Computer Assisted Methods in Engineering and Science, **20**: 289–307, 2013.

TWENTY YEARS OF THE CAMES

Convergence estimates for the acoustic scattering problem approximated by NURBS

Andrzej Karafiat
Institute of Computer Science, Cracow University of Technology
Warszawska 24, 31-155 Kraków, Poland
e-mail: akaraf@pk.edu.pl

The paper contains some estimates of an approximation to the solution of the problem of acoustic waves's scattering by an elastic obstacle in two dimensions. The problem is approximated by the isogeometric adaptive method based on the known NURBS functions. The estimates show how the error of an approximation depends on the size of intervals and the degree of functions.

Keywords: NURBS, adaptive methods, error estimates, acoustic scattering.

1. Introduction

The isogeometric method, inaugurated by T.J.R. Hughes et al. [7] is the subject of intensive development in the last few years. There were organized special conferences and workshops dedicated to this area, e.g., HOFEIM in June 2011 in Cracow, Poland and CIME Summer School "IsoGeometric Analysis: a New Paradigm in the Numerical Approximation of PDEs" in June 2012 in Cetraro, Italy. The method concerns the application of non-uniform rational B-splines (NURBS) to approximation of curves and surfaces. Its undoubtful advantage is the high but adaptable regularity of these functions at the boundary points of intervals and exact reproduction of many curves, e.g., conics, what is unavailable for finite elements, even in their isoparametric version. Some a-priori error estimates for this method were given in [2] and [3]. The aim of the presented paper is the extension of these estimates to the problem of elastic scattering of acoustic waves.

The phenomenon of propagation of acoustic waves in the space is described mathematically by the Helmholtz differential equation in the whole (unbounded) domain and the Sommerfeld condition at infinity. One of used approaches, apart from replacing the problem by the one in a bounded domain, is the boundary integral equation method (BIEM). These equations are discretized with the boundary elements. Theoretical aspects of this method were given e.g., in [5, 8, 9]. There are many publications concerning the adaptive boundary element methods, e.g., [6, 14, 16–18].

The question arises, whether approximation of the shape of the boundary and the solution of the BIEM by NURBS, instead of the boundary elements, could give better results. The research presented below was done to answer this question in a theoretical way. Author is going to complete it in the future by numerical convergence tests.

2. Construction of NURBS

In this part, we present some definitions of NURBS given in [7, 12] and [15]. More detailed explanations and examples may be found there.

- Bernstein polynomials
  For any integer $d \geq 0$, $d+1$ Bernstein polynomials of the degree d on the interval $[0,1]$ are defined by the formula:

$$B_i^d(t) = \binom{d}{i} t^i (1-t)^{d-i}, \qquad t \in [0,1], \qquad i = 0, \ldots, d. \tag{1}$$

For $i \notin \{0, \ldots, d\}$ $B_i^d(t) \equiv 0$. The polynomials fulfil the following recurrence relation:

$$B_i^d(t) = (1-t)B_i^{d-1}(t) + tB_{i-1}^{d-1}(t), \qquad d \geq 1, \qquad i = 0, \ldots, d. \tag{2}$$

The first derivative of this function is

$$\frac{d}{dt} B_i^d = d(B_{i-1}^{d-1} - B_i^{d-1}). \tag{3}$$

Derivatives of higher orders could be evaluated analogously.

- Bezier curves
  To construct such a curve in $\Re^n$ we need a sequence of control points

$$P_i \in \Re^n, \qquad i = 0, \ldots, d.$$

The Bezier curve of a degree d is then defined as

$$C(t) = \sum_{i=0}^{d} B_i^d(t) P_i. \tag{4}$$

Derivatives of Bezier curves have the form:

$$\frac{d}{dt} C(t) = \sum_{i=0}^{d} \frac{d}{dt} B_i^d(t) P_i = \sum_{i=0}^{d-1} d B_i^{d-1}(t) \Delta P_i,$$

$$\frac{d^k}{dt^k} C(t) = \sum_{i=0}^{d-k} \frac{d!}{(d-k)!} B_i^{d-k}(t) \Delta^k P_i,$$

where

$$\Delta^1 P_i = \Delta P_i = P_{i+1} - P_i,$$
$$\Delta^{k+1} P_i = \Delta^k P_{i+1} - \Delta^k P_i.$$

Calculation of the k-th derivative at the endpoint $t = 0$, where only $B_0^d(0) \neq 0$ or $t = 1$, where $B_d^d(1) \neq 0$ involves then $k+1$ neighbouring control points. The Bezier curve can be defined on any $[\alpha, \beta]$ interval. Then, we need an invertible affine transformation $t : [\alpha, \beta] \to [0,1]$. The curve is described by the formula:

$$C(t(\tau)) = \sum_{i=0}^{d} B_i^d(t(\tau)) P_i, \qquad \tau \in [\alpha, \beta]. \tag{5}$$

To increase the degree of the curve we have to take Bernstein polynomials of a degree $d+1$ and add one more control point. To preserve the shape of the curve we have to change all points using the following formula for their coordinates

$$P_i^* = \frac{d+1-i}{d+1} P_i + \frac{i}{d+1} P_{i-1}, \qquad i = 0, \ldots, d+1.$$

A Bezier curve may be stuck together with another Bezier curve keeping C^k-regularity in the sticking point. The derivatives of both functions, including the derivative of order 0, i.e., a function itself, up to the order k, should be equal there. It means that the $k+1$ control points of both curves have to fit on themselves, especially $\Delta^k P_i$ should be equal or proportional to the intervals of their arguments.

- B-spline curves
  B-spline curves of a degree d may be built by sticking Bezier curves, but there is a much simpler method. At first we choose a vector consisting of $r+1$ $(r \geq d)$ real numbers ξ_i, named knots:

$$\Xi = \{\xi_0, \xi_1, \ldots, \xi_n\}, \qquad a = \xi_0 \leq \xi_1 \leq, \ldots, \leq \xi_r = b,$$

where $[a,b] \in \Re$. B-spline basis functions are defined as follows:

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi \in [\xi_i, \xi_{i+1}], \\ 0 & \text{in the other case.} \end{cases} \tag{6}$$

If $\xi_i = \xi_{i+1}$ then $N_{i,0} \equiv 0$. Recursively, for $k = 1, \ldots, d$

$$\begin{aligned} &N_{i,k}(\xi) = \frac{\xi - \xi_i}{\xi_{i+k} - \xi_i} N_{i,k-1}(\xi) + \frac{\xi_{i+k+1} - \xi}{\xi_{i+k+1} - \xi_{i+1}} N_{i+1,k-1}(\xi), \\ &i = 0, \ldots, r(k) = r - k - 1. \end{aligned} \tag{7}$$

It may be easily seen that for $\xi_0 = \ldots = \xi_d = 0$, $\xi_{d+1} = \ldots = \xi_{2d+1} = 1$ functions $N_{i,d}(\xi)$ are simply Bernstein polynomials of a degree d. If we construct a B-spline curve with a knot vector

$$\Xi = \{\xi_0, \xi_1, \ldots, \xi_{3d+2}\},$$

$$\xi_0 = \ldots = \xi_d = 0, \qquad \xi_{d+1} = \ldots = \xi_{2d+1} = 1, \qquad \xi_{2d+2} = \ldots = \xi_{3d+2} = 2,$$

we obtain two independent sets of basis functions, one on $[0,1]$, and the other on $[1,2]$. They may be sticked together by removing one or more equal knots between them. Removal of any such knot increases the sticking regularity by 1. In this way we are able to stick basis functions, and next, the curves with any regularity but smaller than the degree of polynomials on both sides. If both basis functions are of degree d, d is called the degree of the spline. If $\xi_{i+1} - \xi_i =$ const for $i = 0, \ldots, r-1$, we say that the knots are uniform, otherwise they are non-uniform.

Some knots may be repeated. If $\xi_{j-1} < \xi_j = \ldots = \xi_{j+l-1} < \xi_{j+l}$, we say that the multiplicity of the knot ξ_j is equal to l. We associate to the knot vector $\Xi = (\xi_0, \ldots, \xi_r)$ a vector of intervals $Z = (\zeta_0, \ldots, \zeta_\rho)$ in which all knots are of multiplicity 1:

$$\zeta_0 = \xi_0. \tag{8}$$

If $\zeta_i = \xi_j$ then

$$\zeta_{i+1} = \min\{\xi_k \colon \xi_k > \xi_j\}, \qquad i = 0, \ldots, \rho - 1. \tag{9}$$

In this way, $0 = \zeta_0 < \zeta_1 < \ldots < \zeta_\rho = 1$, each ζ_i coincides with an ξ_j and intervals (ζ_i, ζ_{i+1}) are not empty for all $i = 0, \ldots, \rho - 1$.

In the following we assume that the knot vector is open, i.e., if the degree of the corresponding B-spline functions is d then the first and the last knots, ξ_0 and ξ_r, are of multiplicity $d+1$. We introduce the regularity of the knot $k_i = d - l + 1$ being a number of matching values of B-spline functions and their derivatives at ζ_i (l is the multiplicity of the knot ζ_i). By the definition, for each knot $0 \leq k_i \leq d$. 0 means discontinuity, and d means full regularity. We introduce the notation

$$k_{\min} = \min k_i, \qquad k_{\max} = \max k_i \tag{10}$$

for knots $\zeta_1, \dots \zeta_{\rho-1}$, which will be used in the following sections.

