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Mechanics of slackened systems

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The paper concerns slackened systems, i.e. discrete deformable systems with gaps (clearances) at structural joints. The mathematical model of such systems coincides with a FEM-oriented approximation of locking-elastic-plastic bodies. The theory describes a relatively wide class of systems made of time-independent materials. The problem of slackened systems has been developed during the last decade. The work presents the current state of knowledge in this field.

1. INTRODUCTION

Fundamentals of the contemporary mechanics of deformable systems come from the theory of *elasticity*, put forward in the first half of 19th century due to works of A.L. Cauchy (cf. [52]). The idea of continuum medium allowed that author to introduce the concepts of stresses and strains and derive the complete system of equations for elastic bodies. Tens years later G.R. Kirchhoff proved the solution uniqueness of the elasticity problem in the kinematically linear range. The elastic deformation process is reversible and path-independent.

It was de Saint Venant who first formulated the foundations of *plasticity* in 1864 (cf. [52]). However, the fully consistent contemporary theory of plasticity was given by R. Hill in 1950, [27]. The development of the plasticity theory initiated an "inequality" mechanics. The process of plastic deformations is not reversible (dissipative), path-dependent and solutions are in general not unique.

In 1957 W. Prager introduced the idea of *ideal locking materials*, [47]. The Prager's theory corresponds to the "frictionless" formulation of contact problems, presented in 1933 by A. Signiorini [50] and developed mainly by G. Duvaut and J.L. Lions as a problem of unilateral constraints, [5]. The problem of ideal locking systems is not dissipative, non-unique and usually path-independent but, similarly as in the theory of plasticity, its solutions are characterized by variational inequalities.

Elasticity, plasticity and locking are the simplest prototypes of *time-independent materials* (Fig. 1). Various combinations of the particular prototypes lead to more complex physical models of a material.

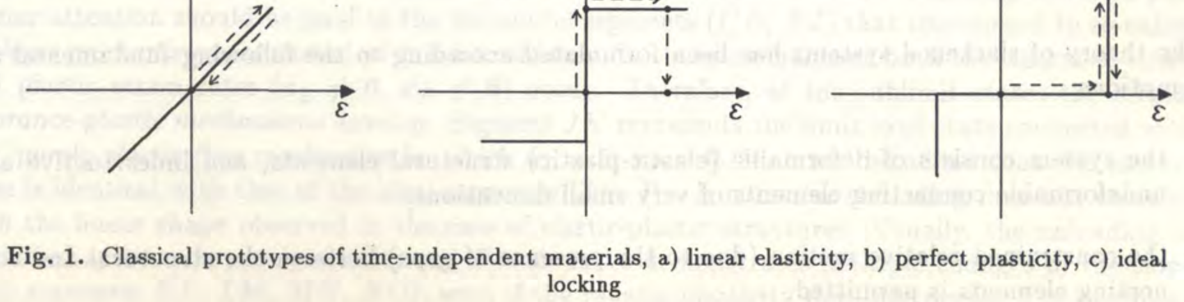

Fig. 1. Classical prototypes of time-independent materials, a) linear elasticity, b) perfect plasticity, c) ideal locking

Among these complex models the model of *elastic-plastic* materials is best recognized. Foundations of the consistent, kinematically linear theory of such materials have been constructed by L. Prandtl [48] and A. Reuss [49], and W.T. Koiter supplemented it by some general theorems, [33]. The combination of elasticity and plasticity resulted in a significant complexity of the theory and essential qualitative changes of the system behaviour. The current state of knowledge in the field of elasto-plasticity is contained in the comprehensive monograph of M. Życzkowski, [56].

To the author's knowledge, there are no sufficiently consistent theories of complex combinations of the remaining prototypes of the material. Nevertheless, an idea of structures with so-called *conditional joints*, proposed by S. Kaliszky in 1978, [32], should be noted. The problem of such joints, where all possible physical components shown in Fig. 1 are taken into considerations, has further been developed by M. Kurutz, using both the continuum and discrete descriptions, [37–39]. A generalization of this problem by means of variational inequalities has been considered by P.D. Panagiotopoulos; fundamental results are collected in monograph [45].

Essential difficulties occurring in the continuum description are due to the presence of locking effects. These difficulties can be relatively simply surmounted by using the FEM-oriented *discrete description*. Then one can assume that the behaviour of the discrete system on the "macro" scale is qualitatively similar to that of the continuum system. Such an approach was employed in the work of L. Corradi and G. Maier [2], published in 1969. The authors constructed the *discrete model of elastic-locking structures*, proved several theorems for this problem and gave a numerical example related to the plane stress state. A more complex model including *elasticity, gaps and friction* was presented in 1983 by W. Ostachowicz, [44]. A "frictionless", kinematically linear theory of discrete systems which describes *elasticity, plasticity and locking* was proposed by this author in 1987, [11]. In the previous works in 1980–1986 the author considered rotation constraints at hinges of bar structures in the presence of elastic and plastic deformations, [6–10]. The general idea of systems considered in [11] comes directly from engineering practice. Locking effects observed on the "macro" scale are caused by the presence of gaps (clearances) at the joints of structural elements. The presence of clearances and their growth is mainly induced by the exploitation process that implies some "slackening" of the structure. Therefore such structures are called *slackened systems.* It should be pointed out that these locking effects do not result from material properties; rather, they represent some imperfections of structural connections. On the other hand, these imperfections can be also treated as a material property when using the FEM-oriented mathematical model. Slackened systems, particularly in the presence of plastic deformations, exhibit a lot of unexpected effects and their behaviour is far from that of common elastic-plastic structures. This observation appeared to be the main reason to further author's studies on slackened systems; it also inspired J. Holnicki-Szulc [29–31] and M. Heinisuo [26] to work in this field.

The main aim of this paper is to present the current state of knowledge in the field of slackened system mechanics. In the subsequent sections we describe the characteristic features of slackened systems, the problems of analysis and synthesis as well as applications and open problems to solve. The problems of synthesis are restricted to the slackened-elastic-perfectly plastic systems.

2. FUNDAMENTAL ASSUMPTIONS, NEW IDEAS AND CHARACTERISTIC FEATURES OF OVERALL BEHAVIOUR OF SLACKENED SYSTEMS

The theory of slackened systems has been formulated according to the following fundamental assumptions:

- the system consists of deformable (elastic-plastic) structural elements, and indestructive and undeformable connecting elements of very small dimensions,
- the constrained relative motion (due to the presence of gaps) between the structural and connecting elements is permitted,

- displacements and gaps (clearances) are so small that the kinematically linear theory can be used,
- all friction effects are neglected,
- only quasi-static processes are considered,
- the ideal structure (i.e. the structure without gaps) is kinematically stable.

The assumption on the frictionless motion requires some additional comments. It is obvious that a mechanical model, in which the friction forces are taken into account, more precisely describes the behaviour of real systems. However, the formulation of such a model is much more complicated. The problems of existence and uniqueness of solutions appear to be very difficult, and only quite recently some significant results in this domain have been obtained by mathematicians (cf., for example, [51]). The solutions of contact problems with friction are not unique, even in the case of the Coulomb friction law. The uniqueness is assured if the friction coefficient is sufficiently small. The non-uniqueness of solutions results from the fact that the friction law is not associated with the friction condition. In the present paper we will show that, even in the simplest case of frictionless deformations the mechanics of slackened systems provides a sufficiently large number of new, non-trivial problems.

We now pass on to present characteristic features of slackened systems and describe new ideas and definitions. A most suggestive illustration of any slackened system behaviour is the "force–displacement" relation for loading and unloading. Let us confine our attention to the elastic-perfectly plastic beam with rotation constraints, shown in Fig. 2a. Deformations of the system consist of elastic strains ($\boldsymbol{\varepsilon}_E$) and concentrated strains: clearance and plastic ones ($\boldsymbol{\varepsilon}_L$, $\boldsymbol{\varepsilon}_P$). The concentrated strains are measured by relative displacements of the end parts of the structural element and the reference connecting element. The model of a slackened connection (i.e. a hinge with rotation constraints), its mechanical characteristic and the geometric meaning of strain components are presented in Figs. 2b, 2c, 2d, respectively. The beam is subjected to the force P which increases up to the limit load P_L. Then, accordingly to the plastic flow mechanism, a deformation process develops. Finally the beam is unloaded. The $P - \Delta$ diagram demonstrates a lot of interesting effects (Fig. 2e).

In the initial state the system is kinematically unstable and takes up a configuration corresponding to the so-called *ideal configuration*, chosen from all kinematically admissible configurations satisfying constraints imposed on clearance strains. The ideal configuration is associated with the kinematically stable ideal structure. Usually, one assumes that displacements of the ideal structure are equal to zero. Then the ideal configuration is defined by the zero-valued *clearance moduli* (i.e. clearance strains that coincide with contacts). The static load acting on the slackened beam implies closing of particular gaps and the system turns into a non-zero stiffness structure, called the *original structure*. The *clearance mechanism*, associated with the formation of such a structure, corresponds to the segment AB on the $P - \Delta$ diagram in which only rates of the clearance strains are non-zero ($\dot{\boldsymbol{\varepsilon}}_L \neq \mathbf{0}$). Obviously, the clearance mechanism does not depend on mechanical properties of the structural element material. When the load increases one observes multiple changes of the stiffness resulting from the occurrence of new contacts or the loss of existing ones. A particular attention should be paid to the horizontal segments (CD, HI) that correspond to so-called *sublimit mechanisms* associated with *sublimit loads*. In these mechanisms both the clearance strain and plastic strain rates ($\dot{\boldsymbol{\varepsilon}}_L \neq \mathbf{0}$, $\dot{\boldsymbol{\varepsilon}}_P \neq \mathbf{0}$) occur. Therefore, at the sublimit states the mixed, *clearance-plastic mechanisms* develop. Segment JK represents the limit load state connected with the *purely plastic flow mechanism* for which $\dot{\boldsymbol{\varepsilon}}_L = \mathbf{0}$, $\dot{\boldsymbol{\varepsilon}}_P \neq \mathbf{0}$. It should be added that this mechanism is identical with that of the ideal structure. The $P - \Delta$ curve for unloading does not coincide with the linear shape observed in the case of elastic-plastic structures. Usually, the unloading in slackened systems is purely elastic, and $P - \Delta$ curve demonstrates multiple changes of the slope (see: segments KL, LM, MN, NO), even if the reverse plasticity does not occur. The nonlinear

