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Energetic approach to direct and inverse heat conduction problems with Trefftz functions used in FEM

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In the paper the stationary 2D inverse heat conduction problems are considered. To obtain an approximate solution of the problems three variants of the FEM with harmonic polynomials (Trefftz functions for Laplace equation) as base functions were used: the continuous FEMT, the non-continuous FEMT and the nodeless FEMT. In order to ensure physical sense of the approximate solution, one of the aforementioned physical aspects is taken into account as a penalty term in the functional, which is to be minimized in order to solve the problem. Three kinds of physical aspects that can smooth the solution were used in the work. The first is the minimization of heat flux jump between the elements, the second is the minimization of the defect of energy dissipation and third is the minimization of the intensity of numerical entropy production. The quality of the approximate solutions was verified on two test examples. The method was applied to solve inverse problem of stationary heat transfer in a rib.

1. Introduction

The key idea of the solving functions' method is to find the functions (polynomials) satisfying a given differential equation. Then the linear combination of these functions is fitted to the governing initial and boundary conditions. In this sense it is a variant of the Trefftz method [19, 20]. The method originates from the paper [18], where one-dimensional heat-conduction problems in the Cartesian coordinate system were solved. The method is continued by the contributions [9, 11], describing heat polynomials for the two- and three-dimensional cases. Application of the heat polynomials in polar and cylindrical coordinates is shown in the papers [7, 8, 10]. The applications of this method for inverse heat conduction problems are described in [1, 5–11]. The paper [1] contains a highly interesting idea of using heat polynomials (Trefftz functions) as a new type of base functions for Finite Element Method (FEMT).

All papers described above refer to the heat conduction equation. The work [2] deals with a lot of other cases, involving other differential equations such as the Laplace, Poisson and Helmholtz equation. The one-dimensional wave equation is solved there as well. The two- and three-dimensional wave polynomials are described in [12–14, 16, 17]. The papers [3, 4] describe energetic approach to direct and inverse heat conduction problems with Trefftz functions used in FEMT. These papers contain the usage of harmonic functions in non - continuous FEMT method and physical regularization for inverse problems for stationary heat conduction. The paper presented here relates to these articles. The authors propose a similar physical regularization in a new approach — nodeless FEMT — and compares all three approaches. The nodeless Finite Elements Method with Trefftz base functions for heat conduction problems has not been used so far. The test examples presented here show that this approach can lead to better results. The nodeless FEMT was used in the last paragraph to solve inverse problem of stationary heat transfer in a rib.

2. Problem description

Let us consider a 2D stationary inverse heat conduction problem described by the mathematical model

- Laplace equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad (x, y) \in (0,1) \times (0,1), \tag{1}$$

- boundary conditions:

$$\frac{\partial T(0,y)}{\partial x} = h_1(y), \tag{2}$$

$$a\,T(x,0) + b\,\frac{\partial T(x,0)}{\partial y} = h_2(x), \tag{3}$$

$$c\,T(x,1) + d\,\frac{\partial T(x,1)}{\partial y} = h_3(x), \tag{4}$$

when $a^2 + b^2 \neq 0$ and $c^2 + d^2 \neq 0$. Additionally we know temperature in discrete K points inside of domain in distance δ from border $x = 1$ (temperature internal response),

$$T(1 - d_b, y_i) = T_i, \qquad i = 1, \ldots, K \tag{5}$$

We are looking for temperature distribution in the whole domain and in particular the temperature and heat flux for $x = 1$. If $d_b = 0$ we have discrete, direct problem.

By solving the problem described by equations (1)–(5) we will use three variants of FEM with Trefftz base functions (FEMT). In each of these cases well known harmonic polynomials are used. We have two sets of these polynomials, which are a real and imaginary part of $\frac{(x+iy)^n}{n!}$,

$$F_n(x,y) = \Re\left(\frac{(x+iy)^n}{n!}\right) = \sum_{k=0,2,\ldots}^{n \geq k} (-1)^{k/2} \frac{x^{n-k} y^k}{(n-k)!k!}, \tag{6}$$

$$G_n(x,y) = \Im\left(\frac{(x+iy)^n}{n!}\right) = \sum_{k=1,3,\ldots}^{n \geq k} (-1)^{(k-1)/2} \frac{x^{n-k} y^k}{(n-k)!k!}. \tag{7}$$

We denote $v_1 = 1$, $v_2 = F_1$, $v_3 = G_1$, $v_4 = F_2$, $v_5 = G_2$, …. All harmonic polynomials v_i fulfill the Laplace equation (1) identically (Trefftz functions). The linear combinations of these functions satisfy Eq. (1) as well. Harmonic polynomials will be used further as base functions for several variants of FEMT.

