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Thermal stresses in two-dimensional model of composite elastic material

Zbigniew Wesołowski
Institute of Fundamental Technological Research, Polish Academy of Sciences, Warsaw, Poland

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A rectangular specimen consists of two kinds of grains. Each kind has a different thermal expansion coefficient. The grains are randomly distributed in J rows and K colums of the specimen. The temperature of the whole specimen is increased and produces internal strains. It is assumed that each grain interacts with its four neighbours. The interaction force is proportional to the relative displacement. If the relative displacement equals the extension due to thermal expansion, the force equals zero.

The relaxation method of calculating the equilibrium strains is used. The average maximum strains are calculated for a large number of numerical experiments. The standard deviations are calculated.

1. Model of the composite material

In homogeneous material, thermal stresses are the result of a non-uniform temperature distribution. A large number of papers and monographs devoted to this kind of stresses appeared about 1960. In composite materials, thermal stresses are the result of local inhomogeneities of the material. In this case even a uniform increase in temperature produces internal stresses. The component with larger thermal expansion pushes other components and all components are strained.

Many papers concern thermal stresses in composites, see e.g. [1]–[5]. In these papers, the stresses are either calculated or measured for a rather accomplished model. Here we intend to expose the statistical aspect of the problem and consider a very simple model of composite elastic material shown in Fig. 1a. The rhombicub-octahedra represent grains. The grains are generally made of different materials. The octagonal prisms (shaded in Fig. 1a) represent elastic joints between the grains. Due to the uniform increase in temperature, each grain expands and there arise internal strains since the thermal properties of the grains are different. Other mechanisms of grain size change like water evaporation in wet medium could be taken into account. In the present paper we confine ourselves to thermal effects only.

Let us consider the plane problem, when all grains in the rows in the z-direction are identical. In this situation, only one plane of grains must be taken into account, Fig. 1b. Initial distances between each two neighbouring grains are denoted by $2r$. A grain in this plane will be identified by two integers (j,k). In general, the expansion of the grain (j,k) is different from the expansion of the grain $(j,k+1)$ and the grain $(j+1,k)$. Prisms responsible for elastic interaction between the grains are situated between two neighbouring grains. In this paper, each grain (j,k) interacts with its four neighbours $(j,k-1)$, $(j,k+1)$, $(j-1,k)$ and $(j+1,k)$.

Let us denote the displacements of the grain (j,k) in the x-direction and the y-direction by $u^x_{j,k}$ and $u^y_{j,k}$, respectively. Let us further denote the expansion coefficient of this grain by $\alpha_{j,k}$. If no stress is present, then $rT\alpha_{j,k} + rT\alpha_{j,k+1}$ is the thermal change of the distance between the grains (j,k) and $(j,k+1)$. The grain (j,k) interacts with the grain $(j,k+1)$ with the longitudinal force $S^{xx}_{j,k}$ and the transverse force $S^{yx}_{j,k}$ and the actual change of distance equals $u^x_{j,k+1} - u^x_{j,k}$. The