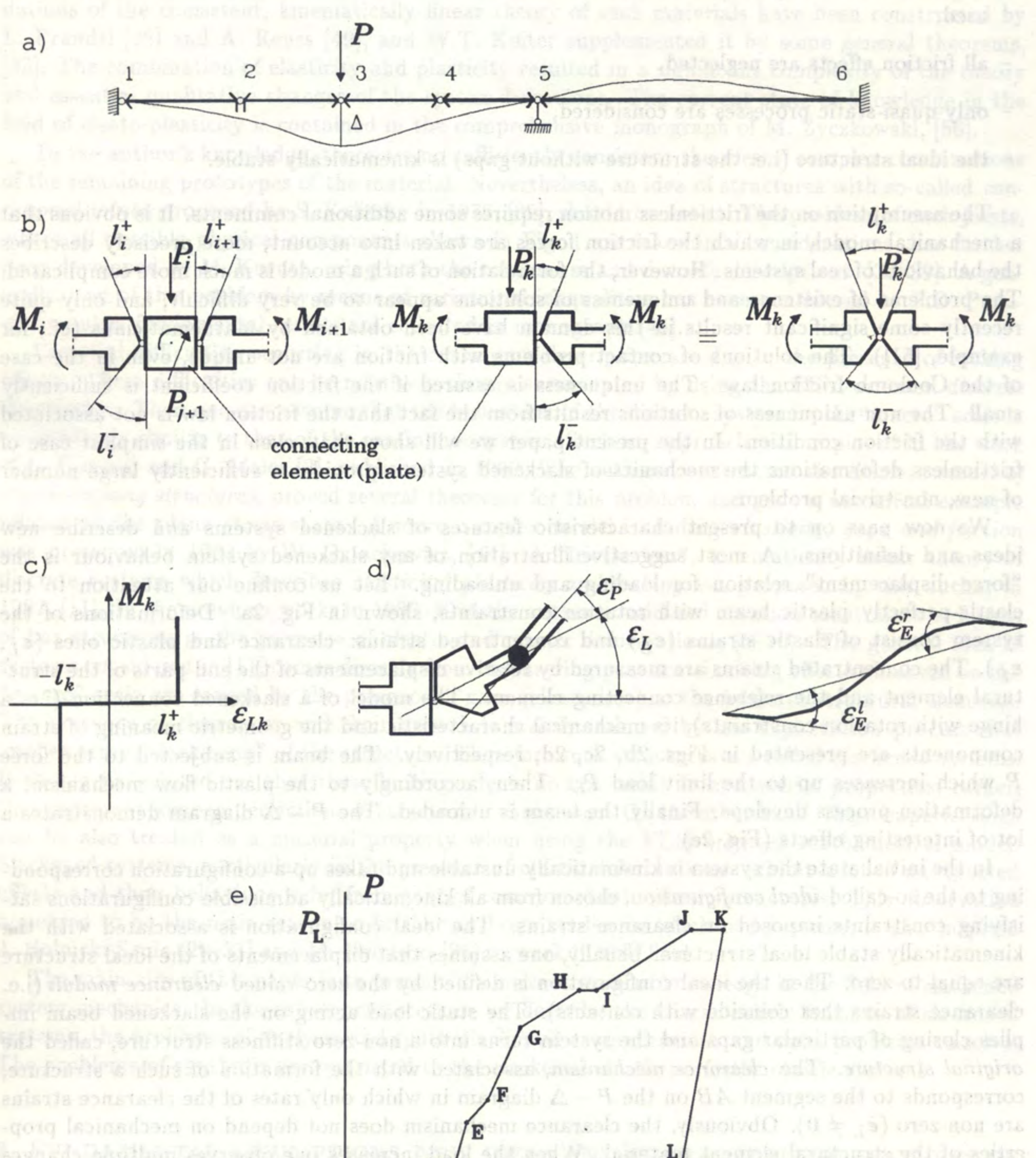

Fig. 2. Loading and unloading of a slackened-elastic-perfectly plastic beam, a) beam and load, b) models of slackened connection, c) mechanical characterictic of hinge with constrained rotation, d) geometric interpretation of strains, e) $P - \Delta$ diagram

course of unloading appears to be a characteristic feature of slackened systems. The $P(\Delta)$ function does not have to be concave for loading and convex for unloading. Energy aspects of the unloading problem will be discussed in Section 4.1.

The occurrence of slackened, slackened-plastic and plastic mechanisms is closely connected with the limit load problem of slackened-perfectly plastic systems. It has been proved in [12] that the ultimate limit load and the corresponding plastic flow mechanism are identical with those obtained for the ideal structure (without gaps). However, reaching the ultimate collapse load is usually preceded by a sequence of sublimit states of lower load multipliers and various clearance-plastic mechanisms. The "load multiplier–displacement" diagram is of step-wise character and the ultimate limit load is attained for non-zero-valued displacements. Figure 3 presents such a diagram for a two-storey frame with rotation clearances ($l_i^+ = l_i^- = 0.01\,\text{rad}$, i=1,2,...,8). The ultimate load multiplier coincides with that obtained by A. Borkowski [3] for the same frame without clearances.

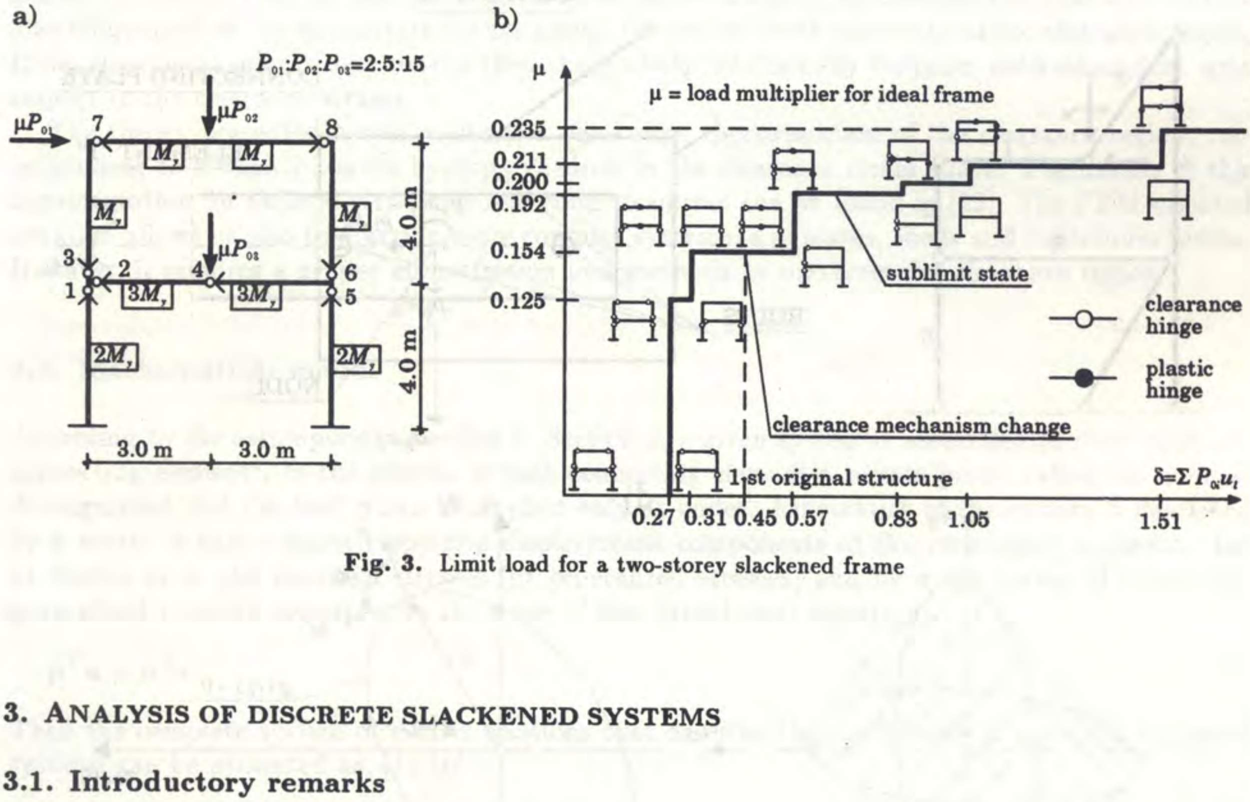

Fig. 3. Limit load for a two-storey slackened frame

3. Analysis of discrete slackened systems

3.1. Introductory remarks

Clearance and plastic deformations can be treated as distortions imposed on a linear-elastic system (cf. [1] and also [24]). Such an approach has been applied to construct the mathematical model of slackened systems [11,13,16]. To this end we use the well-known matrix notation, worked out by G. Maier and his co-workers (see e.g. [40–42]). This FEM-oriented notation has a general character and it allows to describe 1D, 2D and 3D systems. Its essential simplification consists in a linear approximation of both the yield and contact (locking) conditions. In the absence of friction forces one can obtain the Linear Complementarity Problem (LCP), equivalent to the saddle point problem or to dual extremum principles expressed in the form of Quadratic Programming Problems (QPPs).

3.2. Contact condition. Clearance surface

In the theory of slackened systems the contact condition plays a fundamental role. Its meaning is similar to the yield condition provided stresses and plastic strain rate multipliers are replaced by clearance strains and stress multipliers, respectively. The contact condition concerns clearance strains which are bounded by a so called *clearance surface*. This surface corresponds to the locking locus introduced by W. Prager, [47].

In order to illustrate the problem consider a plane model of connection with clearances shown in Fig. 4b as a part of a slackened frame of Fig. 4a (cf. [16]). Two bars are joined by four bolts attached to the connecting element (plate). Due to the gaps between the bolts and the corresponding holes drilled in the end (rigid) parts of structural elements, a relative constrained motion of the bar element and connecting plate can occur. The relative displacements play a role of concentrated strains that appear within the clearance region. Figure 4c illustrates the clearance surface for the clearance hinge of element "1".