2.1. Variants of FEMT

In all variants of FEMT considered here, the domain Ω is divided into subdomains Ω_j, where we take linear combination of harmonic polynomials as an approximation of the solution,

$$T^j(x,y) \approx \widetilde{T}^j(x,y) = \sum_{n=1}^{N} c_n^j v_n \,. \tag{8}$$

Continuous and non-continuous FEMT need to be given base functions, which is not necessary in nodeless FEMT. To obtain FEMT base functions we follow analogously as in [1, 2]. We assume temporarily that for element Ω_j the temperatures $T_1, \ldots, T_N$ in nodes $P_1 = (x_1, y_1), \ldots, P_N = (x_N, y_N)$ are known. The temperature in element Ω_j is approximated by Eq. (8). The continuity of the solution in nodes of elements leads to equations (omitting index j)

$$\begin{bmatrix} v_1(P_1) & v_2(P_1) & \ldots & v_N(P_1) \\ v_1(P_2) & v_2(P_2) & \ldots & v_N(P_2) \\ \vdots & \vdots & \ddots & \vdots \\ v_1(P_N) & v_2(P_N) & \ldots & v_N(P_N) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}$$

or

$$[\mathrm{v}][\mathrm{C}] = [\mathrm{T}],$$

hence

$$[\mathrm{C}] = [\mathrm{v}]^{-1}[\mathrm{T}] = [\mathrm{V}][\mathrm{T}],$$

i.e.

$$c_n = \sum_{i=1}^{N} V_{ni} T_i \,.$$

Substituting c_n into Eq. (8) we have

$$\widetilde{T}(x,y) = \sum_{n=1}^{N} \left(\sum_{i=1}^{N} V_{ni} T_i \right) v_n(x,y) = \sum_{i=1}^{N} \left(\sum_{n=1}^{N} V_{ni} v_n(x,y) \right) T_i = \sum_{i=1}^{N} \phi_i(x,y)\, T_i$$

where

$$\phi_i(x,y) = \sum_{n=1}^{N} V_{ni} v_n(x,y). \tag{9}$$

Functions (9) have properties:

- fulfilling Laplace equation:

$$\Delta \phi_i = 0,$$

- interpolation:

$$\phi_i(P_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

It means that we can control the temperature in the nodes. To find a solution (coefficients of linear combination) we build a functional depending on the variant of method. In continuous FEMT we assume equality of temperature in common nodes between the elements. The condition of the temperature continuity in nodes may be omitted and then we obtain non-continuous FEMT [3]. If the continuity of temperature in the nodes is omitted, then another condition (conditions) has to be introduced in order to ensure the physical sense of the solution. We have to take into account the minimization of inaccuracy (in mean-square sense) of the temperatures in nodes between elements. In the nodeless FEMT the temperature in subdomain Ω_j is approximated by Eq. (8). Here the linear combination does not have the property of interpolation. Therefore we minimize the inaccuracy (in mean-square sense) of the temperature between elements (on the common border). Additionally, in all variants of FEMT we minimize the difference between the approximate solution and given conditions (see the examples).

