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Feasibility of fuzzy controller for efficient iterative method

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In the engineering processes it is important to simulate thermo-hydraulic phenomena numerically in the limited time before design of equipment. In thermo-hydraulic problem, as it may be time-consuming to solve elliptic equation numerically, a heavy burden is imposed on the computer. A SOR method is one of the effective methods to solve elliptic equation. As it is difficult to find the optimum relaxation factor, the value of this factor for the practical problems used to be estimated by the expertise. In this paper, the implications about the relaxation factor are translated into fuzzy control rules on the basis of the expertise of numerical analysts, and then the fuzzy controller is incorporated into the numerical algorithm. A Dirichlet problem of the Poisson's equation and the cavity flow problem are chosen to verify the feasibility of fuzzy controller for relaxation. Numerical experiments with the fuzzy controller resulted in generating a good performance.

1. Introduction

When the flow problem is described by the vorticity and the stream function, it necessitates solving the parabolic and the elliptic differential equation simultaneously. Practical problems in the engineering field require numerical solutions because of their nonlinearity and complexity. Generally the numerical procedure for simulating thermo-hydraulics consists of conducting the explicit scheme for the initial value problem of the parabolic equation and the iterative method for the Dirichlet problem of the elliptic equation. The computational burden comes in about 90% from the iterative solving of the latter problem. It is therefore essential to find the efficient iterative method and/or to optimize the computational algorithm.

The efficiency of the iterative method, SOR, depends on the relaxation factor. As the method to control this factor, an algebraic adaptive control [1] and a model reference adaptive control [2] have been reported. In this paper, the implications concerned with the relaxation factor are translated into fuzzy control rules on the basis of the expertise of the numerical analysts.

Although a lot of examples show that this fuzzy method is effective, it should be observed that the numerical background has not yet been clarified. This apprehension has motivated the authors to write the present paper. A Dirichlet problem of the Poisson's equation is chosen as the reference since the optimum relaxation factor for the SOR method is there derived analytically. That is to say, this problem is the most suitable model to verify the capabilities of the fuzzy controller. Next, the two-dimensional driven cavity flow [3], where the optimum relaxation factor can not be obtained analytically, is considered as well. The latter model problem can serve as a reasonable example of a practical engineering problem to study the effect of the fuzzy controller on the iterative process, which is the main purpose of the article.

2. MATHEMATICAL PRELIMINARIES

2.1. Theorem

There are several implications when solving simultaneous equations with an iterative method, which come from the mathematical formulation [4, 5], the analogy from the mathematical theorem, the experience of numerical calculations [5, 6] and the control theory [6].

The equation $Au = b$ can be rearranged as follows

$$u^{(k+1)} = Gu^{(k)} + f \tag{1}$$

where $u^{(k)}$ is the iterate of the variable approximated by the k-th iteration, f is the known vector and G the iterative matrix. We observe that

① The convergence condition for iterative methods is given by

$$\rho(G) < 1 \tag{2}$$

where $\rho(G)$ is the spectral radius of G. Only when Eq. (2) is satisfied, the solution is obtained from any initial guess $u^{(0)}$.

② The smaller $\rho(G)$, the faster the convergence. The average rate of convergence $R_k(G)$ and the asymptotic rate of convergence $R_\infty(G)$ are defined by

$$R_k(G) = -\frac{\ln \|G\|^k}{k} \tag{3}$$

$$R_\infty(G) = -\ln \rho(G). \tag{4}$$

If $\rho(G)$ can be known, $R_\infty(G)$ is obtained from Eq. (4).

③ As $R_k(G)$ is approximated roughly by $R_\infty(G)$ when k is large enough, we can evaluate reduction of the error as

$$\ln\left(\frac{\|e^{(k_1)}\|}{\|e^{(k_2)}\|}\right) = (k_2 - k_1)R_\infty(G) \tag{5}$$

where $e^{(k_1)}$ and $e^{(k_2)}$ are the errors at the k_1-th and k_2-th iteration, respectively. Equation (5) may be arranged as

$$p = \ln\left(\frac{\|e^{(k_1)}\|}{\|e^{(k_2)}\|}\right) \tag{6}$$

This equation means that the iterate is improved by p figures during the interval of $(k_2 - k_1)$.