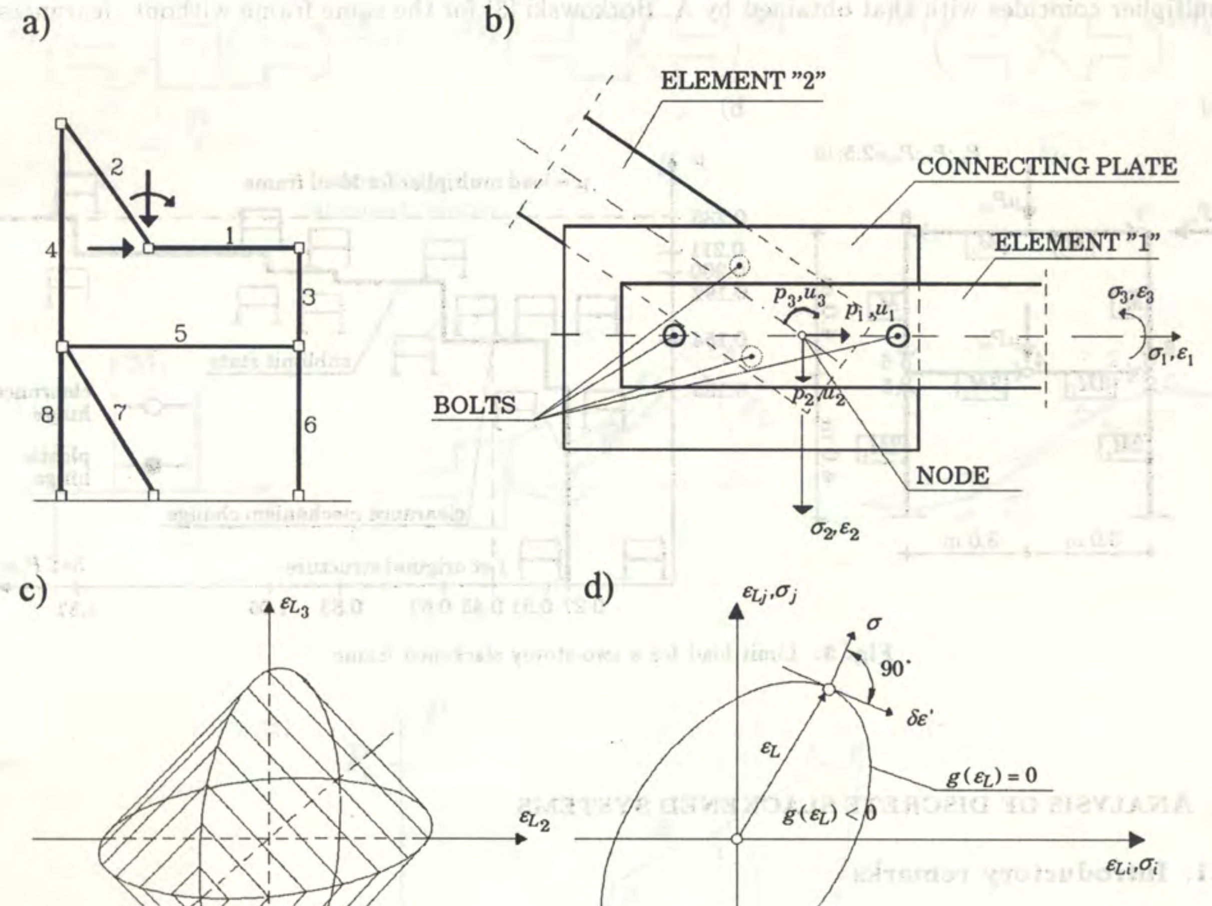

Fig. 4. Model of system with slackened connections, a) plane slackened system, b) slackened connection, c) clearance surface for element "1", d) geometric interpretation of normality rule

The contact condition can be presented by the following inequality

$$g(\varepsilon_L) \leq 0\,. \tag{1}$$

It turns out that neglecting all friction forces a normality rule for stresses is valid (see Fig. 4d), namely

$$\boldsymbol{\sigma} = \psi \left(\frac{\partial g}{\partial \boldsymbol{\varepsilon}_L} \right), \qquad \psi \geq 0\,. \tag{2}$$

Stress multiplier ψ and function $g(\boldsymbol{\varepsilon}_L)$ satisfy the following orthogonality conditions:

$$\psi \cdot g(\boldsymbol{\varepsilon}_L) = 0\,, \qquad \dot{\psi} \cdot g(\boldsymbol{\varepsilon}_L) = \psi \cdot \dot{g}(\boldsymbol{\varepsilon}_L) = 0\,, \tag{3}$$

where the dot denotes the differentiation with respect to time.

Relations (1)–(3) have been derived for plane slackened systems consisting of beam elements and connecting plates (cf. [11,16]).

Non-vanishing stresses can occur only if the corresponding clearance strain point lies on the clearance surface. The uniqueness of the clearance strain state is guaranteed if the clearance region is strictly convex. Only in this case a given stress vector uniquely determines the clearance strains. The uniqueness of the stress state occurs also in the case of weak convexity of the clearance region. If the clearance region is not convex then the problem additionally becomes path-dependent with respect to the clearance strains.

The theory presented herein assumes a piece-wise approximation of the clearance region, corresponding to a weakly convex hyperpolyhedron in the clearance strain space. The details of this approximation for skeletal structures and some examples can be found in [22]. The FEM-oriented notation allows us also to describe more complex systems, e.g. plates, shells and continuum media. However, it requires a proper discretization and methods to construct the clearance region.

3.3. Mathematical model

According to the assumptions specified in Section 2, a given system is assembled of structural and connecting elements. In the interior of each connecting element a certain point, called the node, is distinguished and the load $\mathbf{p}$ can be applied only at nodes. Kinematics of the system is described by a vector $\mathbf{u}$ that collects respective displacement components of the connecting elements. Let us denote by $\boldsymbol{\sigma}$ the vector of stresses (or generalized stresses) and by $\boldsymbol{\varepsilon}$ the vector of strains (or generalized strains), consistent in the sense of the virtual work equation:

$$\mathbf{p}^T \mathbf{u} = \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}\,. \tag{4}$$

Then the complete system of matrix relations that describe the mathematical model of slackened systems can be presented as, [11,16]:

General relations

1) $\mathbf{C}\mathbf{u} = \boldsymbol{\varepsilon}\,,$

2) $\mathbf{C}^T\boldsymbol{\sigma} = \mathbf{p}\,.$

Strain decomposition

3) $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_L + \boldsymbol{\varepsilon}_E + \boldsymbol{\varepsilon}_P\,.$

Linear elasticity

4) $\boldsymbol{\varepsilon}_E = \mathbf{E}^{-1}\boldsymbol{\sigma}\,.$

Plasticity

5) $\mathbf{f} = \mathbf{N}^T\boldsymbol{\sigma} - \mathbf{H}\boldsymbol{\lambda} - \mathbf{k} \leq \mathbf{0}\,,$

6) $\dot{\boldsymbol{\lambda}} \geq \mathbf{0}\,,$

7) $\dot{\boldsymbol{\varepsilon}}_P = \mathbf{N}\dot{\boldsymbol{\lambda}}\,,$

8) $\mathbf{f}^T \cdot \dot{\boldsymbol{\lambda}} = 0\,.$

Slackening (locking)

9) $\mathbf{g} = \mathbf{M}^T\boldsymbol{\varepsilon}_L - \mathbf{l} \leq \mathbf{0}\,,$

10) $\boldsymbol{\psi} \geq \mathbf{0}\,,$

11) $\boldsymbol{\sigma} = \mathbf{M}\boldsymbol{\psi}\,,$

12) $\mathbf{g}^T \cdot \boldsymbol{\psi} = 0\,.$

(5)

In (5) **C** denotes the kinematic compatibility matrix, **E** is the strictly positive definite square and symmetric matrix of elasticity, **N** and **M** are matrices that collect unit external normals to yield and clearance polyhedrons, respectively, and **H** denotes the linear plastic hardening matrix. Vectors **k** and **l** collect positively defined plastic and clearance moduli, respectively. Superscript "*T*" denotes the transpose.

Equations $(5)_1$ and $(5)_2$ are valid for any system which deforms according to the kinematically linear theory. Matrix **C** relates to the ideal structure that has to be kinematically stable which corresponds to the requirement

$$\det\left[\mathbf{C}^T\mathbf{C}\right] \neq 0. \tag{6}$$

Since the problem is considered within the framework of small deformations, the total strain $\boldsymbol{\varepsilon}$ can be assumed as the sum of clearance $\boldsymbol{\varepsilon}_L$, elastic $\boldsymbol{\varepsilon}_E$ and plastic $\boldsymbol{\varepsilon}_P$ components (cf. $(5)_3$).

Matrix equation $(5)_4$ expresses the generalized Hooke law for elastic deformations. Relations $(5)_5$–$(5)_8$, concerning plastic deformations, describe the yield (limit) condition, the non-negativity requirement of plastic strain rate multipliers, the associated flow law and the orthogonality condition, respectively. The contact condition, the non-negativity requirement of stress multipliers, the normality rule for stresses and the orthogonality condition are expressed by $(5)_9$–$(5)_{12}$, respectively.

In the case when the contact and yield conditions are active the following additional orthogonality conditions are valid:

$$\begin{aligned} &\boldsymbol{\psi}^T\dot{\mathbf{g}} = \dot{\boldsymbol{\psi}}^T\mathbf{g} = 0, \\ &\dot{\boldsymbol{\lambda}}^T\dot{\mathbf{f}} = \mathbf{0}. \end{aligned} \tag{7}$$

Conditions $(5)_9$ and $(7)_2$, concerning plastic deformations, lead to the well-known formulae for the plastic dissipation and the material stability. From remaining conditions $(5)_{12}$ and $(7)_1$ one obtains the following relations for clearance deformations:

$$\begin{aligned} &W_L = \boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_L = \boldsymbol{\psi}^T\mathbf{l} \geq 0, \\ &D_L = \boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}}_L \equiv 0, \qquad \boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}}_L = \dot{\boldsymbol{\psi}}^T\mathbf{l}. \end{aligned} \tag{8}$$

We can thus conclude that the *clearance work* W_L is non-negatively defined and *clearance "dissipation"* D_L is always equal to zero.

It is worth mentioning that the mathematical model (5) allows us to modify the boundary conditions of the respective structural elements. Any change of the statical indeterminacy degree can be realized in the natural way by means of a proper choice of clearance modulus components. In particular, the case of $\mathbf{l} = \mathbf{0}$ corresponds to the ideal structure with bilateral constraints.

The mathematical model (5) describes a relatively wide class of time-independent deformable systems, namely:

(a) elastic systems (E): relations $(5)_1$–$(5)_4$, where $\boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon}_E$,

(b) rigid-plastic systems (P): relations $(5)_1$–$(5)_8$, where $\boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon}_P$,

(c) ideal locking systems (S): relations $(5)_1$–$(5)_3$, $(5)_9$–$(5)_{12}$, where $\boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon}_L$,

(d) elastic-plastic systems (EP): relations $(5)_1$–$(5)_8$, where $\boldsymbol{\varepsilon}_L \equiv \mathbf{0}$,

(e) slackened-elastic systems (SE): relations $(5)_1$–$(5)_4$, $(5)_9$–$(5)_{12}$, where $\boldsymbol{\varepsilon}_P \equiv \mathbf{0}$,

(f) slackened-rigid plastic systems (SP): relations $(5)_1$–$(5)_{12}$, where $\boldsymbol{\varepsilon}_E \equiv \mathbf{0}$,

(g) slackened-elastic-plastic systems (SEP): relations $(5)_1$–$(5)_{12}$.