2.2. Energetic approach

We take the physical aspects of the problem into consideration in order to improve the solution. Let us denote Γ_i the common borders between the elements. We can add terms in the functional described in Section 2.1 [3]:

- defect of heat flux

$$J_F = \sum_i \int_{\Gamma_i} \left(\widetilde{q}_+ - \widetilde{q}_- \right)^2 \mathrm{d}\Gamma, \tag{10}$$

where $\tilde{q}_+$, $\tilde{q}_-$ denote the approximation of heat flux on two sides of the border Γ_i,

- defect of entropy production

$$J_E = \sum_i \int_{\Gamma_i} \left(\frac{\widetilde{q}_+}{\widetilde{T}_+} - \frac{\widetilde{q}_-}{\widetilde{T}_-} \right)^2 \mathrm{d}\Gamma, \tag{11}$$

where $\widetilde{T}_+$, $\widetilde{T}_-$ denotes the approximation temperature on two sides of the border Γ_i,

- energy dissipation

$$J_D = \sum_i \int_{\Gamma_i} \left(\ln \widetilde{T}_+ \widetilde{q}_+ - \ln \widetilde{T}_- \widetilde{q}_- \right)^2 \mathrm{d}\Gamma. \tag{12}$$

The additional terms improve the solution. Ciałkowski [4] proved that there is no difference which term we add to the functional — all of them are equivalent and lead to similar results. In fact, the results are similar but not identical. The differences occur especially in numerical practice. The minimization of entropy production or energy dissipation defect leads to nonlinear system of equations, which is more difficult to solve. Nevertheless, the usage of the physical aspects of the problem improves the accuracy of solution (especially for inverse problems).

2.2.1. Error of approximation

Before the method will be applied for practical problems, it should be checked on test problems. If the exact solution $T(x,y)$ is known, the error of approximation can be calculated. In order to prove the quality of the approximation we calculate the errors:

$$\delta L_2 = \left[\frac{\int_\Omega \left(\widetilde{T}(x,y) - T(x,y) \right)^2 \mathrm{d}\Omega}{\int_\Omega (T(x,y))^2 \, \mathrm{d}\Omega} \right]^{\frac{1}{2}} \cdot 100[\%], \tag{13}$$

$$\delta H^1 = \left[\frac{\int_\Omega \left[\left(\widetilde{T}(x,y) - T(x,y) \right)^2 + \left(\frac{\partial \widetilde{T}(x,y)}{\partial x} - \frac{\partial T(x,y)}{\partial x} \right)^2 + \left(\frac{\partial \widetilde{T}(x,y)}{\partial y} - \frac{\partial T(x,y)}{\partial y} \right)^2 \right] \mathrm{d}\Omega}{\int_\Omega \left[(T(x,y))^2 + \left(\frac{\partial T(x,y)}{\partial x} \right)^2 + \left(\frac{\partial T(x,y)}{\partial y} \right)^2 \right] \mathrm{d}\Omega} \right]^{\frac{1}{2}} \cdot 100[\%]. \tag{14}$$

The first shows the accuracy of the temperature approximation. The second takes the error of heat flux approximation into account as well. The error described by Eq. (14) is very important, especially when we approximate the solution by polynomials. The polynomials undulate and when we differentiate them the error of the approximation of heat flux can enlarge.

3. Examples

In the case of inverse problems we do not know the exact solution of the problems. Therefore, the usability of methods presented here should be checked on the test problems, when all conditions are known. We consider two test examples.

3.1. The first test example

Let us consider a 2D stationary inverse heat conduction problem described by Eq. (1) and the conditions

$$\frac{\partial T(0,y)}{\partial x} = h_1(y) = e^y + e^{-y}, \tag{15}$$

$$\frac{\partial T(x,0)}{\partial y} = h_2(x) = 0, \tag{16}$$

$$\frac{\partial T(x,1)}{\partial y} = h_3(x) = (\cos x + \sin x)(e - e^{-1}), \tag{17}$$

Additionally, we know the temperature in eight discrete points arranged uniformly inside the domain in distance d_b from border $x = 1$ (internal temperature responses),

$$T(1 - d_b\,, y_i) = T_i\,, \qquad i = 1, \ldots, 8. \tag{18}$$

The temperature in these points is simulated from the exact solution

$$T(x,y) = (\cos x + \sin x)(e^y + e^{-y}).$$

We solve the problem described by Eq. (1) and conditions (15)–(18) by using the continuous FEMT, the non-continuous FEMT and the nodeless FEMT dividing domain Ω (by straight lines $x = 0.5$ and $y = 0.5$) into four subdomains $\Omega_j = \{(x,y) \in \Omega : x_{dj} \le x \le x_{uj}\,,\, y_{dj} \le y \le y_{uj}\}$ (see Fig. 1). In each element, thirteen harmonic polynomials were used. For each method mentioned above we take three kinds of functional into consideration: the minimizing defect of heat flux, entropy and energy dissipation between the elements. For example for nodeless FEMT and for minimizing of energy dissipation (12) this functional has the form