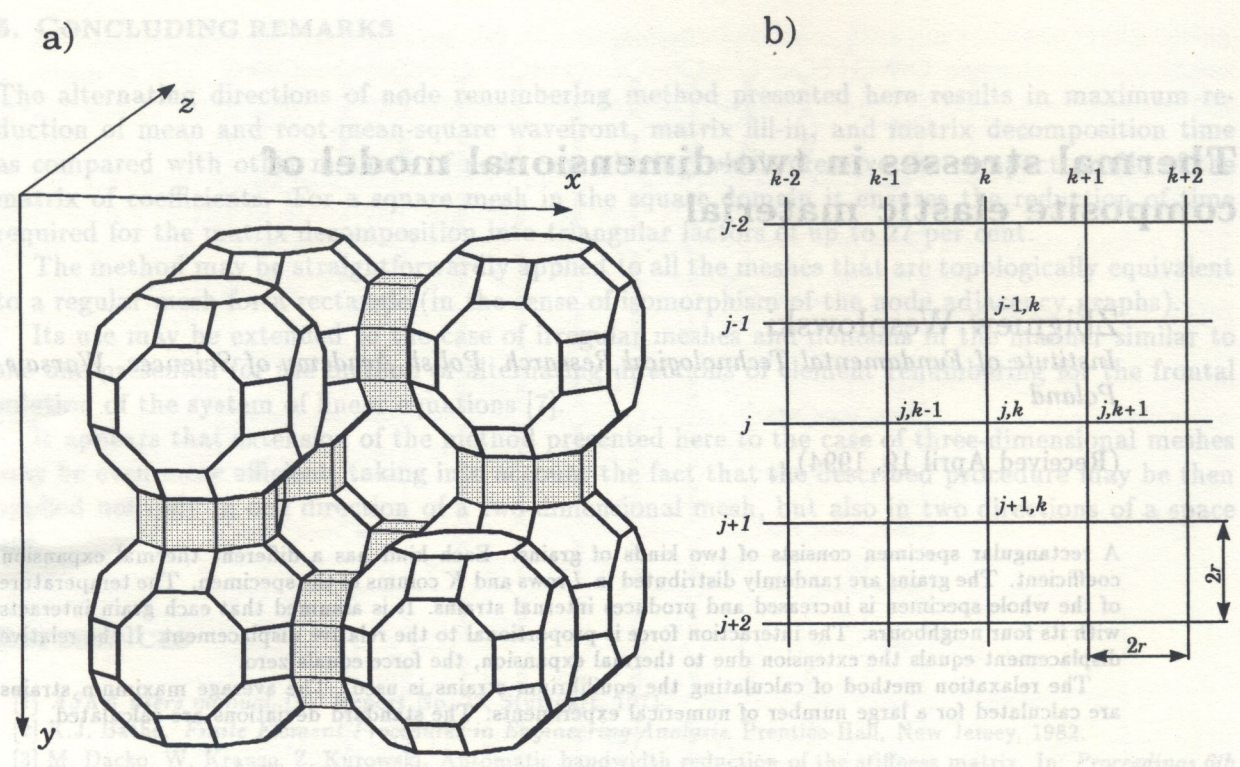

Fig. 1

following relations are assumed to hold in the model

$$S^{xx}_{j,k} = \xi_L \left[u^x_{j,k+1} - u^x_{j,k} - (rT\alpha_{j,k+1} + rT\alpha_{j,k}) \right], \tag{1}$$

$$S^{yx}_{j,k} = \xi_T \left[u^y_{j,k+1} - u^y_{j,k} \right]. \tag{2}$$

In the relation (1), the longitudinal rigidity ξ_L of the elastic octagonal prisms is taken into account. The corresponding force $S^{xx}_{j,k}$ acts in the x-direction. In the absence of the temperature increment T the interaction force $S^{xx}_{j,k}$ is proportional to the displacement difference $u^x_{j,k+1} - u^x_{j,k}$. For T different from zero the natural, stress-free distance equals $2r + rT\alpha_{j,k} + rT\alpha_{j,k+1}$. For $u^x_{j,k+1} - u^x_{j,k} = rT\alpha_{j,k} + rT\alpha_{j,k+1}$ the longitudinal interaction force between the grains (j,k) and $(j,k+1)$ equals zero. In the relation (2), the shear rigidity ξ_T of the elastic prisms is taken into account. The corresponding force $S^{yx}_{j,k}$ acts in the y-direction.

Let us assume that the properties in the y-direction are the same as that for the x-direction. Therefore, the grain $(j+1,k)$ acts on the grain (j,k) with the following force

$$S^{yy}_{j,k} = \xi_L \left[u^y_{j+1,k} - u^y_{j,k} - (rT\alpha_{j+1,k} + rT\alpha_{j,k}) \right], \tag{3}$$

$$S^{xy}_{j,k} = \xi_T \left[u^x_{j+1,k} - u^x_{j,k} \right]. \tag{4}$$

The grain (j,k) interacts additionally with the remaining two neighbours. According to the action-reaction law, the force exerted by the grain $(j,k-1)$ on the grain (j,k) equals minus the force exerted by the grain (j,k) on the grain $(j,k-1)$. Therefore, the above four relations (1)–(4) allow to calculate the forces exerted on the grain (j,k) by its four neighbours. Note that the y-axis in Fig. 1 is directed downwards to fit the convention that the row $j+1$ is situated below the row j.

The specimen is the rectangle $1 \le j \le J$, $1 \le k \le K$. Let us derive the total force acting on the grain (j,k). For the grain situated inside the specimen, the stresses between the grain (j,k) and

four grains $(j,k-1)$, $(j,k+1)$, $(j-1,k)$ and $(j+1,k)$ must be taken into account. According to the above relations, the x-component of this force is given by the relation

$$Q^x_{j,k} = \xi_L \left[u^x_{j,k-1} - 2u^x_{j,k} + u^x_{j,k+1} + rT\alpha_{j,k-1} - rT\alpha_{j,k+1}\right] + \xi_T \left[u^x_{j-1,k} - 2u^x_{j,k} + u^x_{j+1,k}\right]. \tag{5}$$

Other expressions hold in the case of the grains situated on the boundary, since there the number of neighbours is smaller than 4. Here we consider the model in the form of a rectangle with the edges $(1,1)$, $(1,K)$, $(J,1)$, (J,K). The boundary grains are the ones on the sides of the rectangle. We write only the expressions for the forces in the x-direction. For the grains $(1,k)$ and (J,k) situated in the first and last row, respectively, (1) and (4) lead to the following expressions for the force acting on such grains

$$\begin{aligned} Q^x_{1,k} &= \xi_L \left[u^x_{1,k-1} - 2u^x_{1,k} + u^x_{1,k+1} + rT\alpha_{1,k-1} - rT\alpha_{1,k+1}\right] + \xi_T \left[u^x_{2,k} - u^x_{1,k}\right], \\ Q^x_{J,k} &= \xi_L \left[u^x_{J,k-1} - 2u^x_{J,k} + u^x_{J,k+1} + rT\alpha_{J,k-1} - rT\alpha_{J,k+1}\right] + \xi_T \left[u^x_{J-1,k} - u^x_{J,k}\right], \\ & \qquad k = 2,3,\ldots,K-1. \end{aligned} \tag{6}$$

For the grains $(j,1)$ and (j,K) situated in the first and the last columns, respectively, the forces are

$$\begin{aligned} Q^x_{j,1} &= \xi_L \left[u^x_{j,2} - u^x_{j,1} - rT\alpha_{j,2} - rT\alpha_{j,1}\right] + \xi_T \left[u^x_{j-1,1} - 2u^x_{j,1} + u^x_{j+1,1}\right], \\ Q^x_{j,K} &= \xi_L \left[u^x_{j,K-1} - u^x_{j,K} + rT\alpha_{j,K-1} + rT\alpha_{j,K}\right] + \xi_T \left[u^x_{j-1,K} - 2u^x_{j,K} + u^x_{j+1,K}\right], \\ & \qquad j = 2,3,\ldots,J-1. \end{aligned} \tag{7}$$