Example 1. Let the open knot vector be $\Xi = (0, 0, 0, 1, 2, 2, 3, 4, 5, 5, 5)$ and $p = 2$. The B-spline basis functions of degree 2 are

$$N_{0,2}(\xi) = \begin{cases} (1-\xi)^2 & \text{if} \quad \xi \in [0,1), \\ 0 & \text{if} \quad \xi \notin [0,1), \end{cases}$$

$$N_{1,2}(\xi) = \begin{cases} 2\xi - 1.5\xi^2 & \text{if} \quad \xi \in [0,1), \\ 0.5(2-\xi)^2 & \text{if} \quad \xi \in [1,2), \\ 0 & \text{if} \quad \xi \notin [0,2), \end{cases}$$

$$N_{2,2}(\xi) = \begin{cases} 0.5\xi^2 & \text{if} \quad \xi \in [0,1), \\ 4\xi - 1.5\xi^2 - 2 & \text{if} \quad \xi \in [1,2), \\ 0 & \text{if} \quad \xi \notin [0,2), \end{cases}$$

$$N_{3,2}(\xi) = \begin{cases} 0.5(\xi-1)^2 & \text{if} \quad \xi \in [1,2), \\ 0.5(3-\xi)^2 & \text{if} \quad \xi \in [2,3), \\ 0 & \text{if} \quad \xi \notin [1,3), \end{cases}$$

$$N_{4,2}(\xi) = \begin{cases} 8\xi - 1.5\xi^2 - 10 & \text{if} \quad \xi \in [2,3), \\ 0.5(4-\xi)^2 & \text{if} \quad \xi \in [3,4), \\ 0 & \text{if} \quad \xi \notin [2,4), \end{cases}$$

$$N_{5,2}(\xi) = \begin{cases} 0.5(\xi-2)^2 & \text{if} \quad \xi \in [2,3), \\ 7\xi - \xi^2 - 11.5 & \text{if} \quad \xi \in [3,4), \\ 0.5(5-\xi)^2 & \text{if} \quad \xi \in [4,5), \\ 0 & \text{if} \quad \xi \notin [2,5), \end{cases}$$

$$N_{6,2}(\xi) = \begin{cases} 0.5(\xi-3)^2 & \text{if} \quad \xi \in [3,4), \\ 13\xi - 1.5\xi^2 - 27.5 & \text{if} \quad \xi \in [4,5), \\ 0 & \text{if} \quad \xi \notin [3,5), \end{cases}$$

$$N_{7,2}(\xi) = \begin{cases} (\xi-4)^2 & \text{if} \quad \xi \in [4,5), \\ 0 & \text{if} \quad \xi \notin [4,5). \end{cases}$$

The basis functions are shown in Fig. 1.

B-spline curves are built using B-spline basis functions and control points $P_i \in \Re^n$:

$$C(\xi) = \sum_{i=0}^{r(d)} N_{i,d}(\xi) P_i. \tag{11}$$

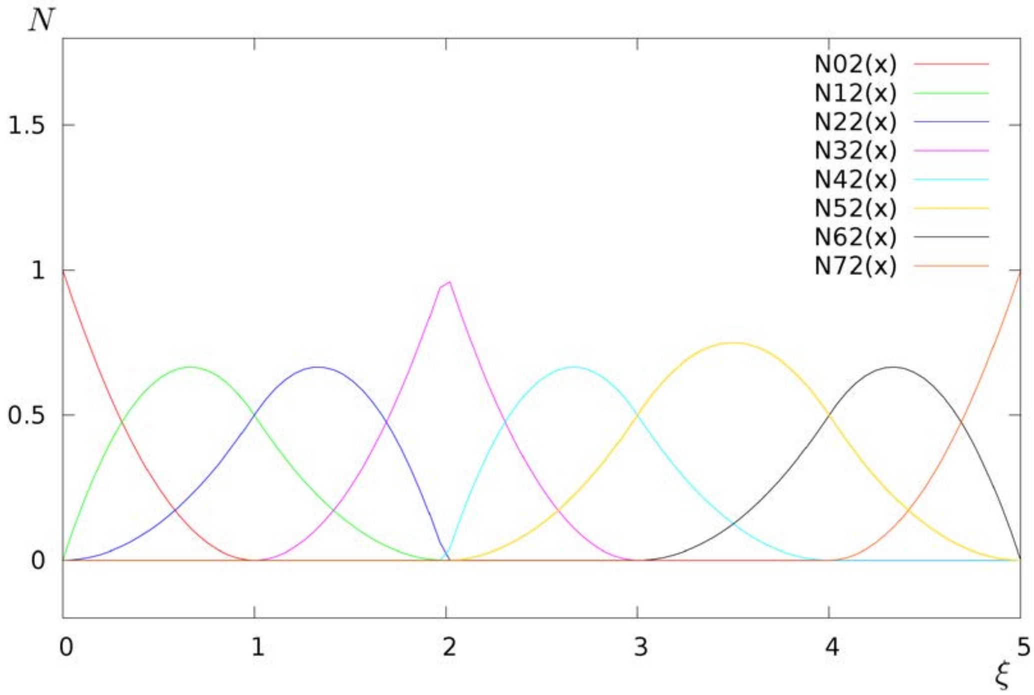

Fig. 1. B-spline basis functions.

Control points may play a part of degrees of freedom in a problem of fitting a B-spline curve to a fixed curve or to a set of points.

Example 2. Let us construct a spline curve using Example 1. We choose the control points $P_0 = (1,3)$, $P_1 = (4,1)$, $P_2 = (5,5)$, $P_3 = (8,7)$, $P_4 = (12,7)$, $P_5 = (15,5)$, $P_6 = (16,1)$, $P_7 = (19,3)$ and construct the curve using the formula (11). The curve is shown in Fig. 2.

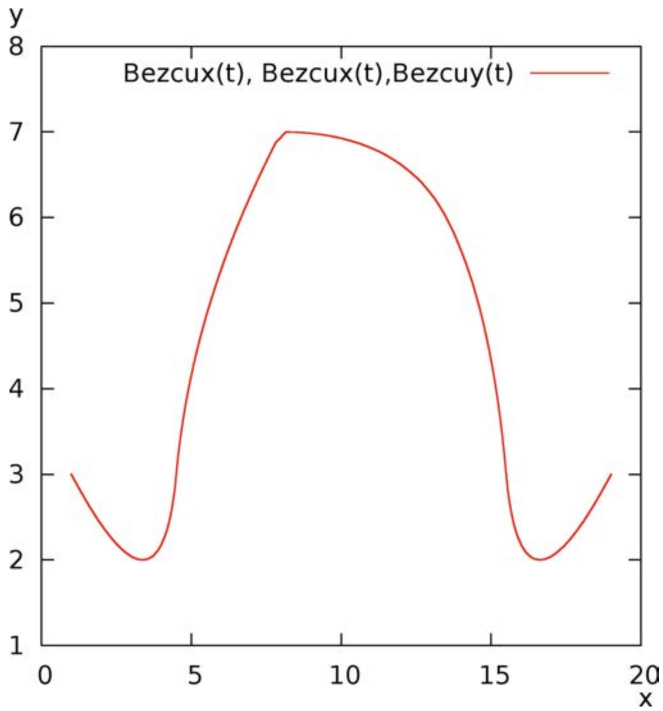

Fig. 2. The B-spline curve for Example 2.

- Non-uniform rational B-spline curves (NURBS)
  Additional shapes of the B-spline curves in $\Re^2$, called rational B-spline curves may be obtained as follows: let control points be given in $\Re^3$, $\widehat{P}_i = (\widehat{p}_{i,1}, \widehat{p}_{i,2}, \widehat{p}_{i,3})$, where $\widehat{p}_{i,3}$ are called weights. A new control point P_i on the plane $x_3 = 1$ is defined by its coordinates $p_{i,j} = \widehat{p}_{i,j}/\widehat{p}_{i,3}$, for $j = 1, 2$. The rational B-spline curve is defined using the B-spline basis functions (7):

$$C(\xi) = \sum_{i=0}^{r(d)} \frac{N_{i,d}(\xi)\widehat{p}_{i,3}}{\sum_{j=0}^{r(d)} N_{j,d}(\xi)\widehat{p}_{j,3}} P_i. \tag{12}$$

  It is a projection of the 3D curve

$$C(\xi) = \sum_{i=0}^{r(d)} N_{i,d}(\xi) P_i \tag{13}$$

  onto the plane $x_3 = 1$. This construction allows one to obtain some other curves, e.g., the conic curves.

Remark 1. *Many regular, closed curves may be presented as NURBS. An example of a circle as a NURBS curve is given in [7].*

3. Adaptation techniques for NURBS

There are three kinds of refinements: knot insertion (h-refinement), order elevation (p-refinement) and k-refinement. In their descriptions we partially follow [7] and [12].

- Knot insertion
  The knot vector $\Xi = \{\xi_0, \xi_1, \ldots, \xi_r\}$, the set of basis functions $N_{i,d}$, control points P_i and the curve $C(\xi)$ of the order d are given. When a new knot $\xi \in [\xi_k, \xi_{k+1})$ is added, the new set of basis functions $N_{i,d}$ is defined by the formulas (7). Redefinition of the control points as follows:

$$\widetilde{P}_i = \alpha_i P_i + (1 - \alpha_i) P_{i-1}, \tag{14}$$

$$\alpha_i = \begin{cases} 1 & \text{for} \quad 1 \le i \le k - d, \\ \dfrac{\xi - \xi_i}{\xi_{i+d} - \xi_i} & \text{for} \quad k - d + 1 \le i \le k, \\ 0 & \text{for} \quad k + 1 \le i \le r(d) + 1. \end{cases} \tag{15}$$

  allows us to keep the shape of the curve without changes.