$$\begin{aligned} J = {} & \sum_{j=1}^{2} \int_{y_{dj}}^{y_{uj}} \left(\frac{\partial \widetilde{T}(0,y)}{\partial x} - h_1(y) \right)^2 dy + \sum_{j=2,4} \int_{x_{dj}}^{x_{uj}} \left(\frac{\partial \widetilde{T}(x,0)}{\partial y} - h_2(x) \right)^2 dx \\ & + \sum_{j=1,3} \int_{x_{dj}}^{x_{uj}} \left(\frac{\partial \widetilde{T}(x,1)}{\partial y} - h_3(x) \right)^2 dx + \sum_{i=1}^{8} \left(\widetilde{T}(1-d_b, y_i) - T^i \right)^2 \\ & + \sum_{i,j} \int_{\Gamma_{ij}} \left(\widetilde{T}_i - \widetilde{T}_j \right)^2 d\Gamma + \sum_{i,j} \int_{\Gamma_{ij}} \left(\frac{\partial \widetilde{T}_i}{\partial x} \cdot \ln(\widetilde{T}_i) - \frac{\partial \widetilde{T}_j}{\partial x} \cdot \ln(\widetilde{T}_j) \right)^2 d\Gamma \\ & + \sum_{i,j} \int_{\Gamma_{ij}} \left(\frac{\partial \widetilde{T}_i}{\partial y} \cdot \ln(\widetilde{T}_i) - \frac{\partial \widetilde{T}_j}{\partial y} \cdot \ln(\widetilde{T}_j) \right)^2 d\Gamma, \end{aligned}$$

where Γ_{ij} are common borders between elements. The functionals for other cases are similar.

Figure 2 shows the error (13) (dependent on the distance d_b) of approximation with using minimization of the defect of heat flux between elements. Figure 3 shows the error (13) (dependent on the distance d_b) of approximation with using minimization of the defect of entropy production between elements. Figure 4 shows the error (13) (dependent on distance d_b) of approximation with

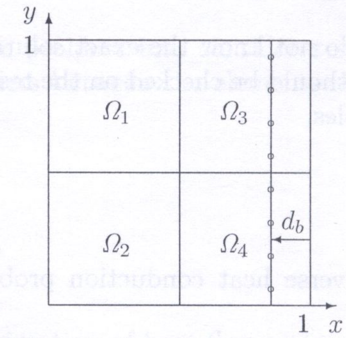


Fig. 1. Division of domain Ω and localization of internal temperature response.