Finally, the forces acting on the four corner grains are determined by the relations

$$\begin{aligned} Q^x_{1,1} &= \xi_L \left[u^x_{1,2} - u^x_{1,1} - rT\alpha_{1,2} - rT\alpha_{1,1}\right] + \xi_T \left[u^x_{2,1} - u^x_{1,1}\right], \\ Q^x_{J,1} &= \xi_L \left[u^x_{J,2} - u^x_{J,1} - rT\alpha_{J,2} - rT\alpha_{J,1}\right] + \xi_T \left[u^x_{J-1,1} - u^x_{J,1}\right], \\ Q^x_{1,K} &= \xi_L \left[u^x_{1,K-1} - u^x_{1,K} + rT\alpha_{1,K-1} + rT\alpha_{1,K}\right] + \xi_T \left[u^x_{2,K} - u^x_{1,K}\right], \\ Q^x_{J,K} &= \xi_L \left[u^x_{J,K-1} - u^x_{J,K} + rT\alpha_{J,K-1} + rT\alpha_{J,K}\right] + \xi_T \left[u^x_{J-1,K} - u^x_{J,K}\right], \end{aligned} \tag{8}$$

Note that for each j,k the force $Q^x_{j,k}$ depends only on $u^x_{j,k}$, and not on $u^y_{j,k}$. The components in the y-direction may be calculated by an appriopriate change of the suffices. In further calculations the expressions for the forces in the y-direction are not needed and we do not quote the corresponding expressions for the y-components $Q^y_{j,k}$. Evidently they depend on $u^y_{j,k}$ only.

We consider the case when only internal forces act on each grain. The whole specimen is free from external load. The equilibrium of the grain (j,k) demands that $Q^x_{j,k} = Q^y_{j,k} = 0$. In the system of equilibrium equations $Q^x_{j,k} = 0$ the unknowns are $u^x_{j,k}$, and in the system of equilibrium equations $Q^y_{j,k} = 0$ the unknowns are $u^y_{j,k}$. It can be seen that the equilibrium equations for the x-direction differ from the equilibrium conditions for the y-direction. Further analysis will be confined to the x-component only. The system of equations for the y-component may be obtained from the system for the x-component by means of an obvious change of suffices.

Let us write the above relations in a slightly different form. Let us denote

$$v^x_{j,k} = \frac{1}{r} u^x_{j,k}, \qquad d_{j,k} = T\alpha_{j,k}, \qquad \kappa = \frac{\xi_T}{\xi_L}, \tag{9}$$

where κ is a material constant, and $d_{j,k}$ is the incompatibility produced by the temperature T.

The equilibrium equations for the x-direction may now be written in the form of $J \times K$ linear non-homogeneous algebraic equations (one for each grain)

$$R_{j,k} \equiv v^x_{j,k-1} - 2v^x_{j,k} + v^x_{j,k+1} + d_{j,k-1} - d_{j,k+1} + \kappa\left[v^x_{j-1,k} - 2v^x_{j,k} + v^x_{j+1,k}\right] = 0, \quad j = 2,3,\ldots,J-1, \quad k = 2,3,\ldots,K-1. \tag{10}$$

$$R_{1,k} \equiv v^x_{1,k-1} - 2v^x_{1,k} + v^x_{1,k+1} + d_{1,k-1} - d_{1,k+1} + \kappa\left[v^x_{2,k} - v^x_{1,k}\right] = 0, \quad k = 2,3,\ldots,K-1.$$

$$R_{J,k} \equiv v^x_{J,k-1} - 2v^x_{J,k} + v^x_{J,k+1} + d_{J,k-1} - d_{J,k+1} + \kappa\left[v^x_{J-1,k} - v^x_{J,k}\right] = 0, \quad k = 2,3,\ldots,K-1. \tag{11}$$

$$R_{j,1} \equiv -v^x_{j,1} + v^x_{j,2} - d_{j,1} - d_{j,2} + \kappa\left[v^x_{j-1,1} - 2v^x_{j,1} + v^x_{j+1,1}\right] = 0, \quad j = 2,3,\ldots,J-1.$$

$$R_{j,K} \equiv v^x_{j,K-1} - v^x_{j,K} + d_{j,K-1} + d_{j,K} + \kappa\left[v^x_{j-1,K} - 2v^x_{j,K} + v^x_{j+1,K}\right] = 0, \quad j = 2,3,\ldots,J-1. \tag{12}$$

$$\begin{aligned} R_{1,1} &\equiv -v^x_{1,1} + v^x_{1,2} - d_{1,1} - d_{1,2} + \kappa\left[v^x_{2,1} - v^x_{1,1}\right] = 0, \\ R_{J,1} &\equiv -v^x_{J,1} + v^x_{J,2} - d_{J,1} - d_{J,2} + \kappa\left[v^x_{J-1,1} - v^x_{J,1}\right] = 0, \\ R_{1,K} &\equiv v^x_{1,K-1} - v^x_{1,K} + d_{1,K-1} + d_{1,K} + \kappa\left[v^x_{2,K} - v^x_{1,K}\right] = 0, \\ R_{J,K} &\equiv v^x_{J,K-1} - v^x_{J,K} + d_{J,K-1} + d_{J,K} + \kappa\left[v^x_{J-1,K} - v^x_{J,K}\right] = 0. \end{aligned} \tag{13}$$

The above system of algebraic equations may be solved, provided the incompatibilities $d_{j,k}$ are known.