Moreover, the mathematical model (5) can be used to formulate the problem of time-independent systems with frictionless unilateral constraints. To this end one should take proper values of clearance moduli in these parts of the boundary where the unilateral constrains are prescribed.

3.4. Dual extremum principles. Solution uniqueness

The matrix relations of the mathematical model (5) represent a discrete form of the Kuhn-Tucker conditions. These relations can be modified to LCP or, after the Legendre transformation, to dual QPPs.

The original structure, SP-structures and limit load problems appear to be the particular cases in which QPPs convert into Linear Programming Problems (LPPs). The remaining problems of slackened systems, i.e. the problem of SE-structures and the problem of SEP-structures (holonomic and incremental analyses) have been formulated by means of QPPs.

The form of QPPs that express the dual extremum principles provides a lot of information concerning properties of the respective solutions. If the clearance region is convex then the problem of SE-structures is path-independent and the incremental analysis is not required. On the other hand, in the general case of systems in which plastic deformations appear the incremental analysis is necessary. However, it should be stressed that in the presence of unilateral constraints the incremental analysis is somewhat different from that usually applied in classical elasto-plasticity. It is due to the fact that stress muliplier rates $\Delta\psi$ can be arbitrary in sign, [16].

Let us turn our attention to the solution uniqueness problem assuming the convexity of the clearance region. In general, the solution uniqueness cannot be expected. It results from the non-uniqueness of LPPs. It is well-known that in the case of structural elements made of the perfectly rigid or rigid-plastic material stress states cannot be in general uniquely determined. This conclusion is of importance in applications of various computational methods.

However, the uniqueness problem is much more complicated and mainly concerns kinematical state variables. From the analysis carried out in [11,16] it follows that the solution uniqueness is only assured for strictly convex clearance regions together with the positive definite plastic hardening; even in this case the non-uniqueness can arise in load-free parts of the slackened system, though. Nevertheless, each solution that satisfies the requirements of the mathematical model (5) is unique in an energy sense. In other words, the non-unique solutions give different stress and strain states but their energies are the same.

3.5. Computational methods

As it has been mentioned problems of slackened systems can be solved by means of the linear and quadratic programming methods. However, large numbers of state variables and constraints require very large computer memories and lead to extremely time-consuming procedures. Therefore, to solve LCPs one also applies the direct method proposed by G.B. Dantzig [4] and C. Panne [46]. Then the required memory and time of calculations are significantly reduced (cf. [36]). For uniparameter contact conditions (e.g. beams and frame systems with rotation constraints or trusses with longitudinal constraints) one can use iterative procedures (more precisely: the trial-and-error method). In these cases the clearance regions are strictly convex and the uniqueness of solutions is assured. Then within each step of calculations the type of the structure is subjected to appropriate changes, [10]. Such an approach is effective and leads to very short computer run-times, [20]. In [14] the tangent matrix and initial stress algorithms have been proposed. In order to obtain a unique solution with respect to kinematical variables, an idea of "small" elasticity introduced into the clearance strains region has been used. However, up to the present applications of such algorithms do not give satisfactory results.

Finally, it should be mentioned that all the computational methods used to solve the problems of unilateral constraints are very sensitive to computer round-off errors which cause a lot of difficulties in formulations of algorithms.

4. Synthesis of discrete slackened-elastic-perfectly plastic systems

4.1. Problems of energy

Energy, being a scalar quantity, is a very useful measure of the current mechanical state of the system and is of great importance in physical problems. In the case of SEP-systems more general principles are valid. They also include, as particular cases, the elementary and simpler complex models of the system (i.e. E-, SE-, SP- and EP-systems). This remark concerns both the dual extremum principles and some energy bounding theorems. An energetic interpretation of unloading problems in SEP-systems appear also to be very interesting.

Consider a deformation process of a system assuming that in the virgin state for time $t = 0$ loads, displacements, stresses and strains are equal to zero. In the case of an unspecified load-path a current mechanical state for $t = t^* > 0$ is described by: $\mathbf{p} = \mathbf{p}(t^*)$, $\mathbf{u} = \mathbf{u}(t^*)$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(t^*)$, $\boldsymbol{\varepsilon}_L = \boldsymbol{\varepsilon}_L(t^*)$, $\boldsymbol{\varepsilon}_E = \boldsymbol{\varepsilon}_E(t^*)$ and $\boldsymbol{\varepsilon}_P = \boldsymbol{\varepsilon}_P(t^*)$. Then

$$
\begin{aligned}
W &= \mathbf{p}^T\mathbf{u} = W_\varepsilon + W_\sigma\,, \\
W_\varepsilon &= \int_0^{t^*} \mathbf{p}^T(t)\dot{\mathbf{u}}(t)\,\mathrm{d}t = \int_0^{t^*} \boldsymbol{\sigma}^T(t)\dot{\boldsymbol{\varepsilon}}(t)\,\mathrm{d}t\,, \\
W_\sigma &= \int_0^{t^*} \dot{\mathbf{p}}^T(t)\mathbf{u}(t)\,\mathrm{d}t = \int_0^{t^*} \dot{\boldsymbol{\sigma}}^T(t)\boldsymbol{\varepsilon}(t)\,\mathrm{d}t\,,
\end{aligned}
\tag{9}
$$

where W_ε is the strain energy and W_σ is the complementary energy when the load is controlled. Energies W_ε and W_σ can be written as

$$
\begin{aligned}
W_\varepsilon &= W_E + W_{R\varepsilon}\,, \\
W_\sigma &= W_E + W_{R\sigma}\,,
\end{aligned}
\tag{10}
$$

where

$$
\begin{aligned}
W_E &= \int_0^{t^*} \boldsymbol{\sigma}^T(t)\dot{\boldsymbol{\varepsilon}}_E(t)\,\mathrm{d}t\,, \\
W_{R\varepsilon} &= \int_0^{t^*} \boldsymbol{\sigma}^T(t)\dot{\boldsymbol{\varepsilon}}_R(t)\,\mathrm{d}t\,, \\
W_{R\sigma} &= \int_0^{t^*} \dot{\boldsymbol{\sigma}}^T(t)\boldsymbol{\varepsilon}_R(t)\,\mathrm{d}t\,, \\
W_E &= \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_E\,, \qquad W_R = W_{R\varepsilon} + W_{R\sigma} = \boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_R\,.
\end{aligned}
\tag{11}
$$

W_E denotes here the current elastic energy of the system, and $\boldsymbol{\varepsilon}_R$ is the distortion vector. For SEP-systems $\boldsymbol{\varepsilon}_R = \boldsymbol{\varepsilon}_L + \boldsymbol{\varepsilon}_P$ and $\boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}}_L \equiv 0$, $\boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}}_P \geq 0$. Then

$$
\begin{aligned}
W_{R\varepsilon} &= D = \int_0^{t^*} \boldsymbol{\sigma}^T(t)\dot{\boldsymbol{\varepsilon}}_P(t)\,\mathrm{d}t \geq 0\,, \\
W_{R\sigma} &= \int_0^{t^*} \dot{\boldsymbol{\sigma}}^T(t)[\boldsymbol{\varepsilon}_P(t) + \boldsymbol{\varepsilon}_L(t)]\,\mathrm{d}t = \boldsymbol{\sigma}^T(\boldsymbol{\varepsilon}_L + \boldsymbol{\varepsilon}_P) - D\,,
\end{aligned}
\tag{12}
$$

where D denotes the total plastic dissipation of the system.

For the holonomic (path-independent) model Eqs. (12) take the following more simple form:

$$
\begin{aligned}
&W_{R\varepsilon} = D = \boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_P\,, \qquad W_{R\sigma} = W_L = \boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_L\,, \\
&\mathbf{p}^T\mathbf{u} = 2W_E + W_L + D\,.
\end{aligned}
\tag{13}
$$

A graphical illustration of Eq. $(13)_2$ is given in Fig. 5.

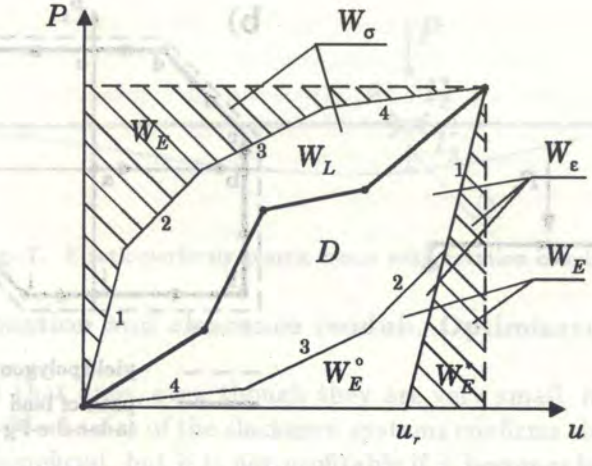

Fig. 5. Partition of the energy for a holonomic case

Using the elastic energy estimates for discrete deformable systems, derived in [17], one can formulate several energy bounding theorems for the slackened systems [21]. The bounds on energy are related to simpler models of systems subjected to the same load.

Theorem 1. *The elastic energy of a linear-elastic system (E) is a* lower *bound on the elastic energy of the identical elastic system with any distortions.* In particular, this intuitively acceptable theorem means that the elastic energy of SE-, EP- or SEP-systems cannot be less than that of the reference E-system. In general, the load-path is unspecified.

Theorem 2. *The elastic energy of the EhP-system is a* lower *bound on the elastic energy of EP- and SEP-systems for an unspecified path of load.* Symbol EhP corresponds to the holonomic model of an elastic-plastic system ("h" stands for "holonomic"). The holonomic behaviour usually occurs for proportional loading.

Theorem 3. *The complementary energy of the E-system is a* lower *bound on the complementary energy of the SE-system for an unspecified path of load.*

Theorem 4. *The complementary energy of the EhP-system is a* lower *bound on the complementary energy of the SEhP-system.*

It turns out that one can also derive the analogy to the Hodge's theorem [28] for SEP-systems, namely

Theorem 5. *The complementary energy of SEhP-system is an* upper *bound on the complementary energy of the SEP-system for any unspecified load-paths.*

The above formulated theorems are related to discrete systems only. However, the author believes that identical results can be obtained using the classical continuum description. The bounding theorems cover a wider class of problems than the variational principles. In general, they determine the upper or lower bounds on the functional (or function). The true solution does not necessarily correspond to the bound specified in the bounding theorem. It is important to add that the theorems of P.G. Hodge [28] (or J.B. Martin [43]) introduce statically admissible stress fields related to proper strain rates fields, and are therefore more general. However, the constraints imposed on these fields are also satisfied in the holonomic model whose properties allow us to obtain the true solution by means of the Haar-Kármán's theorem, [25].