  Example 3. In Example 1 we insert a new knot $\widehat{\xi} = 2.5$. Then $\widehat{\xi} \in [\xi_5, \xi_6)$ then $k = 5$ in the formula (15). We obtain a new knot vector $\widetilde{\Xi} = (\widetilde{\xi}_0, \ldots \widetilde{\xi}_{11}) = (0, 0, 0, 1, 2, 2, 2.5, 3, 4, 5, 5, 5)$. The basis functions after the insertion are $\widetilde{N}_{0,2} = N_{0,2}$, $\widetilde{N}_{1,2} = N_{1,2}$, $\widetilde{N}_{2,2} = N_{2,2}$, $\widetilde{N}_{7,2} = N_{6,2}$, $\widetilde{N}_{8,2} = N_{7,2}$ and

$$\widetilde{N}_{3,2}(\xi) = \begin{cases} 0.5(\xi - 1)^2 & \text{if} \quad \xi \in [1, 2), \\ 4(2.5 - \xi)^2 & \text{if} \quad \xi \in [2, 2.5), \\ 0 & \text{if} \quad \xi \notin [1, 2.5), \end{cases}$$

$$\widetilde{N}_{4,2}(\xi) = \begin{cases} 19\xi - 4\xi^2 - 22 & \text{if} \quad \xi \in [2, 2.5), \\ 2(3 - \xi)^2 & \text{if} \quad \xi \in [2.5, 3), \\ 0 & \text{if} \quad \xi \notin [2, 3), \end{cases}$$

$$\widetilde{N}_{5,2}(\xi) = \begin{cases} 2(\xi-2)^2 & \text{if} \quad \xi \in [2, 2.5), \\ \dfrac{56}{3}\xi - \dfrac{10}{3}\xi^2 - \dfrac{76}{3} & \text{if} \quad \xi \in [2.5, 3), \\ \dfrac{2}{3}(4-\xi)^2 & \text{if} \quad \xi \in [3, 4), \\ 0 & \text{if} \quad \xi \notin [2, 5), \end{cases}$$

$$\widetilde{N}_{6,2}(\xi) = \begin{cases} \dfrac{1}{3}(2\xi-5)^2 & \text{if} \quad \xi \in [2.5, 3), \\ \dfrac{25}{3}\xi - \dfrac{7}{6}\xi^2 - \dfrac{85}{6} & \text{if} \quad \xi \in [3, 4), \\ 0 & \text{if} \quad \xi \notin [2.5, 4). \end{cases}$$

The basis functions are shown in Fig. 3.

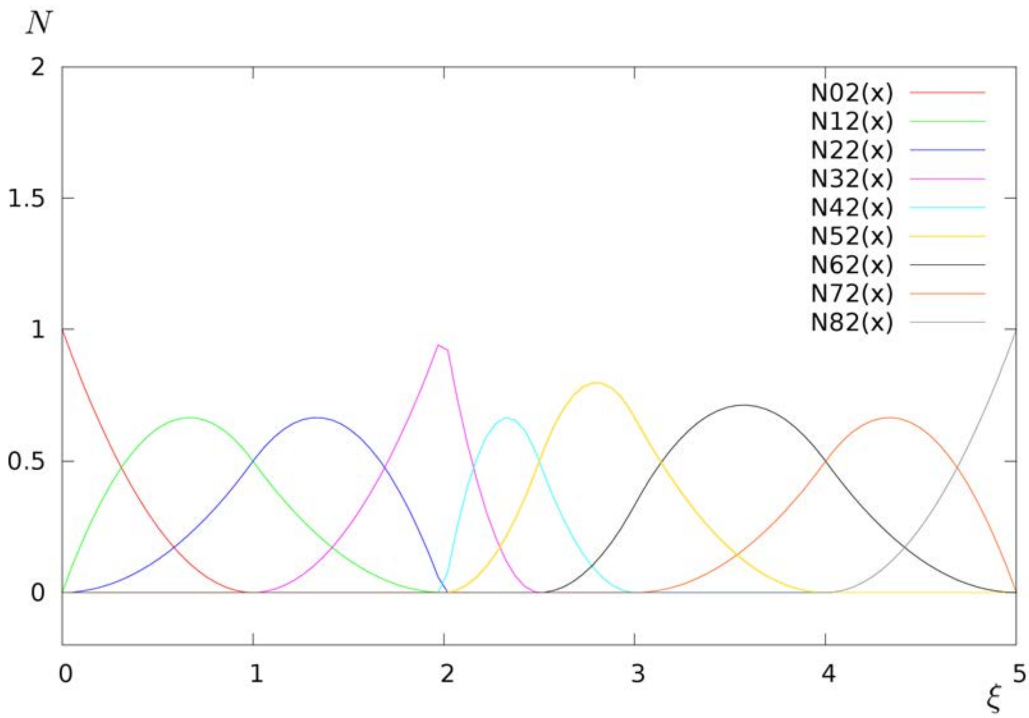

Fig. 3. B-spline basis functions for the inserted knot.

New control points are obtained by the formula (14): $\widetilde{P}_0 = P_0, \ldots, \widetilde{P}_3 = P_3$, $\widetilde{P}_4 = (10, 7)$, $\widetilde{P}_5 = (12.75, 6.50)$, $\widetilde{P}_6 = P_5$, $\widetilde{P}_7 = P_6$, $\widetilde{P}_8 = P_7$. It may be verified that the curve remains the same.

- Knot removal
  Let us assume that a Bezier curve $C(\xi)$ of a degree d is defined by a knot vector $\widetilde{\Xi} = (\widetilde{\xi}_0, \ldots, \widetilde{\xi}_r)$, B-spline basis functions $\widetilde{N}_{i,d}$, $i = 0, \ldots, r(d)$ and control points $\widetilde{P}_0, \ldots, \widetilde{P}_{r(d)}$. Assume that the knot $\widetilde{\xi}_k = \cdots = \widetilde{\xi}_{k+j}$ is repeated $j+1$ times. It implies that $C(\xi)$ may be $d-(j+1)$ times differentiable at this point. Let also assume that C is one more differentiable at this point. This means that at least one knot is not needed there and may be removed. Following Subsec. 5.3.4 [12], we show the removal of the 'last' equal knot, $\xi' = \widetilde{\xi}_{k+j}$. Then, the knot vector becomes

$\Xi = (\xi_0, \dots, \xi_{r-1})$. Functions $N_{i,d}$ should be redefined and control points $P_0, \dots, P_{r(d)-1}$ fulfil the system of equations

$$P_i = \widetilde{P}_i, \qquad \text{for} \quad i = 0, \dots, k + j - d, \tag{16}$$

$$P_i = \widetilde{P}_{i+1}, \qquad \text{for} \quad i = k + j - d + 1, \dots, r(d) - 1. \tag{17}$$

If $C(\xi)$ is $d - j$ times differentiable at the repeated knot, then the shape of the curve remains unchanged. The procedure of the removal of more than one knot is given in [12].

- Order elevation
  The process of order elevation is more complicated. To perform it, a B-spline curve should be divided into independent Bezier curves by successive knot insertion in the places of their sticking, then the degree of each Bezier curve should be elevated. Finally, the additional knots should be removed.

- k-refinement
  k-refinement is an algorithm combining the h- and p-refinements. Shortly speaking, the algorithm elevates the degree first and then inserts knots. This order gives efficient savings of degrees of freedom, in contrary to the inverse order of refinements.

A valuable advantage of NURBS application for curves' approximation or a construction of test function spaces is that higher continuity at the limits of elements is easily attainable in the NURBS modeling, in spite of finite elements, where the C^1-continuity is a maximal and expensive result.

4. Approximation of functions by NURBS

General assumptions: On the reference interval $[0, 1]$ we establish

- the open knot vector $\Xi = (\xi_0, \dots, \xi_r)$;
- the regularity k_i at each knot ζ_i;
- the B-spline basis functions $N_{0,d}(\xi), \dots, N_{r(d),d}(\xi)$ of a degree d;
- their weights $w_0, \dots, w_{r(d)}$ and the global weighting function w, like in the equation (12):

$$w = \sum_{i=0}^{r(d)} w_i N_{i,d};$$

- the NURBS basis functions

$$R_i = \frac{w_i N_{i,d}}{w};$$

- the NURBS space

$$N_h = \left\{ n_h = \sum_{i=0}^{r(d)} v_i R_i \right\};$$

- the bijective parametrization F of the curve $\Gamma \subset \Re^2$, $F\colon [0,1] \to \Gamma$. F – C^1-continuous with its inverse, derivatives of F and its inverse F^{-1} along the curve, i.e., the tangent derivative $\partial F^{-1}/\partial\tau$ are bounded by fixed positive constants m_0, M_0:

$$0 < m_0 \le |\nabla F|, \ \left|\partial F^{-1}/\partial\tau\right| \le M_0; \tag{18}$$

  its approximation $F_h\colon [0,1] \to \Re^2$, $F_h = (F_h^1, F_h^2)$, $F_h^i \in N_h$,
  $m_0 \le \|\nabla F_h^i\| \le M_0$, $i = 1, 2$;

- the NURBS space on Γ i.e. the usual push-forward of N_h through Γ:

$$V_h = \{v_h = n_h \circ F^{-1},\, n_h \in N_h\} \tag{19}$$

(NURBS of V_h and N_h may be different). In what follows, we will denote by c a positive constant which may be different at each occurrence.

On the interval $[0,1]$ divided by the knots $0 = \xi_0 \le \xi_1 \le \ldots \le \xi_{r-1} \le \xi_r = 1$ and the corresponding $\zeta_0 < \zeta_1 < \ldots < \zeta_{\rho-1} < \zeta_\rho = 1$ (cf. (9)) we are able to define the bent Sobolev space

$$\begin{aligned} &\mathcal{H}^m[0,1] = \{v : [0,1] \to \Re :\ \forall i = 0,\ldots,\rho-1,\ v \in H^m(\zeta_i,\zeta_{i+1}),\\ &\zeta_i = \xi_j = \xi_{j+1} = \ldots = \xi_{j+k} \Leftrightarrow v \in C^{d-k}(\zeta_i)\}, \end{aligned} \tag{20}$$

where $H^m(\xi_i,\xi_{i+1})$ is the usual Sobolev space on the interval (ξ_i,ξ_{i+1}). It is assumed that in a knot which is k times repeated, the derivatives up to the order $d-k$ at both sides coincide. The corresponding seminorms are defined as sums of seminorms over each interval

$$|\cdot|^2_{m,[0,1]} = \sum_{i=0}^{\rho-1} |\cdot|^2_{m,[\zeta_i,\zeta_{i+1}]}, \tag{21}$$

$$\|\cdot\|^2_{m,[a,b]} = \sum_{k=0}^{m} |\cdot|^2_{k,[a,b]}, \tag{22}$$

$$|\cdot|_{m,[a,b]} = |\cdot|_{H^m([a,b])}, \tag{23}$$

$$|\cdot|_{\infty,[a,b]} = |\cdot|_{L^\infty([a,b])}. \tag{24}$$

We denote by $Q_i = [\zeta_i,\zeta_{i+1}]$ the i-th subinterval of $[0,1]$, $T_i = F_h(Q_i)$ its range by the approximate mapping F_h, $h_i = diam(T_i)$ length of T_i, $i = 0,\ldots\rho-1$, $h_T = \max\{h_i,\ i = 0,\ldots,\rho-1\}$ its maximal value. Let $\Pi_{N,h}$ be a projection of $L^2[0,1]$ into N_h and $\Pi_{V,h}$ be a projection of $L^2(\Omega)$ into V_h, defined as

$$\Pi_{V,h} = (\Pi_{N,h}(v \circ F)) \circ F^{-1}.$$

We cite the following theorems on approximation, which estimate the interpolation error of NURBS by the Sobolev norm of the approximated function. They are analogous to the corresponding error estimates for the finite element method, cf., e.g., [4], Ch. 3.