Note that in the special case of constant $\alpha_{j,k} = \alpha = \text{const}$ we have $d_{j,k} = \alpha T$. In this case, constant temperature T causes uniform thermal expansion in a free specimen,

$$v^x_{j,k} = 2\alpha k T. \tag{14}$$

2. INTERNAL STRESSES AND THERMAL EXPANSION

Let us confine the calculations to the case when the composite consists of only two different kinds of grains. They may be distributed either randomly or according to a deterministic law. Each kind of grain has a fixed thermal expansion coefficient. Due to interaction between the grains, the expansion of the j-th row of the model is not a sum of the expansions of separate grains in the j-th row.

Generalization to three or more kinds of grains is immediate. In the limiting case all grains may be different.

Let us assume that the equations (10)–(13) have already been solved. Then $u^x_{j,k}$ and $v^x_{j,k}$ are known. This suffices to calculate the thermal expansion of the model. In order to simplify the notation, the x-suffix in the next chapters will be omitted, e.g. instead of $v^x_{j,k}$ we shall write $v_{j,k}$.

The length increment of the j-th row equals $(u_{j,K} - u_{j,1})$. Since the initial length of this row is $2Kr$, the average thermal expansion coefficient is

$$\alpha^{\mathrm{av}}_j = \frac{u_{j,K} - u_{j,1}}{2KrT}. \tag{15}$$

In terms of the function $v_{j,k}$ the above relation reads

$$\alpha^{\mathrm{av}}_j = \frac{v_{j,K} - v_{j,1}}{2KT}. \tag{16}$$

Relations (1)–(4) must be taken into account in order to calculate the stresses. Since we consider only the forces due to the displacement in the x-direction we take into account only $S^{xx}_{j,k}$ and $S^{xy}_{j,k}$. The forces in the y-direction must be calculated separately. The longitudinal stress $\tau^{xx}_{j,k}$ and the transverse stress $\tau^{xy}_{j,k}$ are

$$\tau^{xx}_{j,k} = \frac{1}{2r} S^{xx}_{j,k}, \qquad \tau^{xy}_{j,k} = \frac{1}{2r} S^{xy}_{j,k}, \tag{17}$$

respectively, since the area (unit thickness) equals $2r$. According to (1) and (4) we have

$$\tau^{xx}_{j,k} = \xi_L \varepsilon_{j,k}, \qquad \tau^{xy}_{j,k} = \xi_T \gamma_{j,k}, \tag{18}$$

where

$$\varepsilon_{j,k} = \frac{1}{2}\left[v^x_{j,k+1} - v^x_{j,k} - d_{j,k+1} - d_{j,k}\right], \tag{19}$$

$$\gamma_{j,k} = \frac{1}{2}\left[v^x_{j+1,k} - v^x_{j,k}\right], \tag{20}$$

are the longitudinal and shear strains, respectively, produced by the incompatibility $d_{j,k}$.

Let us assume that we have already calculated the reduced displacement field $v^x_{j,k}$ satisfying Eqs. (10)–(13) for a given $d_{j,k}$. Let us replace $d_{j,k}$ by $d_{j,k} + D$, $D = \text{const}$. The new field $v^x_{j,k} + 2Dk$ satisfies the equilibrium equations. The stress field corresponding to $v^x_{j,k} + 2Dk$ is exactly the same as the stress field corresponding to $v^x_{j,k}$. It follows that the replacement of $d_{j,k}$ by $(d_{j,k} + D)$ does not influence the stress field. This simplifies the calculations. Since we are interested here in the stress field only, by adding an appriopriate D we reduce the calculations to the situation where some $d_{j,k}$ are equal to zero, and the other $d_{j,k} > 0$. Since the model consists of only two kinds of grains and $\alpha_2 < \alpha_1$, then $d_{j,k}$ equals either 0 or $T(\alpha_1 - \alpha_2)$.

3. Numerical simulation

Let us assume that we deal with a set of N numbers x_q, $q = 1, \ldots, N$. For such a set, the mean value x_{av} and the standard deviation σ may be calculated as (each summation from 1 to N)

$$x_{\text{av}} = \frac{1}{N} \sum x_q, \tag{21}$$

$$\sigma = \sqrt{\frac{1}{N-1} \sum (x_q - x_{\text{av}})^2}. \tag{22}$$

Let us consider the rectangular model with free sides. There are J grains on two sides, and K grains on the other two. Let p be a fixed number $0 \leq p \leq 1$. The model consists of pJK grains of the first kind with the expansion coefficient α_1 and $(1-p)JK$ grains of the second kind with the expansion coefficient α_2. Let us generate random distribution of the pJK grains of the first kind. For the points j, k, which the grains of the first kind are situated at, we take $d_{j,k} = 1$. This is equivalent to performing the calculations for $T(\alpha_1 - \alpha_2) = 1$, cf. the remark at the end of the previous Section. If $T(\alpha_1 - \alpha_2) \neq 1$, further results must be multiplied by $T(\alpha_1 - \alpha_2)$.