Let us introduce a model problem in the form of the Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -g \qquad (0 \le x,\, y \le 1) \tag{7}$$

and the Dirichlet conditions

$$u = 0, \qquad (x = 0, 1 \quad 0 \le y \le 1), \qquad (y = 0, 1 \quad 0 \le x \le 1). \tag{8}$$

Equation (7) is approximated by means of the centered spacing to produce the finite difference scheme. This is solved iteratively by the SOR method. This solution is described by

$$a_{i,i}u_i^{(k+1)} = a_{i,i}u_i^{(k)} + \omega\left[-\sum_{j=1}^{i-1} a_{i,j}u_j^{(k+1)} - \sum_{j=i+1}^{n} a_{i,j}u_j^{(k)} + b_i - a_{i,i}u_i^{(k)}\right]. \tag{9}$$

④ SOR method (9) includes the relaxation factor ω which can be adjusted. When ω is optimum, $\rho(L_\omega)$ is the smallest with L_ω being the SOR iterative matrix.

⑤ SOR method is regarded as the process of reducing the residual since the equation $Au = b$ can be written alternatively by

$$u^{(k+1)} = u^{(k)} + D^{-1}\omega(b - A\tilde{u}^{(k)}) \tag{10}$$

where D is a diagonal matrix when matrix A is decomposed into a diagonal, upper and lower triangle matrix. According to the control theory, the transient of variables behaves oscillatory when feedback gain is large. If ω is large, reduction of the error may be dynamically unstable as the iteration proceeds.

⑥ SOR method (9) is rearranged as follows,

$$\begin{aligned} u^{(k+1)} &= (1-\omega)u_i^{(k)} + \omega\tilde{u}_i^{(k+1)}, \\ a_{i,i}\tilde{u}_i^{(k+1)} &= -\sum_{j=1}^{i-1} a_{i,j}u_i^{(k+1)} - \sum_{j=i+1}^{n} a_{i,j}u_j^{(k)} + b_i\,. \end{aligned} \tag{11}$$

This expression means that the iteration matrix becomes diagonally dominant if ω is small enough. That is, convergence becomes faster leading to smaller interaction with the up-dated variables at the neighbour meshes.

⑦ When a Dirichlet problem of the Poisson's equation (7)–(8) is solved by means of SOR method (9) using the 5-point difference scheme, relaxation factor ω_b is given as follows,

$$\omega_b = \frac{2}{1 + \sin \pi\Delta x}\,, \tag{12}$$

to minimize $\rho(L_\omega)$ where Δx is an equally divided mesh. The relation between $\rho(L_\omega)$ and ω_b is given by

$$\rho(L_\omega) = \omega_b - 1\,. \tag{13}$$

In accordance with Eq. (13) we can estimate the spectral radius $\hat{\rho}(L_\omega)$.

2.2. Experience with numerical calculation

A useful idea will be proposed for the design of the controller to accelerate convergence of iterative methods.

⑧ Numerical solution for equation $Au = b$ is improved by iterative operation. It is convenient to represent the converging situation by the error norm $\|e\|_\beta = \|Au^{(k)} - b\|_\beta$, $(\beta = 2, \infty)$. $\|e\|$ and the average convergence rate $R_k(G)$ is the better choice as the inputs to the controller.

⑨ It is common to estimate the controlling performance by the quadratic form of the variables and the manipulates. In this paper, the performance is evaluated by the rate of $d\omega^{(k)}/d(k)$ and $\|e\|$ in addition to the relaxation factor ω.

⑩ Reynolds number and the mesh number are important elements to design controller before starting calculation. That is, we know that if the Reynolds number is large or Δx is small, the iteration number increases. If a value of Δx is given, we can estimate optimum relaxation factor by Eq. (12). Also if iteration number to obtain solution becomes larger, an interval to estimate $R_k(G)$ or to adjust $\omega^{(k)}$ become longer. That is, Reynolds number, the mesh and the sampling interval to estimate the error are utilized for the initial parameter to the controller by the experience.

⑪ The output of the controller is the improved relaxation factor $\omega^{(k+1)}$. We should consider the characteristics of convergence and the terminated condition for this output.

⑫ We know from the point of ⑩ that $\|e\|$ reduces monotonously when ω_0 $(1 \leq \omega < 2)$ is near 1, but the convergence process becomes slow.