Theorem 1. *[2], Th.3.2. Let d be a degree of NURBS, $0 \le t \le s \le d+1$ (integer indices). Suppose that $v \in H^s(\Gamma)$. By the above assumptions*

$$\sum_{T_i} |v - \Pi_{V_h} v|^2_{t,T_i} \le c \sum_{T_i} h_i^{2(s-t)} \sum_{i=0}^{s} \|\nabla F\|^{2(i-s)}_{\infty,Q_i} |v|^2_{i,T_i}\ \forall v \in H^s(\Gamma). \tag{25}$$

This result may be shortly summed up using the definition of the norm in the bent Sobolev space. By the assumption (18) and definitions of Sobolev norms (21)–(24):

Corollary 1.

$$|v - \Pi_{V_h} v|_{t,\Gamma} \le c\, h_T^{s-t}\, m_0^{-s} \|v\|_{s,\Gamma}. \tag{26}$$

In [3] an estimate depending on h and d is given for a two-dimensional physical domain. If we reduce it to one-domain case we obtain, by our general assumptions listed on the beginning of this chapter:

Theorem 2. *[3], Th. 7. For fixed non-negative integers: $d \ge 2k_{\max} - 1$, $t \le k_{\min}$, $s \in [k_{\max}, d+1]$ and for any function $v \in H^s(\Gamma)$*

$$|v - \Pi_{V_h} v|_{t,\Gamma} \le c\,(d - k_{\max} + 1)^{t-s} h_T^{s-t}\, \|v\|_{s,\Gamma}. \tag{27}$$

5. Approximation of the boundary

We consider a bounded domain $\Omega^i \in \Re^2$, its boundary Γ and the external domain $\Omega^e = \Re^2 \setminus (\Omega^i \cup \Gamma)$. The boundary is parametrized by the function $F : [0,1] \to \Gamma$, $F(\tau) = (F^1(\tau), F^2(\tau))$. We assume that $F \in \mathcal{H}^s(0,1) \times \mathcal{H}^s(0,1)$ with the corresponding norm and seminorm $|F|^2_{m,[0,1]} = |F^1|^2_{m,[0,1]} + |F^2|^2_{m,[0,1]}$. Let F_h be an approximation of F, i.e., let F^i_h be an approximation of F^i in the sense of Sec. 4 and $\Gamma_h = F_h[0,1]$ be the range of this approximation. Let W_d be the space of vector interpolants of the order d:

$$W_d = \left\{ F_h = (F^1_h, F^2_h),\ F^i_h \in N_h[0,1] \right\}. \tag{28}$$

We assume next that $dF^i/d\xi$, $dF^i_h/d\xi$ are bounded from below and from above, i.e., there are positive constants m_0 and M_0 independent on $\xi \in [0,1]$ such that

$$0 < m_0 \le \left|\frac{dF^i}{d\xi}\right|, \left|\frac{dF^i_h}{d\xi}\right| \le M_0 \ \ \forall \xi \in [0,1], \tag{29}$$

and they fulfil the Lipschitz condition:

$$\left|\frac{dF^i}{d\xi}(\xi + \Delta\,\xi) - \frac{dF^i}{d\xi}(\xi)\right| \le M_1 |\Delta\,\xi| \ \ \forall \xi, (\xi + \Delta\xi) \in [0,1] \tag{30}$$

and analogous inequality for F^i_h.

In the following some lemmas are proved. They estimate the boundary approximation error in the L^∞ norm (Lemma 1) or give some auxiliary estimates of some elements of the expression for the general approximation error in the proof of the Lemma 6 (Lemmas 2-4).

Lemma 1. *By the general assumptions of Sec. 4 and Theorem 2 we have the following estimates for the boundary approximation as a simple consequence of Theorem 2:*

$$\inf_{F_h \in W_d} \|F - F_h\|_{t,[0,1]} \le c\, h^{s-t} (d - k_{\max} + 1)^{t-s} \|F\|_{s,[0,1]}, \tag{31}$$

$$\inf_{F_h \in W_d} \left\|\frac{dF^i}{d\xi} - \frac{dF^i_h}{d\xi}\right\|_{t,[0,1]} \le c\, h^{s-t-1} (d - k_{\max} + 1)^{1+t-s} \|F\|_{s,[0,1]}. \tag{32}$$

For this "optimal" approximation F_h of F we obtain

$$\sup_{\xi \in [0,1]} |F(\xi) - F_h(\xi)| \le c\, h^{s-1} (d - k_{\max} + 1)^{1-s} \|F\|_{s,[0,1]}, \tag{33}$$

$$\sup_{\xi \in [0,1]} \left|\frac{dF^i}{d\xi}(\xi) - \frac{dF^i_h}{d\xi}(\xi)\right| \le c\, h^{s-2} (d - k_{\max} + 1)^{2-s} \|F\|_{s,[0,1]}. \tag{34}$$

To prove inequalities (33) and (34) on the basis of (31) and (32) it is sufficient to recall, that for the fixed interval $[0,1]$, the space $H^1(0,1)$ is compactly embedded in $L^\infty(0,1)$, i.e., there is a constant c, independent of any function v, for which

$$\|v\|_{\infty,[0,1]} \le c \|v\|_{1,[0,1]} \ \forall v \in H^1(0,1), \tag{35}$$

cf. [4]. Next, we estimate left-hand sides of inequalities (33), (34) by the left-hand sides of inequalities (31), (32) assuming $t = 1$ and applying inequality (35).

We suppose then that any function $v \in C^1(\Gamma)$ may be extended onto some neighbourhood U_Γ of the boundary Γ and therefore we are able to define the tangent derivative of v: $\partial v/\partial \tau$ on Γ. Let in expressions like dv/ds, dx/ds etc. s means the arc length parameter of Γ. We have then at any arbitrary point $x \in \Gamma$

$$\frac{\partial v}{\partial \tau}(\mathbf{x}) = \frac{\partial v}{\partial s}(\mathbf{x}).$$

This function v may be copied on Γ_h using push-forward and pull-back:

$$v(\mathbf{x}_h) = (v \circ F \circ F_h^{-1})(\mathbf{x}_h). \tag{36}$$

It means that

$$v(\mathbf{x}_h(\xi)) = v(\mathbf{x}(\xi))\ \forall \xi \in [0,1]. \tag{37}$$

Let in the same way in expressions like ds_h/ds, dv/ds_h etc. s_h means the arc length parameter of Γ_h. We define similarly $\partial v/\partial \tau_h(x) = \partial v/\partial s_h(x)$. The function $v \circ F$ is approximated by functions $v_h \circ F_h \in V_h$.

Lemma 2. *By the previous assumptions there is a constant $c = c(s)$ such that*

$$\sup_{\xi\in[0,1]} \left|1 - \frac{ds_h}{ds}(\xi)\right| \le c\, h^{s-2}(d - k_{\max} + 1)^{2-s}\|F\|_{s,[0,1]}, \tag{38}$$

$$\sup_{\xi_1,\xi_2\in[0,1]} \left|1 - \frac{ds_h}{ds}(\xi_1)\frac{ds_h}{ds}(\xi_2)\right| \le c\, h^{s-2}(d - k_{\max} + 1)^{2-s}\|F\|_{s,[0,1]}, \tag{39}$$

$$\left\|\frac{dv}{d\tau}(\mathbf{x}) - \frac{dv}{d\tau_h}(\mathbf{x}_h)\right\|_{-1/2,\Gamma} \le c\, h^{s-2}(d - k_{\max} + 1)^{2-s}\|F\|_{s,[0,1]}\|v\|_{1/2,\Gamma}, \tag{40}$$

$$\left\|\frac{d\mathbf{y}_h}{d\tau^y}(y)\frac{dq_h}{d\tau^x}(\mathbf{x}) - \frac{dv_h}{d\tau_h^y}(\mathbf{y}_h)\frac{dq_h}{d\tau_h^x}(\mathbf{x}_h)\right\|_{-1/2,\Gamma}$$

$$\le c\, h^{s-2}(d - k_{\max} + 1)^{2-s}\|F\|_{s,[0,1]}\|v\|_{1/2,\Gamma}. \tag{41}$$