Here we consider square specimens, $J = K$. The numerical experiments have been performed for $0.1 < p < 0.9$. pJK random points at which $d_{j,k} = 1$ were generated first. At the remaining points $d_{j,k} = 0$. In a particular realisation we have e.g.

$$d_{j,k} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & \cdot & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & \cdot & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & \cdot & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & 1 & 0 & \cdot & 1 & 1 \end{bmatrix}.$$

In the next step, the displacements $v_{j,k}$ were calculated from the system (10)–(13). Since the number of unknowns was large (e.g. 3600 for $J = K = 60$), the relaxational approach was adopted instead of solving the linear system (10)–(13) directly. The successive approximations $v_{j,k}(m)$ $(m = 1, 2, 3, \ldots)$ of the displacement field were assumed to satisfy the relations

$$v_{j,k}(m+1) = v_{j,k}(m) + \nu R_{j,k}(m), \tag{23}$$

where $R_{j,k}(m)$ are the left-hand sides of (10)–(13) calculated for $v_{j,k} = v_{j,k}(m)$. The above method of solving the static equations (10)–(13) is based on the assumption that the subsequent approximations of $v_{j,k}$ for $m = 1, 2, 3, \ldots$ are displacement fields for successive time instants. The velocity of the grain is proportional to the instantaneous external force $R_{j,k}(m)$. The numerical results converge, provided ν is sufficiently small. Since there is energy stored and $R_{j,k}$ is the derivative of the stored energy with respect to $v_{j,k}$, the relation (23) is in fact based on the steepest descent method.

Obviously, a possibly large ν must be taken. It has been found that $\nu = 0.2$ leads to numerically converging results. The calculations are performed until the forces $R_{j,k}(m)$ are sufficiently small. Let us denote the maximum value of $R_{j,k}(m)$ (for all j, k) by $R(m)$. For small specimens, e.g. $J = K = 10$, after only about $m = 100$ time steps the value $R(m) < 0.001$ is obtained. For larger specimens, e.g. $J = K = 60$, several thousand steps are necessary to achieve $R(m) < 0.001$.

Let us perform N (fixed N) numerical experiments. In each experiment, strain fields $\varepsilon_{j,k}$ and $\gamma_{j,k}$ are obtained. Their values are functions of the pairs (j, k). The minimum and maximum longitudinal strains are denoted by ε^-, ε^+, respectively. The minimum and maximum transverse strains are denoted by γ^-, γ^+, respectively. In general, ε^+, ε^-, γ^+, γ^- correspond to different grains (j, k). In each experiment a different set of four values ε^+, ε^-, γ^+, γ^- is obtained. On the basis of N experiments (N has the same meaning as in (21)) the average values ε^-, ε^+, γ^-, γ^+ are calculated as

$$\begin{aligned} \varepsilon^-_{\mathrm{av}} &= \frac{1}{N}\sum \varepsilon^-, \qquad \varepsilon^+_{\mathrm{av}} = \frac{1}{N}\sum \varepsilon^+, \\ \gamma^-_{\mathrm{av}} &= \frac{1}{N}\sum \gamma^-, \qquad \gamma^+_{\mathrm{av}} = \frac{1}{N}\sum \gamma^+. \end{aligned} \tag{24}$$

Obviously, we obtain generally different averages ε^-, ε^+, γ^-, γ^+ in each set of N experiments.

Let us first consider a material for which $\xi_L = \xi_T$. Figure 2 shows the average values of the maximum strains for $J = K = 10$. The absolute values are plotted. It can be seen that the longitudinal strain $\varepsilon^-_{\mathrm{av}}$ attains the lowest (negative) value for p equal approximately 0.25. The longitudinal strain $\varepsilon^+_{\mathrm{av}}$ attains the highest value for $p = 0.75$. The curves are symmetric with respect to $p = 0.5$. The absolute values of γ^- and γ^+, are approximately equal to each other and have the maximum at $p = 0.5$, Fig 3.

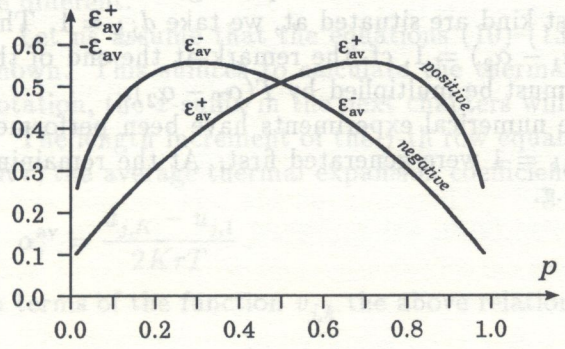

Fig. 2

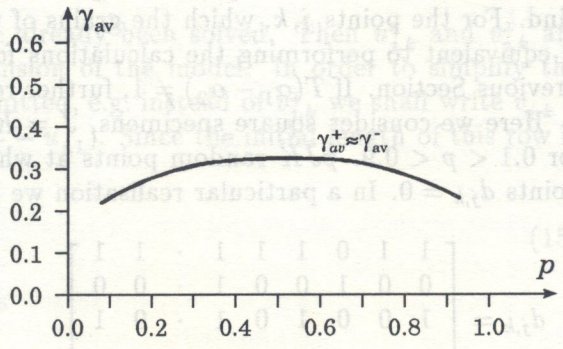

Fig. 3

The numerical experiments allow us to calculate the standard deviations σ_1, σ_2, σ_3, σ_4 (not to be confused with stresses) corresponding to the above results. Let us define

$$\sigma_1 = \sqrt{\frac{1}{N-1}\sum(\varepsilon^- - \varepsilon^-_{\mathrm{av}})^2}\,, \qquad \sigma_2 = \sqrt{\frac{1}{N-1}\sum(\varepsilon^+ - \varepsilon^+_{\mathrm{av}})^2}\,,$$
$$\sigma_3 = \sqrt{\frac{1}{N-1}\sum(\gamma^- - \gamma^-_{\mathrm{av}})^2}\,, \qquad \sigma_4 = \sqrt{\frac{1}{N-1}\sum(\gamma^+ - \gamma^+_{\mathrm{av}})^2}\,. \tag{25}$$

σ_4 equals σ_3 with great accuracy. In Fig. 4, the values of σ_1, σ_2, σ_3 are sketched versus p. For smaller p, the compressive strains have larger standard deviations. In contrast to this, both extensional and shear strains have small standard deviations. In other words, they are better predictable than the negative strains. The opposite situation occurs for large p.