A quantitative illustration of the bounding theorems is shown in Fig. 6c that corresponds to the continuous beam of Fig. 6a. The beam is loaded by two concentrated forces varying according to the load path indicated in Fig. 6b. The details of this example can be found in [17].

Passing now to the problem of unloading we recall again the fundamental inequalities derived in [17]. It will be shown that the unloading curve for EP- or SEP-systems can be nonlinear. Assume that a given state of the system is determined by load vector $\mathbf{p}$, displacement vector $\mathbf{u}$, stress vector $\boldsymbol{\sigma}$, and strain vectors $\boldsymbol{\varepsilon}_L$, $\boldsymbol{\varepsilon}_E$ and $\boldsymbol{\varepsilon}_P$. After unloading we have $\mathbf{p}^o = \mathbf{0}$ and the remaining vectors

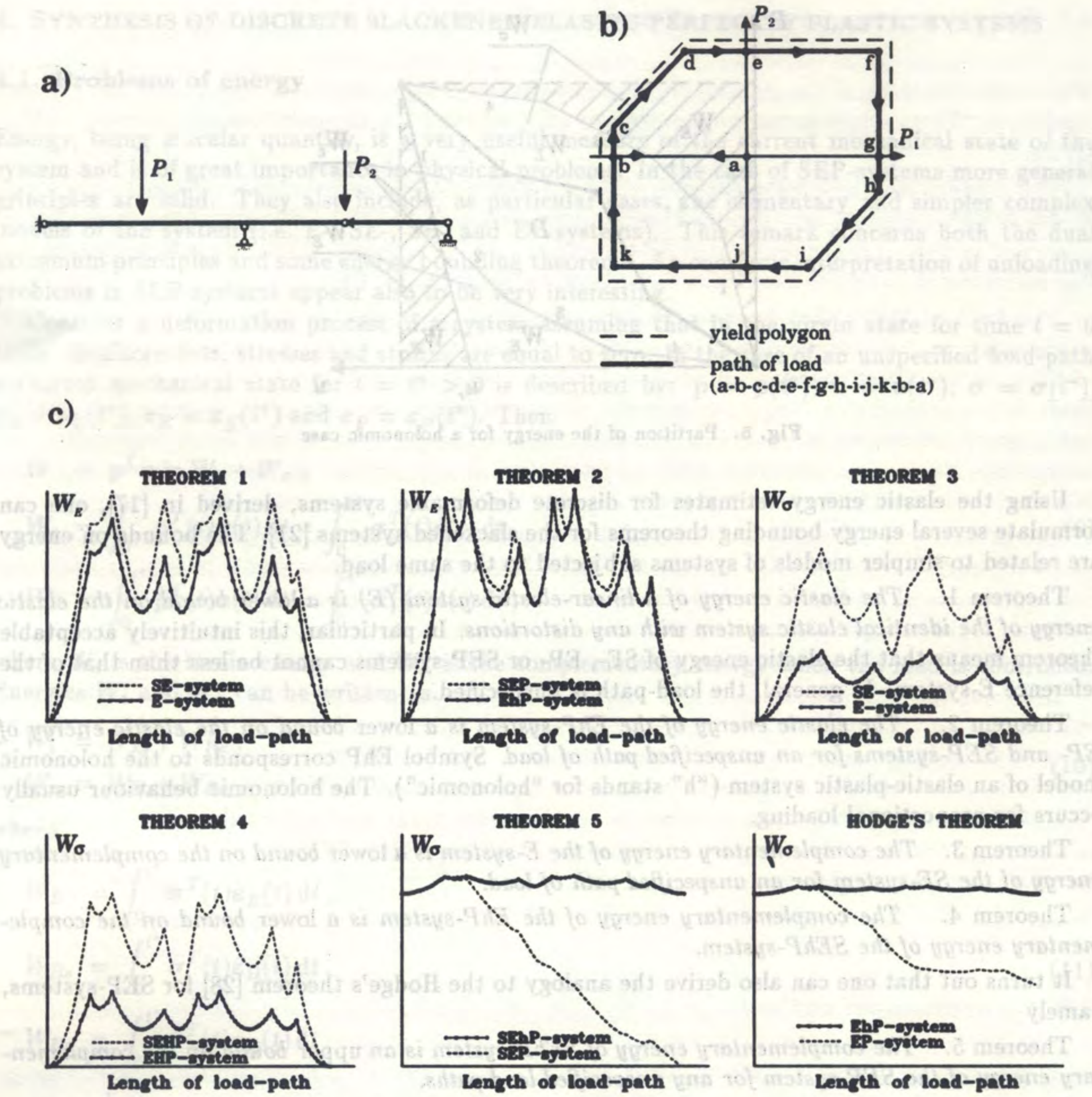

Fig. 6. Graphical illustration of energy bounding theorems, a) beam and load, b) path of load, c) lower and upper bounds on energy

are indicated by σ^o, ε_L^o, ε_E^o, ε_P^o. The recovered elastic energy W_E^* can be expressed as (cf. [18]):

$$W_E^* = \frac{1}{2}\mathbf{p}^T(\mathbf{u}-\mathbf{u}^o) - \frac{1}{2}(\boldsymbol{\sigma}+\boldsymbol{\sigma}^o)^T(\boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P^o + \boldsymbol{\varepsilon}_L - \boldsymbol{\varepsilon}_L^o). \tag{14}$$

From the convexity of the clearance region one obtains the following estimates:

$$\begin{aligned} W_E^* &\geq \frac{1}{2}\mathbf{p}^T(\mathbf{u}-\mathbf{u}^o) - \frac{1}{2}(\boldsymbol{\sigma}+\boldsymbol{\sigma}^o)^T(\boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P^o) - \boldsymbol{\sigma}^T(\boldsymbol{\varepsilon}_L - \boldsymbol{\varepsilon}_L^o), \\ W_E^* &\leq \frac{1}{2}\mathbf{p}^T(\mathbf{u}-\mathbf{u}^o) - \frac{1}{2}(\boldsymbol{\sigma}+\boldsymbol{\sigma}^o)^T(\boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P^o) + \boldsymbol{\sigma}^{oT}(\boldsymbol{\varepsilon}_L^o - \boldsymbol{\varepsilon}_L). \end{aligned} \tag{15}$$

It is seen that, even in the absence of reverse plasticity, i.e. if $\boldsymbol{\varepsilon}_P = \boldsymbol{\varepsilon}_P^o$, the recovered energy W_E^* can be larger or less than the energy of linear unloading (the area of "triangle" formed by $\mathbf{p}$ and $\mathbf{u}-\mathbf{u}^o$). If the residual stresses vanish ($\boldsymbol{\sigma}^o = \mathbf{0}$) then one obtains $W_E^* \leq \mathbf{p}^T(\mathbf{u}-\mathbf{u}^o)/2$. This fact is well-known in mechanics of granular media where the unloading curves are usually convex.

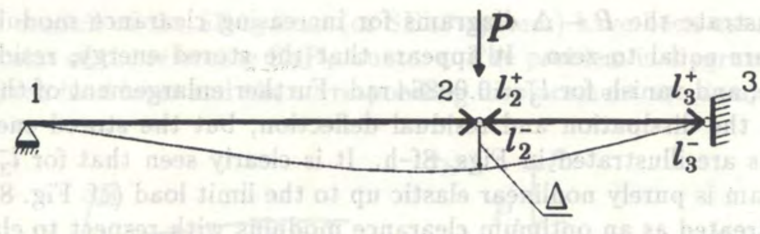

Fig. 7. Elastic-perfectly plastic beam with rotation constraints

4.2. Effects of gaps location and clearance moduli. Optimization

It should be pointed out that gaps, even though they are very small, induce a strongly nonlinear behaviour of the system. Mechanics of the slackened systems confirms the well-known feeling that a sufficiently small gap is beneficial, but it is not profitable if it becomes larger then a certain critical value. This statement can be illustrated by considering the elastic-perfectly plastic I-beam with rotation constraints (see Fig. 7). The beam is loaded by concentrated force P up to limit load P_L and then unloaded.

For the beam with bilateral constraints ($l_i^+ = l_i^- = 0$, $i = 1, 2$) the $P - \Delta$ diagram is shown in Fig. 8a. The following types of the energy are indicated: total dissipation D, total elastic energy W_E together with its partition into recovered energy W_E^* and stored (hidden) energy W_E^o that remains in the system after unloading. The residual deflection Δ_r is also shown therein.

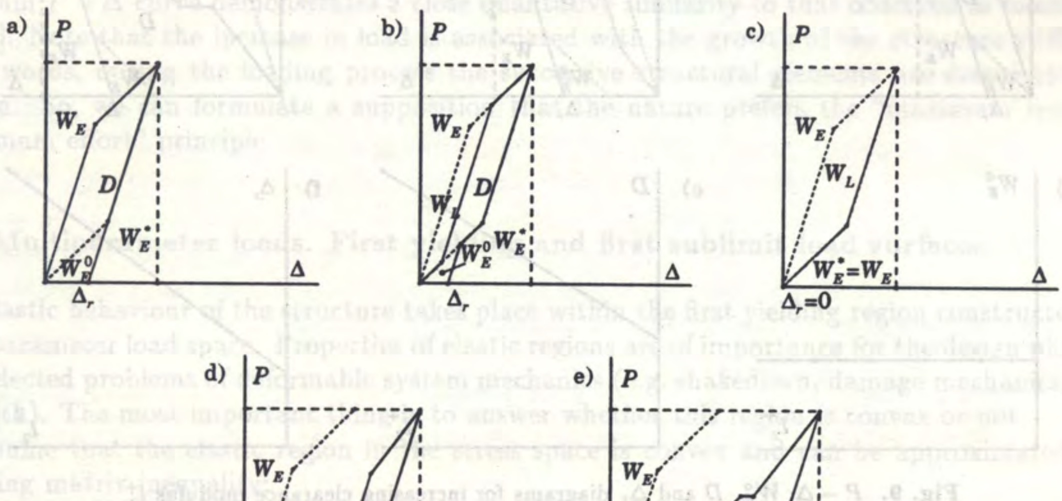

Fig. 8. $P - \Delta$, W_E^o, D and Δ_r diagrams for increasing clearance modulus l_3^-

Figures 8b–e illustrate the $P - \Delta$ diagrams for increasing clearance modulus l_3^- whereas the remaining moduli are equal to zero. It appears that the stored energy, residual deflection and dissipation decrease, and vanish for $l_3^- = 0.00864$ rad. Further enlargement of this modulus implies a linear increase in the dissipation and residual deflection, but the stored energy remains equal to zero. These facts are illustrated in Figs. 8f–h. It is clearly seen that for $l_3^- = 0.00864$ rad the behaviour of the beam is purely nonlinear elastic up to the limit load (cf. Fig. 8c). This particular value of l_3^- can be treated as an optimum clearance modulus with respect to elastic strength. For larger value of l_3^- a significant deterioration of mechanical properties of the structure is observed.