Proof. By the Eqs. (29), (34) and (35)

$$\frac{ds_h}{ds} = \left|\frac{dF_h}{d\xi}\right| \cdot \left|\frac{dF}{d\xi}\right|^{-1},$$

$$\sup_{\xi\in[0,1]} \left|1 - \frac{ds_h}{ds}\right| = \sup_{\xi\in[0,1]} \left|\left(\left|\frac{dF}{d\xi}\right| - \left|\frac{dF_h}{d\xi}\right|\right) \cdot \left|\frac{dF}{d\xi}\right|^{-1}\right|$$
$$\le c\left\|\left(\left|\frac{dF}{d\xi}\right| - \left|\frac{dF_h}{d\xi}\right|\right) \cdot \left|\frac{dF}{d\xi}\right|^{-1}\right\|_{1,[0,1]} \le c\, h^{s-2}(d - k_{\max} + 1)^{2-s}\|F\|_{s,[0,1]},$$

$$\sup_{\xi_1,\xi_2\in[0,1]} \left|1 - \frac{ds_h}{ds}(\xi_1)\frac{ds_h}{ds}(\xi_2)\right| \le \left\|1 - \frac{ds_h}{ds}(\xi_1)\frac{ds_h}{ds}(\xi_2)\right\|_{1,[0,1]}$$
$$= \left\|\left(1 - \frac{ds_h}{ds}(\xi_2)\right) + \left(1 - \frac{ds_h}{ds}(\xi_1)\right) \cdot \frac{ds_h}{ds}(\xi_2)\right\|_{1,[0,1]} \le c\, h^{s-2}(d - k_{\max} + 1)^{2-s}\|F\|_{s,[0,1]},$$

$$\left\| \frac{dv}{d\tau}(\mathbf{x}) - \frac{dv}{d\tau_h}(\mathbf{x}_h) \right\|_{-1/2,\Gamma} = \left\| \frac{dv}{ds}(\mathbf{x}) - \frac{dv}{ds_h}(\mathbf{x}_h) \right\|_{-1/2,\Gamma}$$
$$= \left\| \frac{\partial v}{ds}(\mathbf{x}) \cdot \left(1 - \frac{ds_h}{ds}(\mathbf{x}) \right) \right\|_{-1/2\Gamma} = \sup_{q \in H^{1/2}(\Gamma)} \int_\Gamma \frac{dv}{ds} \left(1 - \frac{ds_h}{ds} \right) (\mathbf{x}) q(\mathbf{x}) ds \cdot \left(\|q\|_{1/2,\Gamma} \right)^{-1}$$
$$\leq \sup_{\xi \in [0,1]} \left| 1 - \frac{ds_h}{ds} \right| \left\| \frac{dv}{ds} \right\|_{-1/2,\Gamma} \leq c\, h^{s-2} (d - k_{\max} + 1)^{2-s} \|F\|_{s,[0,1]} \|v\|_{1/2,\Gamma},$$

$$\left\| \frac{dv}{d\tau}(\mathbf{x}) - \frac{dv}{d\tau_h}(\mathbf{x}_h) \right\|_{-1/2,\Gamma} = \left\| \frac{dv}{ds}(\mathbf{x}) - \frac{dv}{ds_h}(\mathbf{x}_h) \right\|_{-1/2,\Gamma}$$
$$= \left\| \frac{\partial v}{ds}(\mathbf{x}) \cdot \left(1 - \frac{ds_h}{ds}(\mathbf{x}) \right) \right\|_{-1/2\Gamma} = \sup_{q \in H^{1/2}(\Gamma)} \int_\Gamma \frac{dv}{ds} \left(1 - \frac{ds_h}{ds} \right) (\mathbf{x}) q(\mathbf{x}) ds \cdot \left(\|q\|_{1/2,\Gamma} \right)^{-1}$$
$$\leq \sup_{\xi \in [0,1]} \left| 1 - \frac{ds_h}{ds} \right| \left\| \frac{dv}{ds} \right\|_{-1/2,\Gamma} \leq c\, h^{s-2} (d - k_{\max} + 1)^{2-s} \|F\|_{s,[0,1]} \|v\|_{1/2,\Gamma},$$

$$\left\| \frac{dv_h}{d\tau^y}(\mathbf{y}) \frac{dq_h}{d\tau^x}(\mathbf{x}) - \frac{dv_h}{d\tau_h^y}(\mathbf{y}_h) \frac{dq_h}{d\tau_h^x}(\mathbf{x}_h) \right\|_{-1/2,\Gamma}$$
$$\leq \left| \frac{dv_h}{d\tau^y}(\mathbf{y}) \right| \cdot \left| \frac{dq_h}{d\tau^x}(\mathbf{x}) - \frac{dq_h}{d\tau_h^x}(\mathbf{x}_h) \right| + \left| \frac{dq_h}{d\tau_h^x}(\mathbf{x}_h) \right| \cdot \left| \frac{dv_h}{d\tau^y}(\mathbf{y}) - \frac{dv_h}{d\tau_h^y}(\mathbf{y}_h) \right|$$
$$\leq c\, h^{s-2} (d - k_{\max} + 1)^{2-s} \|F\|_{s,[0,1]} \|v\|_{1/2,\Gamma}.$$

At any point $\mathbf{x} \in \Gamma$ we define a local tangential-normal coordinate system with unit vectors τ^x and n^x and at $\mathbf{y}$ analogously τ^y and n^y. We assume that for each such $\mathbf{x}$ there is its neighbourhood U_x in which the boundary Γ may be described in this coordinate system by a function $f \in C^1(a_1, a_2)$ whose derivative is Lipschitz-continuous there:

$$(\mathbf{y}_1, \mathbf{y}_2) \in U_x \Rightarrow y_1 \in (a_1, a_2), \quad \text{where} \quad a_1 < 0 < a_2,$$

$$(\mathbf{y}_1, \mathbf{y}_2) \in U_x \cap \Gamma \Leftrightarrow \mathbf{y}_2 = f(\mathbf{y}_1), \qquad f(0) = 0,$$

$$\left| \frac{df}{dy_1}(\mathbf{y}_1) \right| \leq L_0 \mathbf{y}_1, \qquad \forall \mathbf{y}_1 \in (a_1, a_2).$$

Lemma 3. *For such a function f and vectors τ^x, τ^y, n^x, n^y in the local system, where $\mathbf{x} = (0,0)$:*

$$\left| n^x \cdot \frac{\mathbf{y}}{|\mathbf{y}|} \right| \leq L_0 |\mathbf{y}|, \tag{42}$$

$$|\tau^x - \tau^y| \leq L_0 |\mathbf{y}|, \tag{43}$$

$$|\tau^x (\tau^x - \tau^y)| \leq 0.5 L_0 |\mathbf{y}|^2. \tag{44}$$

The Lemma was proved in [11].

Lemma 4. *We choose two points $\mathbf{x}, \mathbf{y} \in \Gamma$ and their approximations*

$$\mathbf{x}_h, \mathbf{y}_h \in \Gamma_h.$$

We denote

$$\mathbf{r} = \mathbf{x} - \mathbf{y}, \qquad r = |\mathbf{r}|, \qquad \mathbf{r}_h = \mathbf{x}_h - \mathbf{y}_h, \qquad r_h = |\mathbf{r}_h|.$$

Like in Lemma 3 we introduce τ_h^x, $\mathbf{n}_h^x$ τ_h^y, $\mathbf{n}_h^y$ *at the points* $\mathbf{x}_h$, $\mathbf{y}_h$. *By the previous assumptions about approximation the following estimates hold:*

$$|r - r_h| \le |\mathbf{r} - \mathbf{r}_h| \le Crh^{s-1}(d - k_{\max} + 1)^{1-s}\|F\|_{s,[0,1]}, \tag{45}$$

$$|\tau^x - \tau_h^x| \le Ch^{s-2}(d - k_{\max} + 1)^{2-s}\|F\|_{s,[0,1]}, \tag{46}$$

$$|\tau^x\tau^y - \tau_h^x\tau_h^y| \le Ch^{s-2}(d - k_{\max} + 1)^{2-s}\|F\|_{s,[0,1]}, \tag{47}$$

$$|\mathbf{n}^x \cdot \mathbf{r}/r - \mathbf{n}_h^x \cdot \mathbf{r}_h/r_h| \le Crh^{s-2}(d - k_{\max} + 1)^{2-s}\|F\|_{s,[0,1]}, \tag{48}$$

$$\left|\mathbf{n}^y \cdot \mathbf{r}/r - \mathbf{n}_h^y \cdot \mathbf{r}_h/r_h\right| \le Crh^{s-2}(d - k_{\max} + 1)^{2-s}\|F\|_{s,[0,1]}. \tag{49}$$

Proof.
$|r - r_h| \le |\mathbf{r} - \mathbf{r}_h| \le Cr$ by the triangle inequality,

$$|\mathbf{r} - \mathbf{r}_h| = |(\mathbf{x} - \mathbf{y}) - (\mathbf{x}_h - \mathbf{y}_h)| = |(\mathbf{x} - \mathbf{x}_h) - (\mathbf{y} - \mathbf{y}_h)|$$
$$\le \|\mathbf{x} - \mathbf{x}_h\|_{1,[0,1]} + \|(\mathbf{y} - \mathbf{y}_h\|_{1,[0,1]} \le Ch^{s-1}(p - k_{\max} + 1)^{1-s}\|F\|_{s,[0,1]}.$$

$$|\tau^x - \tau_h^x| = \left|\frac{d\mathbf{x}}{ds} - \frac{d\mathbf{x}_h}{ds_h}\right| \le \left|\frac{d\mathbf{x}}{ds} - \frac{d\mathbf{x}}{ds_h}\right| + \left|\frac{d\mathbf{x}}{ds_h} - \frac{d\mathbf{x}_h}{ds_h}\right|$$
$$= \left|\frac{d\mathbf{x}}{ds_h}\frac{ds_h}{ds} - \frac{d\mathbf{x}}{ds_h}\right| + \left|\left(\frac{d\mathbf{x}}{d\xi} - \frac{d\mathbf{x}_h}{d\xi}\right) \cdot \frac{d\xi}{ds_h}\right| \le \left|\frac{d\mathbf{x}}{ds_h}\right| \cdot \left|\frac{ds_h}{ds} - 1\right| + \left|\frac{d\mathbf{x}}{d\xi} - \frac{d\mathbf{x}_h}{d\xi}\right| \left|\frac{d\xi}{ds_h}\right|$$
$$\le C\left(1 + \frac{1}{m}\right) h^{s-2}(d - k_{\max} + 1)^{2-s}\|F\|_{s,[0,1]}.$$