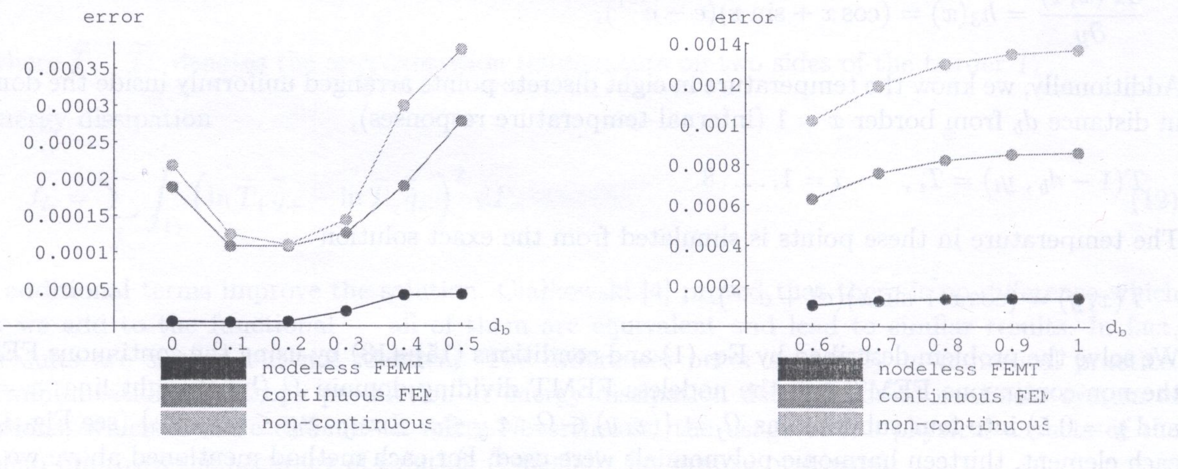


Fig. 2. The error δL_2[%] dependent on distance d_b (minimization of defect of heat flux between elements)

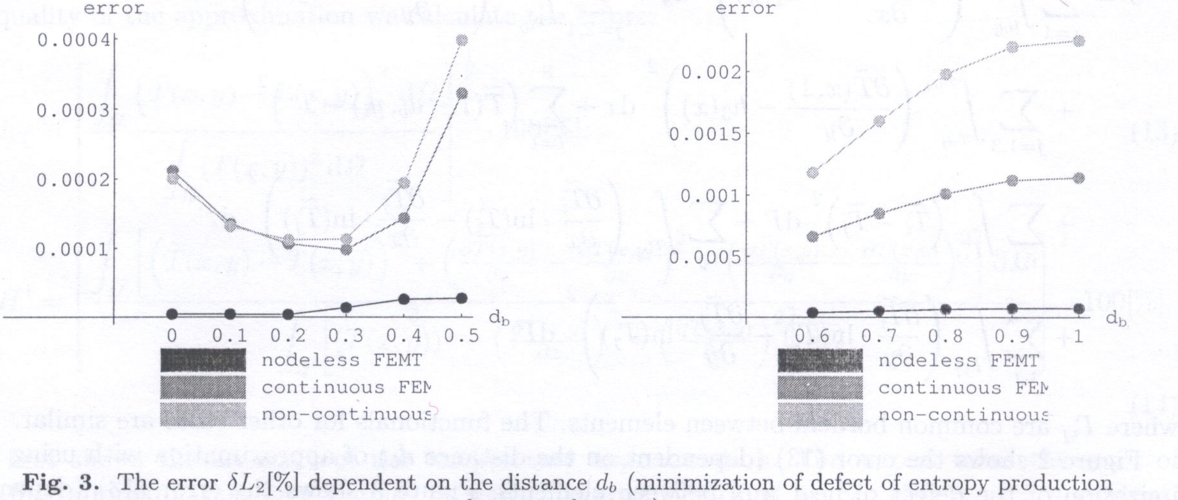
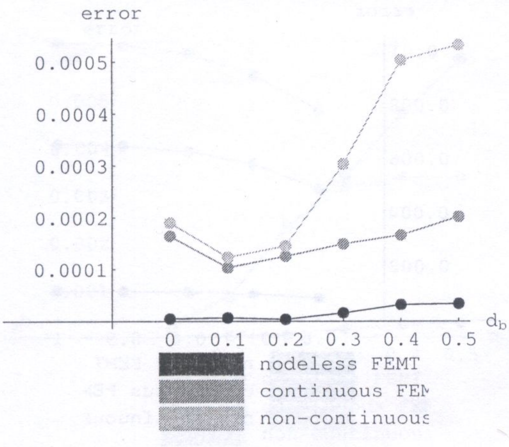


Fig. 3. The error δL_2[%] dependent on the distance d_b (minimization of defect of entropy production between elements)



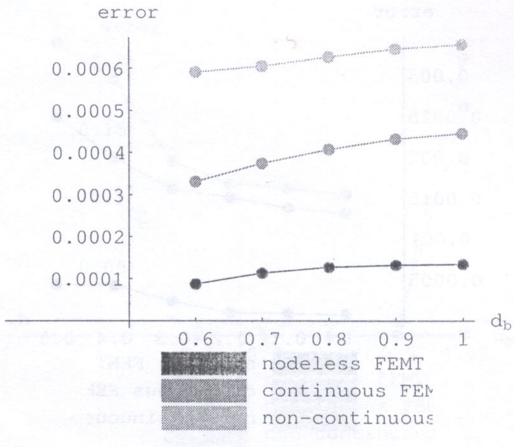


Fig. 4. The error $\delta L_2[\%]$ dependent on the distance d_b (minimization of defect of energy dissipation between elements)