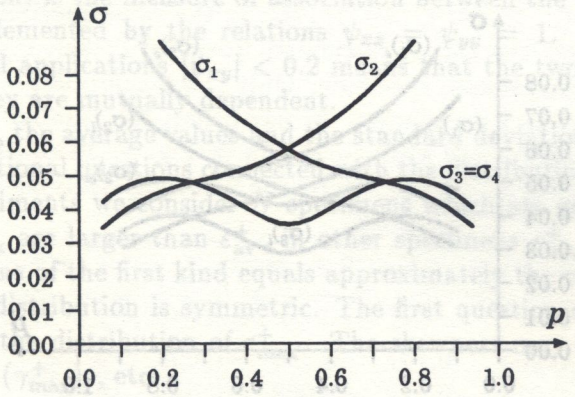

Fig. 4

The inspection of the strains $\varepsilon_{j,k}$, $\gamma_{j,k}$ (in one typical experiment) in the whole specimen proves that the maximum strains on the boundary grains are generally different from the maximum strains inside the rectangle. Let us calculate two sets of maximum strains separately. The strains for the grains situated in a single boundary layer (boundary row or column of the whole $J \times K$ rectangle) will be marked with the suffix 'e'. Let us disregard the two outermost rows and columns of the specimen and thus obtain the inner $(J-4) \times (K-4)$ rectangle. The maximum strains for the grains situated in this inner rectangle will be marked with the suffix 'i'. Note that strains in some grains, e.g. those in the second row and the second column, are disregarded. The eight sets $(\varepsilon^-)_i$, $(\varepsilon^+)_i$, $(\gamma^-)_i$, $(\gamma^+)_i$, $(\varepsilon^-)_e$, $(\varepsilon^+)_e$, $(\gamma^-)_e$ and $(\gamma^+)_e$ will further be numbered $1, 2, \ldots, 8$. According to this convention, the standard deviations σ have the suffices 1, 2, 3, 4 for the inner grains as in (25), while for the outer ones they have the suffices 5, 6, 7, 8.

Figure 5 shows the maximum strains for the outer and inner grains for $J = K = 20$. The maximum strains for the grains situated inside the rectangle are shown separately; two outermost rows and columns are not taken into account. Two results are essential. For $p < 0.5$, the maximum compressive strains, $(\varepsilon^-)_i$ and $(\varepsilon^-)_e$, are larger than the maximum dilatational strains $(\varepsilon^+)_i$ and $(\varepsilon^+)_e$, respectively. For $p > 0.5$ the situation is opposite: the maximum compressive strains are smaller than the maximum dilatational ones. The second result is the observation that the internal longitudinal strains, $(\varepsilon^-)_i$ and $(\varepsilon^+)_i$, are larger than the external longitudinal ones, $(\varepsilon^-)_e$ and $(\varepsilon^+)_e$, respectively. In contrast to the above results for longitudinal strains, the maximum positive and negative shear strains $(\gamma^-)_e$, $(\gamma^+)_e$ are almost the same (see Fig. 6). Moreover, the maximum shear strains on the boundary grains and inside the specimen are almost the same, $(\gamma^-)_e = (\gamma^-)_i$. Figure 7 gives the corresponding standard deviations.

For all strains the standard deviations σ_1, σ_2, σ_3, σ_4 for the inner grains are much lower than σ_5, σ_6, σ_7, σ_8 for the outer grains. In general, the standard deviations for transverse strains are smaller than those for longitudinal strains. It follows that the maximum shear strains are better predictable, i.e. in all specimens the maximum shear strains are closer to each other than the

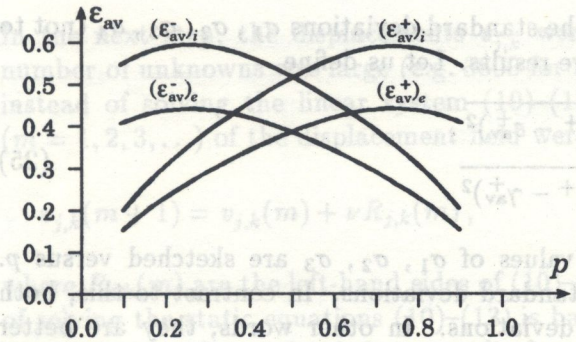

Fig. 5

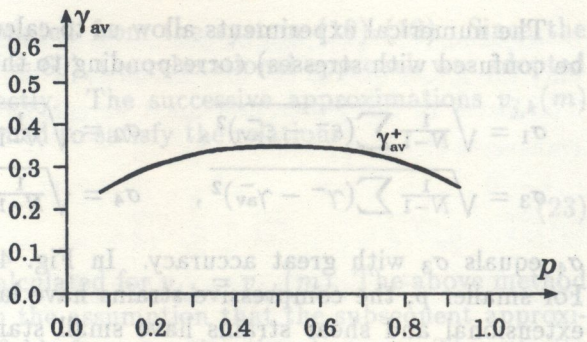

Fig. 6

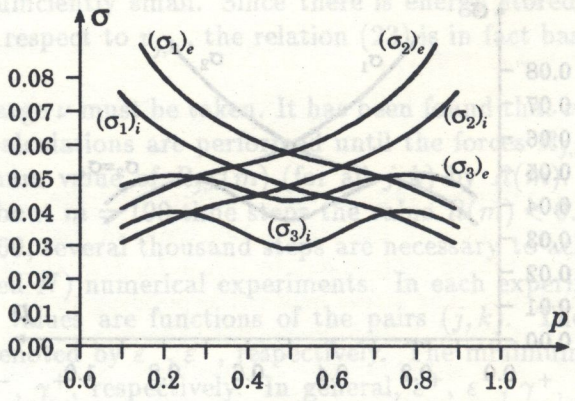

Fig. 7

maximum longitudinal ones. It is necessary to note that this result has been obtained for $\gamma_L = \gamma_T$, i.e. for $\kappa = 1$.