$$|\tau^x\tau^y - \tau_h^x\tau_h^y| = |\tau^x\tau^x + \tau^x\left(\tau^y - \tau^x\right) - \tau_h^x\tau_h^x - \tau_h^x\tau_h^x - \left(\tau_h^y - \tau_h^x\right)|$$
$$= |\tau^x\left(\tau^y - \tau^x\right) - \tau_h^x\left(\tau_h^y - \tau_h^x\right)| = |\left(\tau^x - \tau_h^x\right)\left(\tau^y - \tau^x\right) + \tau_h^x\left[\left(\tau^y - \tau_h^y\right) - \left(\tau^x - \tau_h^x\right)\right]|$$
$$\le Ch^{s-2}(d - k_{\max} + 1)^{2-s}\|F\|_{s,[0,1]}.$$

$$\left|\frac{\mathbf{n}^x \cdot \mathbf{r}}{r} - \frac{\mathbf{n}_h^x \cdot \mathbf{r}_h}{r_h}\right| \le \left|\frac{\mathbf{n}^x \cdot \mathbf{r}}{r} - \frac{\mathbf{n}^x \cdot \mathbf{r}_h}{r_h}\right| + \left|\frac{\mathbf{n}^x \cdot \mathbf{r}_h}{r_h} - \frac{\mathbf{n}_h^x \cdot \mathbf{r}_h}{r_h}\right|$$
$$\le |\mathbf{n}^x| \cdot \left|\frac{\mathbf{r}r_h - r\mathbf{r}_h}{rr_h}\right| + |\mathbf{n}^x - \mathbf{n}_h^x|\frac{|\mathbf{r}_h|}{r_h} \le \frac{1}{rr_h}\left[|\mathbf{r}| \cdot |r_h - r| + r \cdot |\mathbf{r} - \mathbf{r}_h|\right]$$
$$\le Ch^{s-2}(d - k_{\max} + 1)^{2-s}\|F\|_{s,[0,1]}.$$

It was proved in Corollary 3.1.2 [9], that $|\mathbf{n}^x \cdot \mathbf{r}| \le Cr^2$ for $f \in C^1(a_1, a_2)$ with Lipschitz continuous derivatives. This completes the proof of (48). The estimate (49) can be proved analogously.

6. The acoustic wave scattering problem in the 2-D space

Let us recall the classical formulation of the acoustic wave scattering problem on an elastic body immersed in a fluid. We denote by $\Omega^i \in \Re^2$ a regular, bounded domain occupied by the body, Γ is its boundary and Ω^e is the unbounded external domain around the body filled with a fluid. We look for a function $p : \Omega^e \to \Re$ which describes the total pressure of the wave in the fluid. It is a sum of the incident and scattered wave pressure:

$$p(\mathbf{x}) = p^{inc}(\mathbf{x}) + p^s(\mathbf{x}). \tag{50}$$

The total pressure fulfils the Helmholtz differential equation (k is the wavenumber):

$$-\Delta p(\mathbf{x}) - k^2 p(\mathbf{x}) = 0 \qquad \text{for each} \qquad \mathbf{x} \in \Omega^e, \tag{51}$$

a boundary condition on Γ:

$$\frac{\partial p}{\partial n_x} = \varepsilon p(\mathbf{x}) \qquad \text{for} \qquad \mathbf{x} \in \Gamma \tag{52}$$

and the scattered pressure fulfils the Sommerfeld condition at infinity:

$$\left| \frac{\partial p^s}{\partial r} - ikp^s \right| = o(r^{-0.5}) \qquad \text{for} \qquad r = |\mathbf{x}| \to \infty, \tag{53}$$

where i is the imaginary unit. The boundary condition (52) corresponds to a spring-like scatterer, which models a rubber layer on the surface of the body. For details of this model see [8] or [9]. The fundamental solution of the Helmholtz equation (51) in the plane is

$$\Phi(\mathbf{x}, \mathbf{y}) = 0.25 i H_0^1(k|\mathbf{x} - \mathbf{y}|), \tag{54}$$

where $H_n^1(x)$ is the Hankel function of the first kind of the order n.

The elastic scattering problem (50)–(53) may be formulated in a function space $V = H^{0.5}(\Gamma)$ being the Sobolev space on the boundary Γ with the corresponding scalar product and norm. This boundary-value problem is replaced there by a variational boundary integral equation using the Burton-Miller formulation with coefficient $\alpha \in [0, 1]$. It is expected to find a function $p \in \Gamma$ such that

$$a(p, q) = l(q), \qquad \forall q \in V. \tag{55}$$

The continuous, sesquilinear form $a : V \times V \to C$ is defined by the formula

$$\begin{aligned} a(p,q) = 0.5\alpha \int_\Gamma p(\mathbf{x})q(\mathbf{x})ds(\mathbf{x}) + 0.5(1-\alpha)k^{-1}\varepsilon i \int_\Gamma p(\mathbf{x})q(\mathbf{x})ds(\mathbf{x}) \\ + \alpha \int_\Gamma \int_\Gamma \left[\varepsilon\Phi(\mathbf{x},\mathbf{y})p(\mathbf{y})q(\mathbf{x}) - \frac{\partial \Phi}{\partial n^y}(\mathbf{x},\mathbf{y})p(\mathbf{y})q(\mathbf{x}) \right] ds(\mathbf{y})ds(\mathbf{x}) \\ + (1-\alpha)k^{-1}i \int_\Gamma \int_\Gamma \left[\Phi(\mathbf{x},\mathbf{y}) \frac{\partial p}{\partial \tau^y}(\mathbf{y}) \frac{\partial q}{\partial \tau^x}(\mathbf{x}) - k^2 \Phi(\mathbf{x},\mathbf{y})p(\mathbf{y})q(\mathbf{x})\tau^y\tau^x \right. \\ \left. + \varepsilon \frac{\partial \Phi}{\partial n^y}(\mathbf{x},\mathbf{y})p(\mathbf{y})q(\mathbf{x}) \right] ds(\mathbf{y})ds(\mathbf{x}) \end{aligned} \tag{56}$$

and the continuous, semilinear form $l : V \to C$ by

$$l(q) = \alpha \int_\Gamma p^{inc}(\mathbf{x})q(\mathbf{x})ds(\mathbf{x}) + (1-\alpha)k^{-1}i \int_\Gamma \frac{\partial p^{inc}}{\partial n^x}(\mathbf{x})q(\mathbf{x})ds(\mathbf{x}). \tag{57}$$

a fulfils the Gårding inequality

$$Re[a(q,q)+c(q,q)] \geq \gamma \|v\|^2, \qquad \forall v \in V, \tag{58}$$

where c is a fixed sesquilinear, compact form and γ a fixed positive constant. For details see [1, 5, 8, 9, 18].

7. Convergence estimates

We know that the Hankel functions are interrelated by the equation

$$H_1^1(x) = -\frac{d}{dx} H_0^1(x), \tag{59}$$

cf. [13]. For $n = 0, 1$, $x, y > 0$ the following estimate was proved in Lemma 5 [11]:

$$\left| 1 - \frac{H_n^1(y)}{H_n^1(x)} \right| \leq C \frac{|x-y|}{y}. \tag{60}$$

Lemma 5. *There is a constant $c(s)$ independent on different points $\mathbf{x}$, $\mathbf{y} \in \Gamma$ such that for these points and their approximations $\mathbf{x}_h$, $\mathbf{y}_h \in \Gamma_h$:*

$$|\Phi(\mathbf{x},\mathbf{y}) - \Phi(\mathbf{x}_h,\mathbf{y}_h)| \leq c|\Phi(\mathbf{x},\mathbf{y})| \cdot c\, h^{s-1}(d-k_{\max}+1)^{1-s}\|F\|_{s,[0,1]}, \tag{61}$$

$$\left| \frac{\partial \Phi}{\partial n^y}(\mathbf{x},\mathbf{y}) - \frac{\partial \Phi}{\partial n_h^y}(\mathbf{x}_h,\mathbf{y}_h) \right| \leq c \left[\left| \frac{\partial \Phi}{\partial n^y}(\mathbf{x},\mathbf{y}) \right| + 1 \right] h^{s-2}(d-k_{\max}+1)^{2-s}\|F\|_{s,[0,1]}. \tag{62}$$

Proof. To prove the inequality (61) we need the above (60) formula. To the numerator of its right-hand side we apply the estimate (45), what provides the expected result.

To prove (62) we need the definition (54) of the function Φ and the property (59).

$$\frac{\partial \Phi}{\partial n^y}(\mathbf{x},\mathbf{y}) = \frac{i}{4}\frac{d}{dr}H_0^1(kr) \cdot \frac{\mathbf{r}\cdot\mathbf{n}^y}{r} = \frac{k}{4i}H_1^1(kr)\frac{\mathbf{r}\cdot\mathbf{n}^y}{r}. \tag{63}$$

Then we are able to estimate the following

$$\begin{aligned} \left| \frac{\partial \Phi}{\partial n^y}(\mathbf{x},\mathbf{y}) - \frac{\partial \Phi}{\partial n_h^y}(\mathbf{x}_h,\mathbf{y}_h) \right| &\leq \frac{k}{4} \left| \left[H_1^1(kr) - H_1^1(kr_h) \right] \frac{\mathbf{r}\cdot\mathbf{n}^y}{r} \right| \\ &+ \frac{k}{4} H_1^1(kr) \left| \frac{\mathbf{r}\cdot\mathbf{n}^y}{r} - \frac{\mathbf{r}_h\cdot\mathbf{n}_h^y}{r_h} \right| \leq cH_1^1(kr) \left| \frac{\mathbf{r}\cdot\mathbf{n}^y}{r} \right| \frac{|r-r_h|}{r_h} \\ &+ cH_1^1(kr) r h^{s-2}(d-k_{\max}+1)^{2-s}\|F\|_{s,[0,1]}. \end{aligned} \tag{64}$$

$H_1^1(kr)r$ is bounded, cf. (61). Equations (45) and (49) imply the inequality (62).

In computations we use approximate values of the forms a and l obtained on the approximate boundary Γ_h using numerical integration. In fact we obtain an approximate solution $p_h \in V_h$ of the equation

$$a_h(p_h,q_h) = l_h(q_h), \qquad \forall q_h \in V_h. \tag{65}$$

V_h is defined by (19).