Table 1. The error $\delta L_2[\%]$ dependent on distance d_b

d_b	Heat flux			Entropy			Energy dissipation		
	Nodeless	Con	N-Con	Nodeless	Con	N-Con	Nodeless	Con	N-Con
0	0.000006	0.00019	0.00022	0.000004	0.00021	0.00020	0.000005	0.00016	0.00019
0.1	0.000005	0.00011	0.00012	0.000004	0.00014	0.00013	0.000007	0.00010	0.00012
0.2	0.000006	0.00011	0.00011	0.000004	0.00011	0.00011	0.000004	0.00013	0.00015
0.3	0.000020	0.00013	0.00014	0.000014	0.00010	0.00011	0.000016	0.00015	0.00030
0.4	0.000042	0.00019	0.00030	0.000026	0.00014	0.00019	0.000032	0.00017	0.00050
0.5	0.000043	0.00028	0.00038	0.000028	0.00032	0.00040	0.000034	0.00020	0.00053
0.6	0.000104	0.00063	0.00102	0.000038	0.00066	0.00118	0.000087	0.00033	0.00059
0.7	0.000125	0.00076	0.00119	0.000050	0.00085	0.00160	0.000113	0.00037	0.00061
0.8	0.000135	0.00082	0.00130	0.000066	0.00101	0.00198	0.000127	0.00041	0.00063
0.9	0.000139	0.00085	0.00135	0.000073	0.00112	0.00220	0.000132	0.00043	0.00065
1	0.000140	0.00086	0.00137	0.000075	0.00114	0.00225	0.000134	0.00045	0.00066

using minimization of the defect of energy dissipation between elements. The total error of temperature approximation observed in Figs. 2–4 is very small even for large values of d_b. Generally, these figures suggest that all functionals (minimization of heat flux, entropy and energy dissipation) lead to similar results for $d_b \leq 0.5$. It is the confirmation of theoretical results obtained in [4] (in this paper the nodeless FEMT were not considered). But for $d_b > 0.5$ the best results are given by the minimization of energy dissipation between the elements. The calculations showed clearly that the nodeless FEMT leads to far better results than the continuous and the non-continuous FEMT.

Table 1 contains the results presented in Figs. 2–4 which show that the approximation of temperature is very good both for direct and inverse problem (the biggest error observed in Table 1 does not cross 0.0025%). On the whole, in the inverse problem the error increases when the distance of internal temperature response from the border $x = 1$ is increased. In the cases considered above, even for $d_b = 1$ (two boundary conditions on the border $x = 0$ and no condition for $x = 0$) the error is very small. Generally, the quality of approximation improves when we take more polynomials in Eq. (8) and more subdomains Ω_j. In the example presented here four subdomains and thirteen polynomials are sufficient to obtain the error (for temperature in the whole domain) below 0.0025% for all methods and smaller than 0.00015% for the nodeless FEMT.

Table 2. The error $\delta L_2[\%]$ in dependence on the distance d_b

d_b	Heat flux			Entropy			Energy dissipation		
	Nodeless	Con	N-Con	Nodeless	Con	N-Con	Nodeless	Con	N-Con
0	0.000025	0.00023	0.00027	0.000026	0.00017	0.00020	0.000021	0.00027	0.00030
0.1	0.000023	0.00084	0.00091	0.000026	0.00045	0.00050	0.000021	0.00029	0.00030
0.2	0.000056	0.00230	0.00233	0.000047	0.00152	0.00172	0.000042	0.00092	0.00088
0.3	0.000171	0.00338	0.00316	0.000117	0.00267	0.00292	0.000094	0.00261	0.00164
0.4	0.000268	0.00471	0.00337	0.000154	0.00369	0.00333	0.000112	0.00498	0.00311
0.5	0.000286	0.00584	0.00334	0.000160	0.00470	0.00339	0.000112	0.00516	0.00338
0.6	0.000288	0.00834	0.00458	0.000172	0.00759	0.00503	0.000124	0.00568	0.00385
0.7	0.000293	0.00894	0.00649	0.000175	0.00851	0.00776	0.000130	0.00737	0.00490
0.8	0.000304	0.00918	0.00865	0.000177	0.00887	0.01120	0.000129	0.01090	0.00733
0.9	0.000475	0.00968	0.01030	0.000190	0.00919	0.01370	0.000193	0.01380	0.00981
1	0.001460	0.16600	0.20100	0.000728	0.12700	0.16000	0.000399	0.02310	0.01810