For many materials (metals), the ratio of the shear and longitudinal elastic moduli equal approximately $\kappa = 0.4$. For such materials, stresses have a very similar dependence on p, as that shown in Fig. 5 and Fig. 6 for κ=1. The differences are only quantitative. In order to save space we do not quote the corresponding curves and give the data for $p = 0.3$, $\kappa = 0.4$, $J = K = 20$. The internal strains are

$$(\varepsilon^-)_i = -0.507, \quad (\varepsilon^+)_i = 0.382, \quad (\gamma^-)_i = -0.495, \quad (\gamma^+)_i = 0.497.$$

The corresponding standard deviations are

$$\sigma_1 = 0.061, \quad \sigma_2 = 0.043, \quad \sigma_3 = 0.056 \quad , \sigma_4 = 0.048.$$

The boundary maximum strains are

$$(\varepsilon^-)_e = -0.353 \quad (\varepsilon^+)_e = 0.272, \quad (\gamma^-)_e = -0.522, \quad (\gamma^+)_e = 0.521.$$

The coresponding standard deviations are

$$\sigma_5 = 0.065, \quad \sigma_6 = 0.043, \quad \sigma_7 = 0.088 \quad , \sigma_8 = 0.092.$$

The important qualitative result is: for $p = 0.4$ the maximum internal longitudinal strains are larger than the maximum boundary longitudinal strains. The internal shear strains are smaller than the boundary shear strains.

The standard deviations for the internal and external longitudinal strains for $p = 0.4$ are of the same order of magnitude. In contrast to this, the standard deviations for the internal shear strains are smaller than for the external ones. The external shear strains are more scattered, than the internal ones. Since the averages are almost the same, the actual maximum shear strains on the boundary are much larger for some realisations than the maximum internal shear strains. This conclusion may be of great importance for the explanation of crack formation.

4. SKEWNESS AND CORRELATION

For a set of N numbers x_q, the skewness ζ is defined by the relation

$$\zeta = \frac{1}{N}\sum\left[\frac{x_q - x_{\mathrm{av}}}{\sigma}\right]^3, \tag{26}$$

where σ is the corresponding standard deviation.

For two sets x_q, y_q of N numbers the correlation coefficient ψ_{xy} is, [7],

$$\psi_{xy} = \frac{\sum(x_q - x_{\mathrm{av}})(y_q - y_{\mathrm{av}})}{\sqrt{\sum(x_q - x_{\mathrm{av}})^2}\sqrt{\sum(y_q - y_{\mathrm{av}})^2}}. \tag{27}$$

The correlation coefficient is the measure of association between the variables x and y. Usually the definition (27) is suplemented by the relations $\psi_{xx} = \psi_{yy} = 1$. Note that ζ and ψ_{xy} are dimensionless. In practical applications $|\psi_{xy}| < 0.2$ means that the two sets are independent and $|\psi_{xy}| > 0.8$ means that they are mutually dependent.

In the previous chapter, the average values and the standard deviations of the maximum strains were calculated. Two additional questions connected with the distribution of the maximum stresses arise. In each set of experiments we consider N specimens which are generally different. In some specimens the strains $\varepsilon^+_{\max}$ are larger than $\varepsilon^+_{\mathrm{av}}$, in other specimens $\varepsilon^+_{\max}$ are smaller than $\varepsilon^+_{\mathrm{av}}$. If the number of the specimens of the first kind equals approximately the number of the specimens of the second kind then the distribution is symmetric. The first question concerns skewness, i.e. the measure of asymmetry in the distribution of $\varepsilon^+_{\max}$. The skewness may be calculated not only for $\varepsilon^+_{\max}$ but also for $(\gamma^-_{\max})_e$, $(\gamma^+_{\max})_i$, etc.

The second question is connected with the relation of the maximum strains $(\varepsilon^-)_i$, $(\varepsilon^+)_i$, $(\gamma^-)_i$, $(\gamma^+)_i$, $(\varepsilon^-)_e$, $(\varepsilon^+)_e$, $(\gamma^-)_e$ and $(\gamma^+)_e$ in a single fixed specimen. For example, if the strain $(\varepsilon^-)_i$ in a specimen labelled as specimen no. 126 is large, what are the expected values of $(\varepsilon^+)_i$, $(\gamma^-)_i$, $(\gamma^+)_i$, etc. in the same specimen? In particular, is there a tendency for $(\gamma^-_{\max})_i$ to be large in specimen no. 126, if $(\varepsilon^-)_i$ is large in that specimen? The components of the correlation matrix are the measure of this tendency.