Theorem 3. *We accept the following regularity assumptions and approximation tools:*

- *the solution p of the equation (55) belongs to the space $H^\sigma(\Gamma)$ for an integer $\sigma \geq 2$;*

- *the parametrization* $\mathbf{F}$ *of* Γ *belongs to* $H^s[0,1] \times H^s[0,1]$ *for an integer* $s > \sigma$*;*
- *its approximation* $F_h \in N_h \times N_h$ *with B-spline basis functions of degree* $d \geq s$*;*
- *the approximate shape- and test functions* p_h*,* $q_h \in V_h$*;*
- *the assumptions of Theorem 2 are fulfilled;*
- *incident pressure functions* p^{inc} *and* $\partial p^{inc}/\partial n^x$ *are replaced by their approximations* p_h^{inc} *and* $\partial p_h^{inc}/\partial n_h^x$ *on* Γ_h*.*

Then there is a positive constant $c = c(s,\sigma)$*, for which*

$$\|p - p_h\|_{0.5,\Gamma} \leq c\left[\|p^{inc} - p_h^{inc}\|_{-0.5,\Gamma} + \left\|\frac{\partial p^{inc}}{\partial n^x} - \frac{\partial p_h^{inc}}{\partial n_h^x}\right\|_{-0.5,\Gamma}\right]$$
$$+ c\left[h^{s-2}(d - k_{\max} + 1)^{2-s}\left(\|F\|_{s,[0,1]}\|p\|_{0.5\Gamma} + \|p\|_{s,\Gamma}\right)\right]. \quad (66)$$

Proof. We can use the second Strang Lemma in the version proved in Theorem 4.3 [10].

Lemma 6. *By the assumptions of Theorem 3, for* $p \in V$*,* $p_h \in V_h$ *there is a constant* c *for which*

$$\|p - p_h\| \leq c\left\{\inf_{v_h\in V_h}\left[\|p - v_h\| + \sup_{q_h\in V_h}\frac{|a_h(v_h,q_h) - a(v_h,q_h)|}{\|q_h\|}\right] + \sup_{q_h\in V_h}\frac{|l_h(q_h) - l(q_h)|}{\|q_h\|}\right\}. \quad (67)$$

Proof of Theorem 3

We start from the first term

$$|a_h(v_h,q_h) - a(v_h,q_h)| \leq \Bigg|\int_{\Gamma_h}\left[\alpha v_h(\mathbf{x}_h) + i(1-\alpha)k^{-1}\varepsilon v_h(\mathbf{x}_h)\right]q_h(\mathbf{x}_h)ds(\mathbf{x})$$
$$+ \int_{\Gamma_h}\int_{\Gamma_h}\Big\{\alpha\varepsilon\Phi(\mathbf{x}_h,\mathbf{y}_h)v_h(\mathbf{y}_h)q_h(\mathbf{x}_h) - \alpha\frac{\partial\Phi}{\partial n_h^y}(\mathbf{x}_h,\mathbf{y}_h)v_h(\mathbf{y}_h)q_h(\mathbf{x}_h)$$
$$+ i(1-\alpha)k^{-1}\varepsilon\frac{\partial\Phi}{\partial n_h^x}(\mathbf{x}_h,\mathbf{y}_h)v_h(\mathbf{y}_h)q_h(\mathbf{x}_h) - i(1-\alpha)k\Phi(\mathbf{x}_h,\mathbf{y}_h)\tau_h^x\tau_h^y v_h(\mathbf{y}_h)q_h(\mathbf{x}_h)$$
$$+ i(1-\alpha)k^{-1}\Phi(\mathbf{x}_h,\mathbf{y}_h)\frac{\partial v_h}{\partial\tau_h^y}(\mathbf{y}_h)\frac{\partial q_h}{\partial\tau_h^x}(\mathbf{x}_h)\Big\}ds(\mathbf{y})ds(\mathbf{x})$$
$$- \int_{\Gamma}\left[\alpha v_h(\mathbf{x}) + i(1-\alpha)k^{-1}\varepsilon v_h(\mathbf{x})\right]q_h(\mathbf{x})ds(\mathbf{x})$$
$$- \int_{\Gamma}\int_{\Gamma}\Big\{\alpha\varepsilon\Phi(\mathbf{x},\mathbf{y})v_h(\mathbf{y})q_h(\mathbf{x}) - \alpha\frac{\partial\Phi}{\partial n^y}(\mathbf{x},\mathbf{y})v_h(\mathbf{y})q_h(\mathbf{x})$$
$$+ i(1-\alpha)k^{-1}\varepsilon\frac{\partial\Phi}{\partial n^x}(\mathbf{x},\mathbf{y})v_h(\mathbf{y})q_h(\mathbf{x}) - i(1-\alpha)k\Phi(\mathbf{x},\mathbf{y})\tau^x\tau^y v_h(\mathbf{y})q(\mathbf{x})$$
$$+ i(1-\alpha)k^{-1}\Phi(\mathbf{x},\mathbf{y})\frac{\partial v_h}{\partial\tau^y}(\mathbf{y})\frac{\partial q_h}{\partial\tau^x}(\mathbf{x})\Big\}ds(\mathbf{y})ds(\mathbf{x})\Bigg|. \quad (68)$$

We remember that $v_h(\mathbf{y}) = v_h(\mathbf{y})_h,\ q_h(\mathbf{x}) = q_h(\mathbf{x}_h)$, cf. (37).

$$
\begin{aligned}
(68) \leq \Bigg| &\int_{\Gamma_h} \big(\alpha v_h(\mathbf{x}) + i(1-\alpha)k^{-1}\varepsilon v_h(\mathbf{x})\big)\, q_h(\mathbf{x}) \left[1 - \frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})}\right] ds(\mathbf{x}) \\
&+ \int_{\Gamma}\int_{\Gamma} \Bigg\{ \alpha\varepsilon\Phi(\mathbf{x},\mathbf{y}) v_h(\mathbf{y}) q_h(\mathbf{x}) \left[1 - \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})}\right] \\
&+ \alpha\varepsilon \left[\Phi(\mathbf{x}_h,\mathbf{y}_h) - \Phi(\mathbf{x},\mathbf{y})\right] v_h(\mathbf{y}) q_h(\mathbf{x}) \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})} \\
&- \alpha \frac{\partial\Phi}{\partial n^y}(\mathbf{x},\mathbf{y}) v_h(\mathbf{y}) q_h(\mathbf{x}) \left[1 - \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})}\right] \\
&- \alpha \left[\frac{\partial\Phi}{\partial n^y}(\mathbf{x},\mathbf{y}) - \frac{\partial\Phi}{\partial n_h^y}(\mathbf{x}_h,\mathbf{y}_h)\right] v_h(\mathbf{y}) q_h(\mathbf{x}) \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})} \\
&+ i(1-\alpha)k^{-1}\varepsilon \Bigg(\frac{\partial\Phi}{\partial n_h^x}(\mathbf{x}_h,\mathbf{y}_h) v_h(\mathbf{y}) q_h(\mathbf{x}) \left[1 - \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})}\right] \\
&+ \left[\frac{\partial\Phi}{\partial n^x}(\mathbf{x},\mathbf{y}) - \frac{\partial\Phi}{\partial n_h^x}(\mathbf{x}_h,\mathbf{y}_h)\right] v_h(\mathbf{y}) q_h(\mathbf{x}) \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})} \Bigg) \\
&- i(1-\alpha)k \Bigg(\Phi(\mathbf{x},\mathbf{y}) \tau^x\tau^y v_h(\mathbf{y}) q_h(\mathbf{x}) \left[1 - \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})}\right] \\
&- \Phi(\mathbf{x},\mathbf{y}) \left[\tau^x\tau^y - \tau_h^x\tau_h^y\right] v_h(\mathbf{y}) q_h(\mathbf{x}) \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})} \\
&+ \left[\Phi(\mathbf{x}_h,\mathbf{y}_h) - \Phi(\mathbf{x},\mathbf{y})\right] \tau_h^x\tau_h^y v_h(\mathbf{y}) q_h(\mathbf{x}) \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})} \Bigg) \\
&+ i(1-\alpha)k^{-1} \Bigg(\Phi(\mathbf{x},\mathbf{y}) \frac{\partial v_h}{\partial \tau^y}(\mathbf{y}) \frac{\partial q_h}{\partial \tau^x}(\mathbf{x}) \left[1 - \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})}\right] \\
&+ \Phi(\mathbf{x},\mathbf{y}) \left[\frac{\partial v_h}{\partial \tau^y}(\mathbf{y}) \frac{\partial q_h}{\partial \tau^x}(\mathbf{x}) - \frac{\partial v_h}{\partial \tau_h^y}(\mathbf{y}_h) \frac{\partial q_h}{\partial \tau_h^x}(\mathbf{x}_h)\right] \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})} \\
&+ \left[\Phi(\mathbf{x}_h,\mathbf{y}_h) - \Phi(\mathbf{x},\mathbf{y})\right] \frac{\partial v_h}{\partial \tau^y}(\mathbf{y}) \frac{\partial q_h}{\partial \tau^x}(\mathbf{x}) \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})} \Bigg) \Bigg\} ds(\mathbf{y}) ds(\mathbf{x}) \Bigg|. \qquad (69)
\end{aligned}
$$

Terms in square brackets in each line are bounded by $c\,h^{s-2}(d - k_{\max} + 1)^{2-s}\|F\|_{s,[0,1]}$. The single and double layer potentials with the adjoint one are linear and continuous in respective spaces:

$$V(v)(\mathbf{x}) = \int_{\Gamma} \Phi(\mathbf{x},\mathbf{y}) v(\mathbf{y}) ds(\mathbf{y}), \qquad V\colon H^{-0.5}(\Gamma) \to H^{0.5}(\Gamma), \tag{70}$$

$$K(v)(\mathbf{x}) = \int_{\Gamma} \frac{\partial\Phi}{\partial n^y}(\mathbf{x},\mathbf{y}) v(\mathbf{y}) ds(\mathbf{y}), \qquad K\colon H^{0.5}(\Gamma) \to H^{0.5}(\Gamma), \tag{71}$$

$$K'(v)(\mathbf{x}) = \int_{\Gamma} \frac{\partial\Phi}{\partial n^x}(\mathbf{x},\mathbf{y}) v(\mathbf{y}) ds(\mathbf{y}), \qquad K'\colon H^{0.5}(\Gamma) \to H^{0.5}(\Gamma). \tag{72}$$

