0026	1.0026	1.0025	1.0024	1.0026
0.4	1.0106	1.0106	1.0106	1.0104	1.0097	1.0105
0.6	1.0237	1.0238	1.0237	1.0234	1.0218	1.0235
0.8	1.0416	1.0419	1.0416	1.0412	1.0383	1.0414
1.0	1.0641	1.0647	1.0641	1.0638	1.0592	1.0639
1.5	1.1378	1.1404	1.1378	1.1378	1.1284	1.1379
2.0	1.2319	1.2386	1.2318	1.2319	1.2179	1.2319
3.0	1.4605	1.4838	1.4603	1.4605	–	1.4605

5. Conclusion

In the paper, the DQ method based on a piecewise polynomial has been applied to the vibration analysis of geometrically nonlinear beams. The initial–boundary-value problem has been solved, where the emphasis has been put on the stability of the system of ordinary differential equations, resulting from the DQ discretisation. To verify the accuracy of the method, the fundamental frequency has been computed and compared with the values from other methods.

The results indicate that the SSDQ method is much more stable then the conventional one. Apart from the stability, the method is characterized by the well known features of the conventional DQ method like: high rate of convergence, high accuracy and simplicity of formulation. Its advantage is also in the fact, that the degree of the interpolation function does not depend on the number of nodes. Suitably high degree of spline ensures high rate of convergence and by increasing the number of nodes the results are more accurate.

In view of these remarks, one can say that the method has a potential to be an effective numerical technique in linear or nonlinear analysis.

The main disadvantage of the method is the lack of the explicit formulas for the weighting coefficients. Symbolic computing has to be involved in the stage of the determination of these coefficients, what worsen the efficiency of the method.

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