The results presented on figures 6–8 contains Table 2. The results presented in Table 2 show that the approximation of temperature is very good both for direct and inverse problem. Even for $d_b = 1$ the error is smaller than 0.21% for all methods and less than 0.0015% for the nodeless FEMT.

To sum up, the test examples lead to three important conclusions. The approximate solution of stationary inverse heat conduction problem is very good (temperature and heat flux). All functionals (minimization of heat flux, entropy and energy dissipation) leads to similar results. The nodeless FEMT leads to far better results then the continuous and the non-continuous FEMT.

The second test example presented here refers to the paper [3]. The results presented in this paper are similar for continuous and the non-continuous FEMT, while the results for nodeless FEMT are much better — this is the advantage of the method proposed here.

3.3. Inverse problem of stationary heat transfer in a rib

In the paper [15], the inverse heat conduction problem for stationary heat transfer in a rib was solved. There the harmonic polynomials were used but the domain was not divided into subdomains (solution in whole domain). Of course the energetic approach was not used. The same problem was solved in this section.

Let us consider a 2D stationary inverse heat conduction problem in a rectangular rib $\Omega = \{(x,y) \in (0, 0.08) \times (0, 0.003)\}$ described by the Laplace equation (1) and the conditions

$$T(0,y) = 29, \tag{23}$$

$$\frac{\partial T(0.08,y)}{\partial x} = 0, \tag{24}$$

$$\frac{\partial T(x,0)}{\partial y} = 0, \tag{25}$$

$$T(x_i,0) = T_i, \qquad i = 1,\ldots,67. \tag{26}$$

The temperature in discrete points T_i was measured with THV 550 thermovision camera. We are looking for a temperature distribution in the whole domain and in particular the temperature and heat flux for $y = 3$ mm. On the border $y = 3$ mm there is no condition but for $y = 0$ there are two conditions (inverse problem). Here the domain was divided into four subdomains and thirteen harmonic polynomials in each element were used (analogously as in test examples). The nodeless FEMT with minimization of the defect of heat flux between the elements (see Eq. (10)) was applied.