Instead of two sets x_k, y_k (as in (27)) we have eight sets $(\varepsilon^-)_i$, $(\varepsilon^+)_i$, $(\gamma^-)_i$, $(\gamma^+)_i$, $(\varepsilon^-)_e$, $(\varepsilon^+)_e$, $(\gamma^-)_e$ and $(\gamma^+)_e$. These sets are numbered from 1 to 8, respectively. In this convention, the skewness of $(\varepsilon^-)_i$ is ζ_1. According to (26), it is defined by the formula

$$\zeta_1 = \frac{1}{N}\sum\left[\frac{(\varepsilon^-)_q - (\varepsilon^-_{\mathrm{av}})_i}{\sigma_1}\right]^3. \tag{28}$$

Analogous formulae hold for the remaining seven skewnesses $\zeta_2, \zeta_3, \ldots, \zeta_8$.

Let us proceed to correlation coefficients. According to our convention regarding the numbering of the sets $(\varepsilon^-)_i$, $(\varepsilon^+)_i$, $(\gamma^-)_i$, $(\gamma^+)_i$, $(\varepsilon^-)_e$, $(\varepsilon^+)_e$, $(\gamma^-)_e$ and $(\gamma^+)_e$, the correlation coefficient between $(\varepsilon^-)_i$ and $(\varepsilon^+)_i$ is ψ_{12}. According to (27), we have

$$\psi_{12} = \frac{\sum[\varepsilon^-_i - (\varepsilon^-_{\mathrm{av}})_i][\varepsilon^+_i - (\varepsilon^+_{\mathrm{av}})_i]}{\sqrt{\sum[\varepsilon^-_i - (\varepsilon^-_{\mathrm{av}})_i]^2}\sqrt{\sum[\varepsilon^+_i - (\varepsilon^+_{\mathrm{av}})_i]^2}}. \tag{29}$$

Analogous formulae hold for the remaining 55 correlation coefficients $\psi_{13}, \psi_{14}, \psi_{15}, \ldots, \psi_{78}$. Due to symmetry $\psi_{xy} = \psi_{yx}$, there are only 23 different coefficients.

For the specimen discussed above, the following values of the skewness and correlation coefficients are obtained for $p = 0.3$

$$\begin{aligned} &\xi_1 = -1 * 10^{-6}, \quad \xi_2 = 64 * 10^{-6}, \quad \xi_3 = -0.4 * 10^{-6}, \quad \xi_4 = -0.5 * 10^{-6}, \\ &\xi_5 = -4 * 10^{-6}, \quad \xi_6 = 0.4 * 10^{-6}, \quad \xi_7 = -9 * 10^{-6}, \quad \xi_8 = 33 * 10^{-6}. \end{aligned} \tag{30}$$

The above data must be compared with a known distribution. The conclusion is that the distribution is symmetric with great accuracy.

$$\psi_{RS} = \begin{vmatrix} 1 & 0.058 & 0.084 & -0.019 & 0.023 & -0.023 & -0.016 & -0.123 \\ & 1 & 0.120 & -0.108 & -0.076 & -0.014 & -0.007 & -0.107 \\ & & 1 & -0.147 & -0.094 & -0.063 & 0.084 & -0.151 \\ & & & 1 & 0.035 & -0.014 & -0.121 & -0.086 \\ & & & & 1 & 0.151 & -0.072 & -0.026 \\ & & & & & 1 & 0.110 & -0.056 \\ & & & & & & 1 & 0.178 \\ & & & & & & & 1 \end{vmatrix}. \quad (31)$$

The matrix is symmetric, the values on the main diagonal equal 1. The above values suggest that $(\varepsilon^-)_i$, $(\varepsilon^+)_i$, $(\gamma^-)_i$, $(\gamma^+)_i$, $(\varepsilon^-)_e$, $(\varepsilon^+)_e$, $(\gamma^-)_e$ and $(\gamma^+)_e$ are not correlated with one another. In other words, the fact that e.g. $(\gamma^-)_i$ is large or small in a specimen does not give any information about the values of $(\gamma^+)_i$, $(\varepsilon^-)_e$, $(\varepsilon^+)_e$, $(\gamma^-)_e$, etc. in the same specimen.

It should be stressed that the actual calculated values of ζ_k and ψ_{RS} are not connected with the physical properties of the random specimen. Only the magnitudes of ζ_R and ψ_{RS} do give some information about skewness and correlation. In particular, for another N numerical experiments the maximum strains are almost the same as shown in Fig. 5. In contrast to this, skewness and correlation are not given by (30), (31) but e.g. by the following relations (data obtained for another numerical experiment)

$$\xi_1 = -1 * 10^{-6}, \quad \xi_2 = 6 * 10^{-6}, \quad \xi_3 = -0.2 * 10^{-6}, \quad \xi_4 = 1 * 10^{-6},$$
$$\xi_5 = 34 * 10^{-6}, \quad \xi_6 = 3 * 10^{-6}, \quad \xi_7 = -8 * 10^{-6}, \quad \xi_8 = 0.1 * 10^{-6}.$$

$$\psi_{RS} = \begin{vmatrix} 1 & 0.778 & -0.005 & 0.147 & -0.154 & -0.008 & -0.138 & 0.098 \\ & 1 & -0.018 & -0.014 & -0.092 & -0.041 & -0.099 & 0.076 \\ & & 1 & 0.071 & -0.164 & -0.130 & 0.120 & -0.034 \\ & & & 1 & -0.048 & -0.015 & 0.095 & -0.066 \\ & & & & 1 & 0.096 & 0.109 & -0.376 \\ & & & & & 1 & -0.080 & -0.111 \\ & & & & & & 1 & 0.002 \\ & & & & & & & 1 \end{vmatrix}.$$

Only the orders of magnitude of ζ_R and ψ_{RS} do coincide with those of (30) and (31).

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