Consider now the increase of clearance modulus l_2^+ with the remaining moduli equal to zero. The corresponding $P - \Delta$ diagrams are shown in Figs. 9a–c, and Figs. 9d–f illustrate relations $W_E^o(l_2^+)$, $D(l_2^+)$, $\Delta_r(l_2^+)$. The growth of l_2^+ induces a permanent deterioration of mechanical properties of the beam. The qualitative differences between the cases where clearances occur at points "2" and "3" are evident. The presence of clearances at point "2" is not profitable, but sufficiently small clearances at point "3" are beneficial. Note that it is not sufficient to specify the proper location of gaps; equally important is to determine the suitable sign of clearance strains.

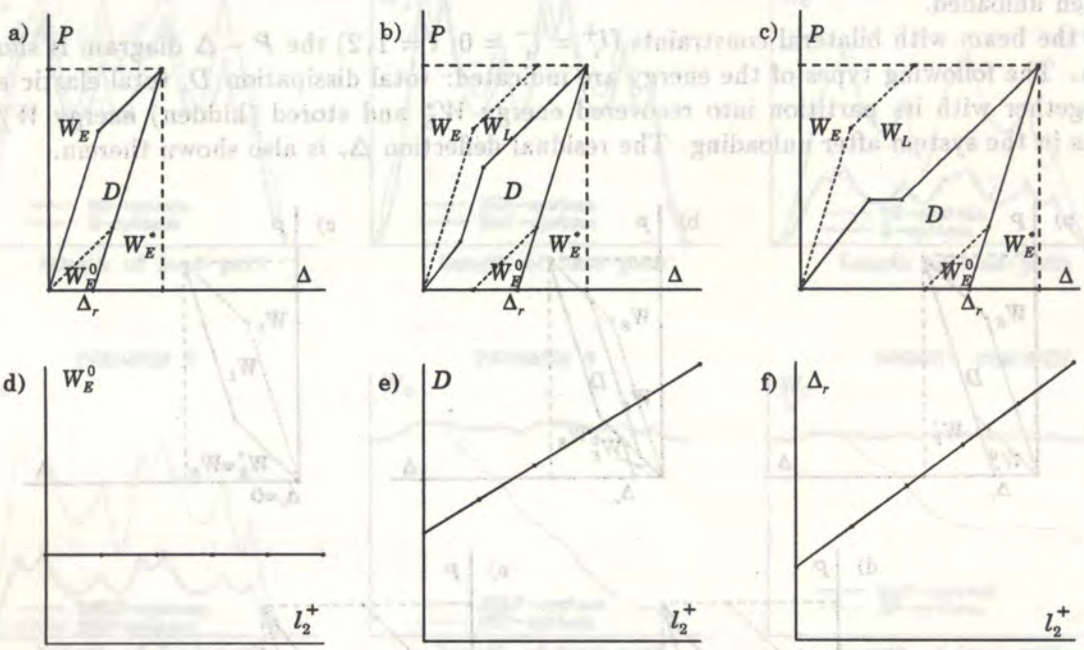

Fig. 9. $P - \Delta$, W_E^o, D and Δ_r diagrams for increasing clearance modulus l_2^+

We can conclude that the behaviour of the system under a given load depends both on the location of gaps and their dimensions (moduli). On the other hand it can be shown that a change of the load location affects the optimum location and dimensions of gaps. Moreover, if the occurrence of clearance mechanisms is excluded then for a certain location of the load the optimum values of the clearance moduli can be negative. It means that the optimum slackened structure should be prestressed (cf. [10]).

The optimization problem for proportional loading can be solved by using the theorem derived in [11] (see also [16]): "Any EhP-system can be associated with the SE-system (or SEhP-system) which, for the same load, exhibits the same stress, displacement and total strain states". Clearance strains in the corresponding SE-system (or SEhP-system) should be equal to plastic strains in EhP-system, i.e. $\epsilon_L^{se} \equiv \epsilon_P^{ep}$. In general, such an attractive solution is difficult to achieve in practice, especially in the case of multiparameter contact conditions when clearances interaction occur. However, some results have been already obtained for non-interacting clearance strains (e.g. rotation constraints in beams and frames, longitudinal gaps in trusses). In those cases the non-zero clearance moduli of the SE-system (or SEhP-system) have been assumed to be identical with the plastic strains of the reference EhP-structure. The problem is diagrammatically explained in Fig. 10a. The given load level and the corresponding displacement are denoted by P_A and u_A.

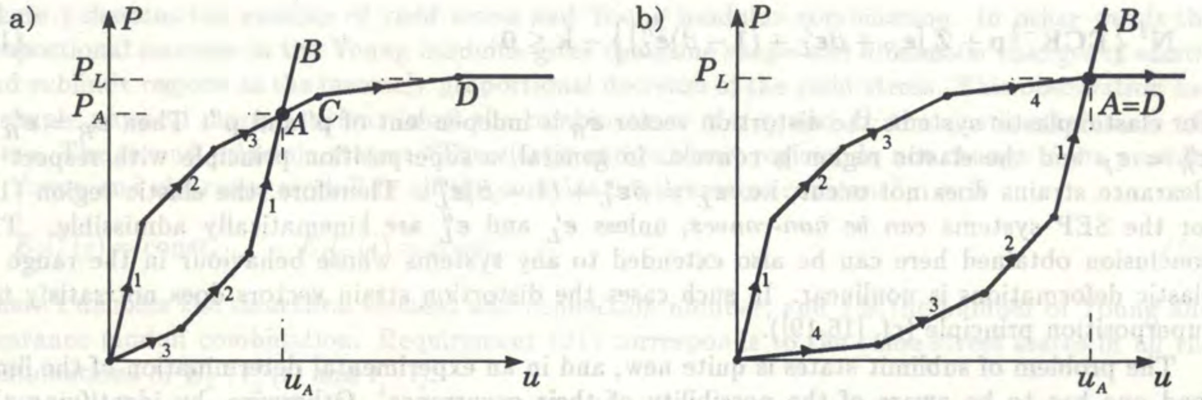

Fig. 10. $P-\Delta$ **diagrams for EP-SE- and SEP-structures, a) SE-and SEP-structures associated with an EP-structure, b) optimum SEP-structure**

If the point A coincides with the point D, corresponding to the limit load level (i.e. $P_A = P_L$), then one obtains the optimum solution with respect to the elastic strength (cf. Fig. 10b). This optimum $P-\Delta$ curve demonstrates a close qualitative similarity to that observed in biomaterials, [54,55]. Note that the increase in load is associated with the growth of the structure stiffness. In other words, during the loading process the successive structural elements are drawn into cooperation. So, we can formulate a supposition that the nature prefers the "maximum reserve" or "minimum effort" principle.

4.3. Multiparameter loads. First yielding and first sublimit load surfaces

The elastic behaviour of the structure takes place within the first yielding region constructed in the multiparameter load space. Properties of elastic regions are of importance for the design philosophy and selected problems of deformable system mechanics (e.g. shakedown, damage mechanics, fatigue strength). The most important thing is to answer whether this region is convex or not.

Assume that the elastic region in the stress space is convex and can be approximated by the following matrix inequality:

$$\mathbf{N}^T\boldsymbol{\sigma} - \mathbf{k} \leq \mathbf{0}\,. \tag{16}$$

In the case of the linear elastic system with imposed distortions $\boldsymbol{\varepsilon}_R$, inequality (16) can be rewritten in the form:

$$\mathbf{N}^T(\mathbf{ECK}^{-1}\mathbf{p} + \mathbf{Z}\boldsymbol{\varepsilon}_R) - \mathbf{k} \leq \mathbf{0}\,, \tag{17}$$

where $\mathbf{K} = \mathbf{C}^T\mathbf{EC}$ and $\mathbf{Z} = \mathbf{ECK}^{-1}\mathbf{C}^T\mathbf{E} - \mathbf{E}$. In (17) $\mathbf{K}$ denotes the common stiffness matrix for the linear elastic system without distortions, and $\mathbf{Z}$ is the distortion influence matrix (cf. [40,3,16]). The inequality (17) describes the elastic region S in the load space. Let us assume that $\mathbf{p}'$ and $\mathbf{p}''$ are elements of the set S, i.e.

$$\begin{aligned}\mathbf{N}^T(\mathbf{ECK}^{-1}\mathbf{p}' + \mathbf{Z}\boldsymbol{\varepsilon}'_R) - \mathbf{k} &\leq \mathbf{0}\,,\\ \mathbf{N}^T(\mathbf{ECK}^{-1}\mathbf{p}'' + \mathbf{Z}\boldsymbol{\varepsilon}''_R) - \mathbf{k} &\leq \mathbf{0}\,,\end{aligned} \tag{18}$$

where $\boldsymbol{\varepsilon}'_R$ and $\boldsymbol{\varepsilon}''_R$ denote distortions connected with $\mathbf{p}'$ and $\mathbf{p}''$, respectively.

If $\mathbf{p} = \beta\mathbf{p}' + (1-\beta)\mathbf{p}''$, $\beta \in \langle 0,1 \rangle$, belongs to S, then S is convex. In the case of a slackened-linear elastic-plastic system with a steady plastic strain vector $\boldsymbol{\varepsilon}_P$ and clearance strain vectors $\boldsymbol{\varepsilon}'_L$ and $\boldsymbol{\varepsilon}''_L$, depending on the loads $\mathbf{p}'$ and $\mathbf{p}''$, from (18) one obtains

$$\mathbf{N}^T\left\{\mathbf{ECK}^{-1}\mathbf{p} + \mathbf{Z}\left[\boldsymbol{\varepsilon}_P + \beta\boldsymbol{\varepsilon}'_L + (1-\beta)\boldsymbol{\varepsilon}''_L\right]\right\} - \mathbf{k} \leq \mathbf{0}. \tag{19}$$

For elastic-plastic systems the distortion vector $\boldsymbol{\varepsilon}_R$ is independent of $\mathbf{p}'$ and $\mathbf{p}''$. Then $\boldsymbol{\varepsilon}_R = \boldsymbol{\varepsilon}'_R = \boldsymbol{\varepsilon}''_R = \boldsymbol{\varepsilon}_P$ and the elastic region is convex. In general, a superposition principle with respect to clearance strains does not occur, i.e. $\boldsymbol{\varepsilon}_L \neq \beta\boldsymbol{\varepsilon}'_L + (1-\beta)\boldsymbol{\varepsilon}''_L$. Therefore, the elastic region (19) for the SEP systems *can be non-convex*, unless $\boldsymbol{\varepsilon}'_L$ and $\boldsymbol{\varepsilon}''_L$ are kinematically admissible. The conclusion obtained here can be also extended to any systems whose behaviour in the range of elastic deformations is nonlinear. In such cases the distortion strain vectors does not satisfy the superposition principle (cf. [16,19]).