To estimate the first term of the previous expression (69) we use the Schwarz inequality

$$\left| \int\limits_{\Gamma_h} \left(\alpha v_h(\mathbf{x}) + i(1-\alpha)k^{-1}\varepsilon v_h(\mathbf{x})\right) q_h(\mathbf{x}) \left[1 - \frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})}\right] ds(\mathbf{x}) \right|$$
$$\leq c\, h^{s-2}(d-k_{\max}+1)^{2-s}\|F\|_{s,[0,1]}\|v_h\|_{0.5,\Gamma}\|q_h\|_{0.5,\Gamma} \quad (73)$$

and to the next ones we use properties of the potentials (70)–(72), e.g.,

$$\left| \int\limits_{\Gamma}\int\limits_{\Gamma} \alpha\varepsilon\Phi(\mathbf{x},\mathbf{y})v_h(\mathbf{y})q_h(\mathbf{x}) \left[1 - \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})}\right] ds(\mathbf{y})ds(\mathbf{x}) \right|$$
$$\leq c\, h^{s-2}(d-k_{\max}+1)^{2-s}\|F\|_{s,[0,1]}\|V(v_h)\|_{0,\Gamma}\|q_h\|_{0,\Gamma}$$
$$\leq c\, h^{s-2}(d-k_{\max}+1)^{2-s}\|F\|_{s,[0,1]}\|v_h\|_{0.5,\Gamma}\|q_h\|_{0.5,\Gamma}, \quad (74)$$

$$\left| \int\limits_{\Gamma}\int\limits_{\Gamma} \alpha\frac{\partial\Phi}{\partial n^y}(\mathbf{x},\mathbf{y})v_h(\mathbf{y})q_h(\mathbf{x}) \left[1 - \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})}\right] ds(\mathbf{y})ds(\mathbf{x}) \right|$$
$$\leq c\, h^{s-2}(d-k_{\max}+1)^{2-s}\|F\|_{s,[0,1]}\|K(v_h)\|_{0,\Gamma}\|q_h\|_{0,\Gamma}$$
$$\leq c\, h^{s-2}(d-k_{\max}+1)^{2-s}\|F\|_{s,[0,1]}\|v_h\|_{0.5,\Gamma}\|q_h\|_{0.5,\Gamma}, \quad (75)$$

$$\left| \int\limits_{\Gamma}\int\limits_{\Gamma} \frac{\partial\Phi}{\partial n_h^x}(\mathbf{x}_h,\mathbf{y}_h)v_h(\mathbf{y})q_h(\mathbf{x}) \left[1 - \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})}\right] ds(\mathbf{y})ds(\mathbf{x}) \right|$$
$$\leq c\, h^{s-2}(d-k_{\max}+1)^{2-s}\|F\|_{s,[0,1]}\|K'(v_h)\|_{0,\Gamma}\|q_h\|_{0,\Gamma}$$
$$\leq c\, h^{s-2}(d-k_{\max}+1)^{2-s}\|F\|_{s,[0,1]}\|v_h\|_{0.5,\Gamma}\|q_h\|_{0.5,\Gamma}, \quad (76)$$

$$\left| \int\limits_{\Gamma}\int\limits_{\Gamma} \Phi(\mathbf{x},\mathbf{y})\frac{\partial v_h}{\partial\tau^y}(\mathbf{y})\frac{\partial q_h}{\partial\tau^x}(\mathbf{x}) \left[1 - \frac{ds_h(\mathbf{y}_h)}{ds(\mathbf{y})}\frac{ds_h(\mathbf{x}_h)}{ds(\mathbf{x})}\right] ds(\mathbf{y})ds(\mathbf{x}) \right|$$
$$\leq c\, h^{s-2}(d-k_{\max}+1)^{2-s}\|F\|_{s,[0,1]}\left\|V\left(\frac{\partial v_h}{\partial\tau^y}\right)\right\|_{0.5,\Gamma}\left\|\frac{\partial q_h}{\partial\tau^x}\right\|_{-0.5,\Gamma}$$
$$\leq c\, h^{s-2}(d-k_{\max}+1)^{2-s}\|F\|_{s,[0,1]}\|v_h\|_{0.5,\Gamma}\|q_h\|_{0.5,\Gamma}. \quad (77)$$

We have obtained then

$$|a_h(v_h,q_h) - a(v_h,q_h)| \leq c\, h^{s-2}(d-k_{\max}+1)^{2-s}\|F\|_{s,[0,1]}\|v_h\|_{0.5,\Gamma}\|q_h\|_{0.5,\Gamma}. \quad (78)$$

The term $|l_h(q_h) - l(q_h)|$ may be estimated analogously. Substituting it to the Eq. (67) and applying the general estimate for approximation by NURBS (27) we obtain the expected (66) estimate.

Corollary 2. *If p^{inc} and $\partial p^{inc}/\partial n^x$ are hp-approximated like p, the final estimate of Theorem 3 will have the form*

$$\|p-p_h\|_{0.5,\Gamma} \leq c\, h^{s-2}(d-k_{\max}+1)^{2-s}$$
$$\times\left(\|F\|_{s,[0,1]}\|p\|_{0.5\Gamma} + \|p\|_{s,\Gamma} + \|p^{inc}\|_{s,\Gamma} + \left\|\frac{\partial p^{inc}}{\partial n^x}\right\|_{s,\Gamma}\right). \quad (79)$$

8. Conclusions

In comparison to an analogous estimate obtained for the considered problem approximated in two dimensions by the boundary element method [11], it can be easily seen that the predicted order of convergence of both methods, i.e., the isogeometric method and the boundary element method, is the same if $k_{\max} = 1, 2$ (cf. (10)). If $k_{\max}$ depends on the degree of approximation in the sense that $d - k_{\max} = \text{const.}$, then the estimate (79) does not give the errors reduction velocity as high as the boundary element method. Numerical tests should verify the effectivity of both methods, comparing the accuracy, number of degrees of freedom, time of execution and size of the problem. They may clearly answer the question, whether the isogeometric method is in practice more effective for the problem described above.

Acknowledgment

The research was supported by the scientific grant of the Ministry of Science and Higher Education of the Republic of Poland under contract No F3/104/DS/2012.

References

[1] S. Amini. On the choice of the coupling parameter in boundary integral formulations of the the exterior acoustic problem. *Applicable Anal.*, **35**: 79–92, 1990.
[2] Y. Basilevs, L. Beirao da Veiga, J.A. Cottrell, T.J.R. Hughes, G. Sangalli. Isogeometric analysis: approximation, stability and error estimates for h-refined meshes. *Math. Models in Appl. Sci.*, **16**(7): 1031–1090, 2006.
[3] L. Beirao da Veiga, A. Buffa, J. Rivas, G. Sangalli. Some estimates for h-p-k-refinement in isogeometric analysis. *Numer. Math.*, **118**: 271–305, 2011.
[4] P.G. Ciarlet, *The finite element method for elliptic problems*, North Holland, Amsterdam, 1978.
[5] L. Demkowicz, J.T. Oden, M. Ainsworth, P. Geng. Solution of elastic scattering problems in linear acoustics using h-p boundary element methods. *J. Comput. Appl. Math.*, **36**: 29–63, 1991.
[6] N. Heuer, E.P. Stephan. The hp-version of the boundary element method. *ZAMM*, **74**: T511–T513, 1994.
[7] T.J.R. Hughes, J.A. Cottrell, Y. Basilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Methods Appl. Mech. Eng.*, **194**: 4135–4195, 2005.
[8] A. Karafiat, J.T. Oden, P. Geng. Variational formulations and hp boundary element approximations for hyper-singular integral equations for Helmholtz exterior boundary value problems in two dimensions. *Int. J. Eng. Sci.*, **31**(4): 649–672, 1993.
[9] A. Karafiat. *An analysis of the boundary element method for the acoustic scattering problem* [in Polish]. Cracow University of Technology, Kraków, 1996.
[10] A. Karafiat. On *hp* - error estimation in the BEM for a three-dimensional Helmholtz exterior problem. *Comput. Methods Appl. Mech. Eng.*, **150**: 199–214, 1997.
[11] A. Karafiat. A-priori estimates of the *hp*-adaptive BEM in elastic scattering of acoustic waves. *Comput. Assisted Mech. Eng. Sci.*, **7**: 559–570, 2000.
[12] P. Kiciak. *Curves and surfaces modelling* [in Polish]. WNT, Warszawa, 2005.
[13] N.N. Lebedev. *Special functions and their applications* [in Russian]. Fizmatgiz, Moscow, Leningrad, 1963.
[14] M. Maischak, P. Mund, E.P. Stephan. Adaptive multilevel BEM for acoustic scattering. *Comput. Methods Appl. Mech. Eng.*, **150**: 351–367, 1997.
[15] L. Piegl, W. Tiller. *The NURBS book*. Springer, New York, 1997.
[16] F.V. Postell, E.P. Stephan. On the h, p, and h-p versions for boundary element method – numerical results. *Comput. Methods Appl. Mech. Eng.*, **83**: 69–90, 1990.
[17] E.P. Stephan. The h-p boundary element method for solving 2- and 3-dimensional problems. *Comput. Methods Appl. Mech. Eng.*, **133**: 183–208, 1996.
[18] W.L. Wendland. Boundary element methods and their asymptotic convergence. In: P. Filippi [Ed.] *Theoretical acoustics and numerical treatment, CISM Courses and Lectures*, pp. 135–216, Springer, Vienna, 1983.