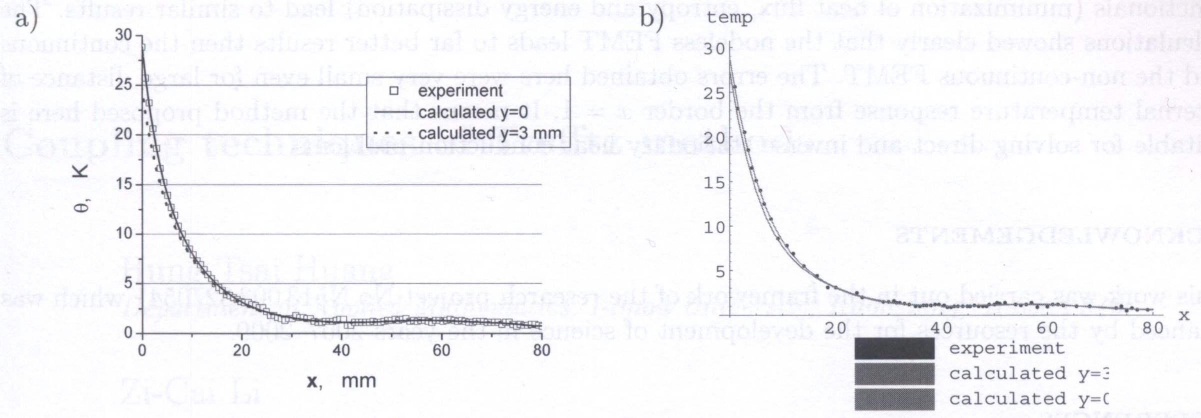


Fig. 9. Comparison of the temperature distribution

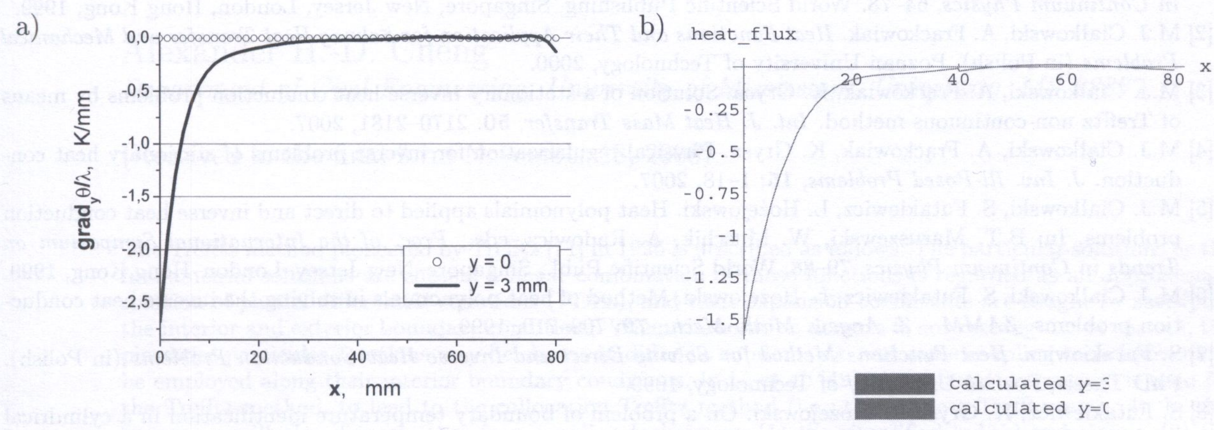


Fig. 10. Comparison of the component $\frac{\partial T(x,y)}{\partial y}$ of the heat flux distribution

Figure 9 shows the comparison of the temperature distribution for $y = 0$ and for $y = 3$ mm obtained in the paper [15] (Fig. 9a) and with the nodeless FEMT (Fig. 9b). The pictures had to be scanned and so that their quality is low. The approximation is similar in both cases.

Figure 10 shows comparison of the component $\frac{\partial T(x,y)}{\partial y}$ of the heat flux distribution for $y = 0$ and $y = 3$ mm obtained in the paper [15] (Fig. 10a) and with the nodeless FEMT (Fig. 10b). The approximations of $\frac{\partial T(x,y)}{\partial y}$ for $y = 0$ in both cases are very good. But for $y = 3$ mm the approximation in Fig. 10b is more physical than that in Fig. 10a. For $y = 3$ mm the component $\frac{\partial T(x,y)}{\partial y}$ of the heat flux should be negative. In Fig. 10b the distribution of $\frac{\partial T(x,y)}{\partial y}$ undulates more. It is especially visible near the end of the rib $x = 80$ mm. Moreover, comparing the heat flux distribution in Fig. 10 with the temperature distribution in Fig. 9 we can observe that the rise of the curve in Fig. 10a for $x < 20$ mm is too fast — not proportional to the decreasing of the temperature. Summarizing the energetic approximation improved the physical aspects of solution.

CONCLUSIONS

As a rule, the inverse problems are ill-posed and difficult to solve. In this paper a new approach was proposed — the nodeless Finite Elements Method with harmonic polynomials as base functions. Additionally, the energetic approach was taken into consideration — minimization of the defect of the heat flux, entropy production and energy dissipation. The examples presented here show that all functionals (minimization of heat flux, entropy and energy dissipation) lead to similar results. The calculations showed clearly that the nodeless FEMT leads to far better results then the continuous and the non-continuous FEMT. The errors obtained here were very small even for large distance of internal temperature response from the border $x = 1$. It means that the method proposed here is suitable for solving direct and inverse stationary heat conduction problems.

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