The problem of sublimit states is quite new, and in an experimental determination of the limit load one has to be aware of the possibility of their occurrence. Otherwise, by identifying the sublimit state as the limit one, we can reach errenous conclusions. In the examples considered by the author it follows that the region of the first sublimit load appears to be a sum of convex sets. Therefore the sublimit region can be non-convex. Passing now to micromechanical scale we can imagine that for an elastic-perfectly plastic body with internal gaps a sequence of numerous sublimit states can suggests the occurrence of a plastic hardening, a plastic anisotropy, and even a non-convexity of yield condition as well as a non-associated flow law.

It should be mentioned that the convexity and dimensions of elastic and sublimit regions are strongly influenced by initial distortions resulting from previous loading cycles.

The problems discussed here are illustrated in Fig. 11 prepared for the SEP-structure that is subjected to two-parameter loads (cf. [23]). The yield polygon, the polygon of a first sublimit load and the initial elastic region (without plastic deformations) are shown in Fig. 10a. The same regions are visualized in Figs. 11b,c but the structure includes plastic distortions due to previous loading cycles. These cycles correspond to the proportional loading and unloading from points "2" and "3", respectively. In all the cases the shape and dimensions of the yield polygon are the same because they are independent of gaps and permanent plastic deformations.

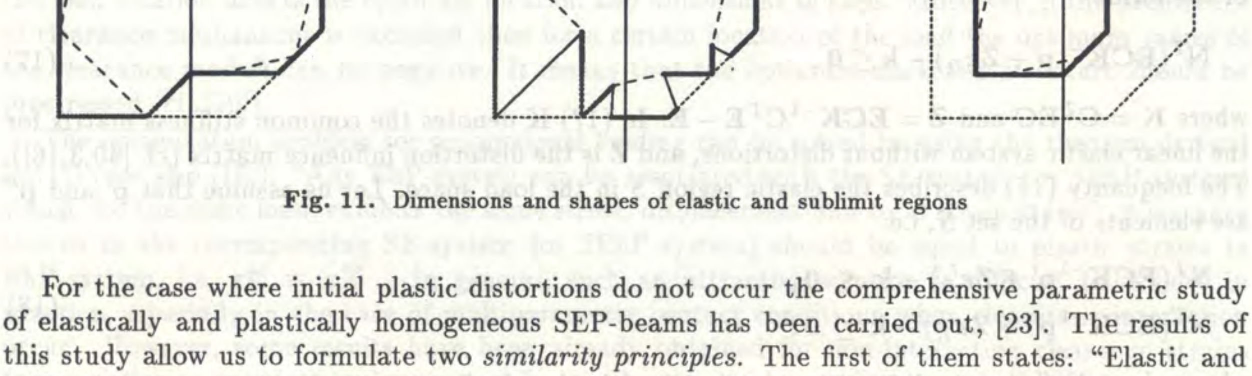

Fig. 11. Dimensions and shapes of elastic and sublimit regions

For the case where initial plastic distortions do not occur the comprehensive parametric study of elastically and plastically homogeneous SEP-beams has been carried out, [23]. The results of this study allow us to formulate two *similarity principles*. The first of them states: "Elastic and sublimit regions are homotetic with respect to the origin of load plane if the ratio of Young modulus and yield stress remains constant", i.e. if

$$\frac{\sigma_{Pj}}{E_j} = \text{const} \tag{20}$$

where j denotes the number of yield stress and Young modulus combination. In other words the proportional increase in the Young modulus gives the same shape and dimension changes of elastic and sublimit regions as the inversely proportional decrease of the yield stress. This observation has a simple physical interpretation: in all the combinations of σ_{Pj} and E_j the elastic strains are the same. The second principle states: "The elastic and sublimit regions do not change if the product of Young and clearance moduli in all the combinations remains constant", i.e. if

$$E_j l_j^+(i) = \text{const}, \qquad E_j l_j^-(i) = \text{const}, \tag{21}$$

where i denotes the structural element and connection number, and j is the number of Young and clearance moduli combination. Requirement (21) corresponds to the same stress states in all the combinations of E_j , $l_j^+(i)$ and $l_j^-(i)$.

The similarity principles have been here specified for the very particular case. However, the clear and suggestive physical interpretation allows us to suppose that they can be extended to much more complex systems.

4.4. Cyclic loading. Shakedown

The problem of cyclic loading of slackened systems appears to be very complicated, particularly when plastic deformations occur. The complexity of this problem can be illustrated by $P-\Delta$ diagrams for the alternate load ($-P^* \leq P \leq P^*$) acting on the SEP- and SP-beam of Fig. 2a (cf. Fig. 12). It is obvious that the SP-system is more stiff than the SEP one. Therefore the number of sublimit states in the SEP-beam cannot be greater than that of the SP-beam.

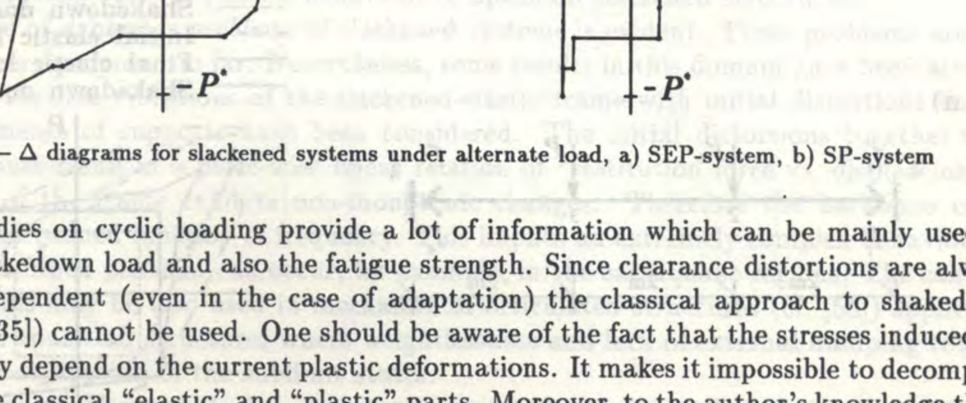

Fig. 12. $P-\Delta$ diagrams for slackened systems under alternate load, a) SEP-system, b) SP-system

Numerical studies on cyclic loading provide a lot of information which can be mainly used to determine the shakedown load and also the fatigue strength. Since clearance distortions are always load- and time-dependent (even in the case of adaptation) the classical approach to shakedown problems (cf. [34,35]) cannot be used. One should be aware of the fact that the stresses induced by clearances strongly depend on the current plastic deformations. It makes it impossible to decompose of stresses into the classical "elastic" and "plastic" parts. Moreover, to the author's knowledge there is no theorem concerning the shakedown problem of SEP-structures. Therefore, the studies on this problem are only restricted to analyses of cyclic loading by means of the "step by step" method. Results obtained using such a approach show that the shakedown load depends on the gap location and clearance moduli [15,18].

It follows from the study on cyclic loading of the beam with rotation constraints located at the support (point "3" in Fig. 13a) that for the alternate loads the shakedown load P_S depends only on the sum of clearance moduli: $\phi = l_3^- + l_3^+$. The corresponding diagram of $P_S(\phi)$ is presented in Fig. 13b (cf. [18]). It is clearly seen that for small values of clearance moduli the shakedown load is larger than that of the beam fully fixed at the support "3". For a certain value of ϕ the shakedown load reaches its maximum. In the case of symmetric load cycles the most interesting aspect is that $P_S(\phi)$ exhibits a discontinuity for large clearances.

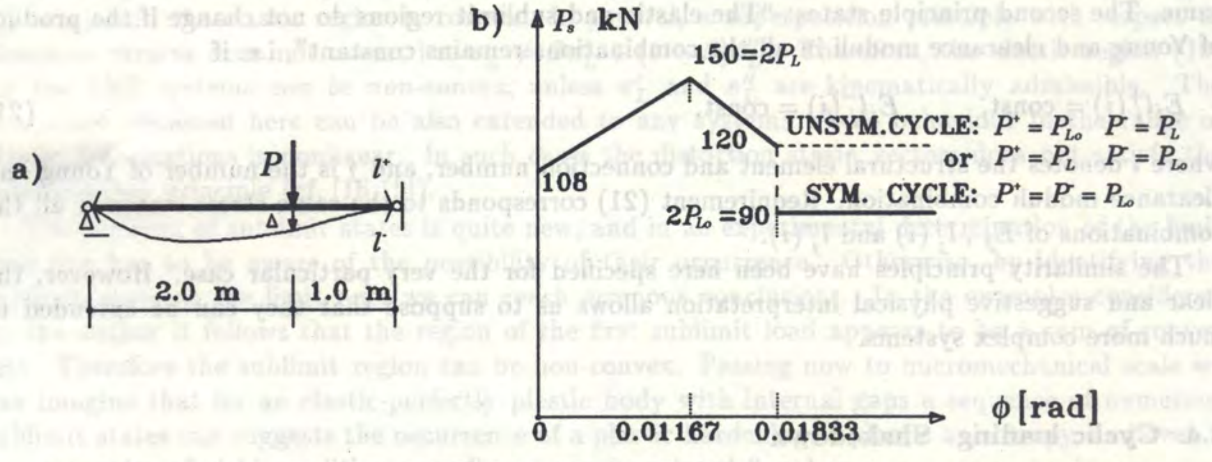

Fig. 13. Shakedown of a slackened beam subjected to the alternate load, a) beam, load and rotation constraints, b) shakedown load versus the sum of limit rotations ϕ

According to the Coffin-Miner formula it turns out that for a relatively wide range of clearance moduli, a significant increase in critical number of load cycles to failure can be expected.

Usually, an improper location of the clearances leads to a decrease in the shakedown load. If the shakedown load is constrained by the incremental collapse one can show some examples in which the presence of clearances does not affect the load multiplier. On the other hand it is very difficult to determine such a clearance distribution that gives an increase in shakedown load. However, in the case of alternate loads a profitable gap distribution can be easily determined using the theorem for associated structures (cf. Section 4.2). This problem is explained in Fig. 14 where a proper

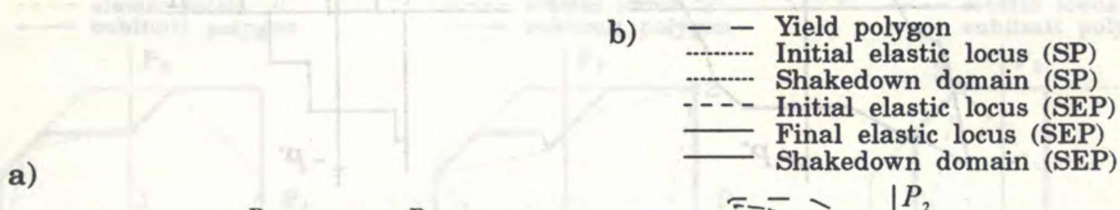

Fig. 14. Shakedown of an optimally slackened beam under alternate loads, a) beam, loads and rotation constraints, b) optimum solution

choice of limit rotations leads to the absolute maximum of shakedown load which coincides with the ultimate collapse load.

It should be pointed out that all the effects presented above arise for very small deformations and displacements.

The presence of gaps allow us also to formulate a *shakedown problem of slackened-perfectly plastic systems,* [18]. In the case of variable loads the current shape and dimensions of the first sublimit region depend on plastic deformations accumulated in previous load cycles. The necessary condition for adaptation is the formation of time-independent plastic strains such that the corresponding sublimit region contains the prescribed load domain. A link between this formulation and the classical one is clearly seen. The sublimit region and steady plastic deformations correspond to the elastic region and the residual stresses, respectively. The concept of residual stresses is useless because the stress state in rigid systems cannot be determined. The shape and dimensions of the sublimit region demonstrate *step-wise changes* when the plastic deformations develop. The problem discussed is illustrated in Fig. 15 where the cases of incremental collapse and adaptation of the structure are shown.

Thus, for slackened systems made of an elastic-plastic and rigid-plastic materials some modification of the classical definition of shakedown can be proposed, namely:

"A slackened system shakes down if after a finite number of load cycles *a time-independent plastic strain field* forms, and the stress states do not violate the yield condition".

5. APPLICATIONS

Results obtained for *quasi-static problems of analysis* are of practical and theoretical significance. Engineers many times face situations in which the behaviour of real structures is far from that theoretically predicted when using classical methods of analysis. The theory of slackened systems allow us to describe more precisely real structures and explain some unexpected effects observed in experiments. For instance, inexplicable reserves of strength, non-typical $P - \Delta$ relations or non-convexity of elastic regions can result from the presence of clearances due to manufacturing or exploitation imperfections. These observations are of importance for both the design and the theory of deformable systems. The non-convexity of yield conditions or the violation of normality rules can be interpreted on the micromechanical level as effects of internal slackening of the material. Slackened systems exhibit a close relationship with *cracking bodies* (unilateral constraints) and *granular media* ($P - \Delta$ relations for loading and unloading, pre-consolidation). The author is convinced that the fundamental assumptions used in the theory of slackened systems can be also applied to *biomechanical systems* (elastic behaviour of optimum slackened structures).

The significance of *dynamics problems* of slackened systems is evident. These problems are not covered by the theory proposed so far. Nevertheless, some results in this domain have been already obtained. In [10] the free vibrations of the slackened-elastic frame with initial distortions due to enforced displacements of supports have been considered. The initial distortions together with unilateral constraints result in a piece-wise linear relation of "restitution force vs. displacement", and the stiffness of the frame exhibits non-monotonic changes. Therefore the backbone curve appears to be a multivalued function of frequency. This implies an extremely complex behaviour of the frame. The described problem can occur, for example, in the earthquake regions. The concept of slackened systems may be also used in mechanics of articulated structures (cf. [55]) applied in biomechanics or in space-lab structures where weightlessness and lack of external damping require an internal energy dissipation in the sublimit states.

The *optimization problems* of slackened systems seem to be the most interesting ones. These problems correspond to the case in which the distributions of clearance moduli are programmable. It may be mentioned that the author has initiated his studies on slackened systems in this field. The author considered the following complex system: "elastic frame and reinforced concrete foundation" (cf. [11]). A minimum cost of the system subjected to dead and variable loads (snow and

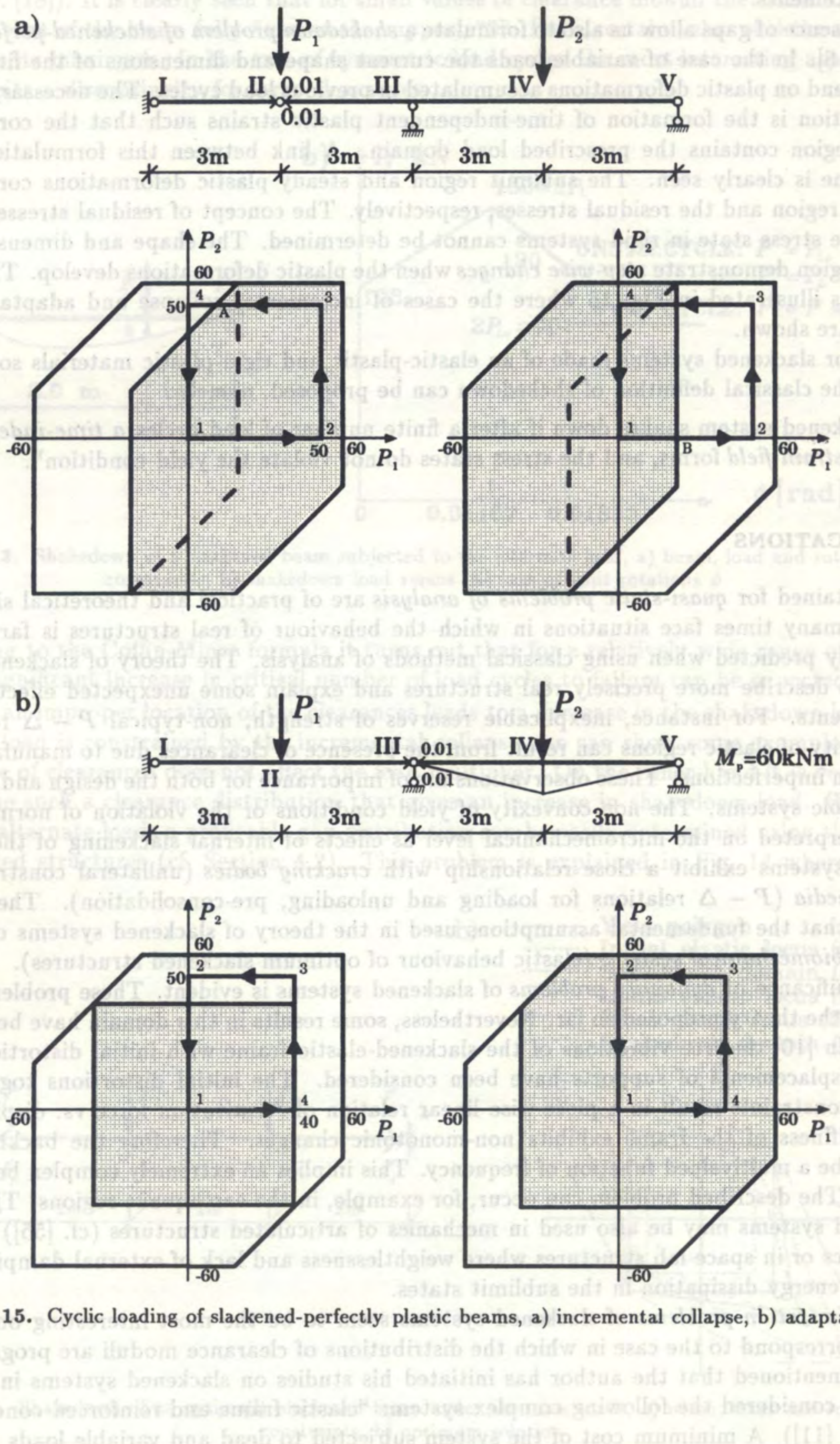

Fig. 15. Cyclic loading of slackened-perfectly plastic beams, a) incremental collapse, b) adaptation

wind) corresponds to a certain optimum constraints imposed on the rotation angle of the columns at supports. For slackened-elastic-plastic beams the elastic envelope in P, Δ plane has been also analyzed. The concept of slackened systems provides a lot of new, non-trivial questions. The optimum solution with respect to elastic strength exhibits a relatively small stiffness. Therefore the displacement constraints are usually more restrictive than those in the standard cases. However, up to now the optimization problem of slackened systems has not been precisely formulated. Nevertheless, taking into considerations the already obtained results, one can suppose that optimization would be particularly profitable in *shakedown* (mainly for alternating loads), *fatigue strength, damage mechanics*, as well as in *robotics* and *civil engineering* (bridges) where clearance moduli may be actively controlled. It follows from initial, not yet published studies on the *optimization* of slackened-elastic structures *with respect to elastic stability*, that a proper choice of clearance moduli leads to an essential increase in the critical load.

6. Problems to solve

Mechanics of slackened systems is relatively new, and a lot of problems remain open. The most important of them are listed below:

- consistent theory of locking-elastic-plastic continuum,
- shakedown (problem formulation, theoretical foundations, solution methods),
- theory of slackened-elastic-visco-plastic systems,
- problem of initial distortion,
- theory allowing for the clearance surface evolution,
- the clearance surface for complex systems,
- dynamics,
- theory allowing for friction effects,
- theory of large deformation,
- elastic stability,
- optimization theory (formulation and solution methods),
- theory for non-convex clearance surfaces.

7. Final remarks

In the author's opinion the problem of locking-elastic systems is qualitatively equivalent to the problem of elastic-plastic systems. From the mathematical point of view both the problems are equally difficult and lead to a variational inequality formulation. The locking-elastic-plastic (LEP) model appears to be extremely complicated and its properties are far from that of the standard elastic-plastic model. The LEP model can explain a lot of phenomena observed in practice and should be included into the general theory of time-independent materials. It is worth indicating that slackened systems demonstrate a strong nonlinearity even in the range of small deformations. The author believes that the results presented in this article confirm the statements specified above.

From the engineering point of view the presence of clearances can be sometimes very beneficial. It is so for small clearance moduli. Large values of these moduli are not profitable and confirm the well-known fact that a small gap may be useful, but the one larger than a certain critical value is not needed. It follows from these remarks that in order to obtain optimum properties of the mechanical system, clearance moduli should be programmable or actively controlled. This concept can be applied in civil and mechanical engineering design as well as in robotics. The most interesting fact is that optimum distributions of clearances with respect to elastic strength correspond to the "minimum effort" or "maximum reserve" principle. We believe that this principle, according to results of experiments in biomechanics, holds also true in nature.

ACKNOWLEDGEMENTS

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