⑬ A variable of $\omega^{(k,s)}$ means the relaxation factor at the k-th iteration during the s-th interval. A value of ω_0 stands for $\omega^{(0,0)}$. If the value of $|\omega^{(k+1,s+1)} - \omega^{(k,s)}|$ is large, the feedback becomes large enough to induce an oscillatory transient. Introduction of the restriction may be a solution in accordance with ⑤. This restriction stands for

$$g_1(\omega) \le \omega^{(,s+1)}/\omega^{(,s)} \le g_2(\omega)\,, \tag{14}$$

$$\Delta\omega^{(,s)} = \left|\omega^{(,s+1)} - \omega^{(,s)}\right| < g_3(\omega) \tag{15}$$

where g_1, g_2, g_3 are positive constants. Equations (14)–(15) are shown in Fig. 1. It is meaningless that g_1 is too small because ω is always equal to ω_0 . g_3 is defined by the small limiting value of g_2.

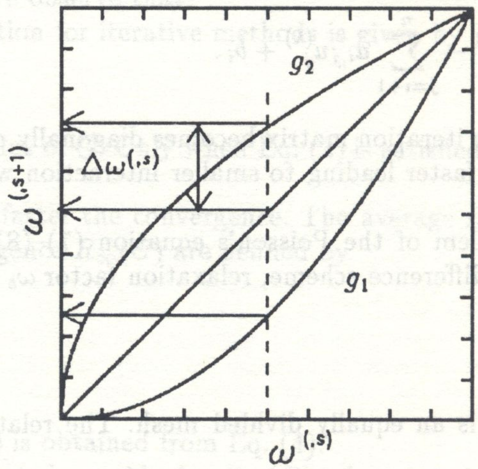

Fig. 1. Restriction for relaxation factor

3. FUZZY CONTROLLER

Feasibility of fuzzy controller for the iterative method will be examined in this section. The present controller has an advantage of making convergence faster in updating the variables $u^{(k)}$ with adjusting the relaxation factor. A benchmark problem is defined by Eqs. (7)–(9).

The converged solution can be obtained after satisfying the reasonable criteria for error norm of improving $u^{(k)}$'s. A decreasing error is estimated by the relations ①, ② and ③. Consider the discrete approximation of the Poisson's equation, Eq. (9) with the Dirichlet condition. The optimum relaxation factor is shown in Eq. (12). Consider the problem of numerically solving the generalized Poisson's equation, which describes a non-linear engineering system. An alternative for adjusting the relaxation factor should be introduced to the nonlinear system. Consider a fully-experienced taxi-driver. His skill may be attributed to his refined expertise. Designing fuzzy controller for efficient iterative method takes an example for the taxi-driver. The controller for relaxation may be derived from our expertise. For instance, the implications may be assembled to adjust a manipulate of the relaxation factor by estimating the output $\|e\|$ of the numerical system; as follows

"IF error norm is large and relaxation factor is small,
THEN relaxation factor shall be controlled as larger than the current value." (16)

"IF error norm is small, average convergence rate is rather large and relaxation factor is large, THEN relaxation factor should keep unchanged." (17)

These points ①, ②, ..., ⑬ will be utilized to device the controlling rules.

3.1. Inputs and output for fuzzy controller

The error norm, the average convergence rate and the present relaxation factor are chosen as inputs for the fuzzy controller according to ⑧ and ⑨, while the improved relaxation factor is chosen as output by ⑪. Such an implication as Eq. (16) is required to serve as the important base of the fuzzy controller.

Basic concept will now be explained to transform the linguistic expression into the fuzzy subset. Consider the error norm $\|e\|$ which is defined by

$$\|e^{(k)}\| = \left\| \frac{u^{(k)} - u^{(k-1)}}{u^{(k)}} \right\|_2, \tag{18}$$

$$\|u\|_2 = (u, u) = \left\{ \sum_i |u_i|^2 \right\}^{\frac{1}{2}}. \tag{19}$$

Computation by 32 bits permits that variables are adequately precise up to the 7 significant figures. This induces the authors to say that the convergence of the k-th iterates is unsatisfactory in the error norm exceeding 10^{-3}. Figure 2 shows the membership functions of "la" (large) and "sm" (small), which describe the fuzzy subset.

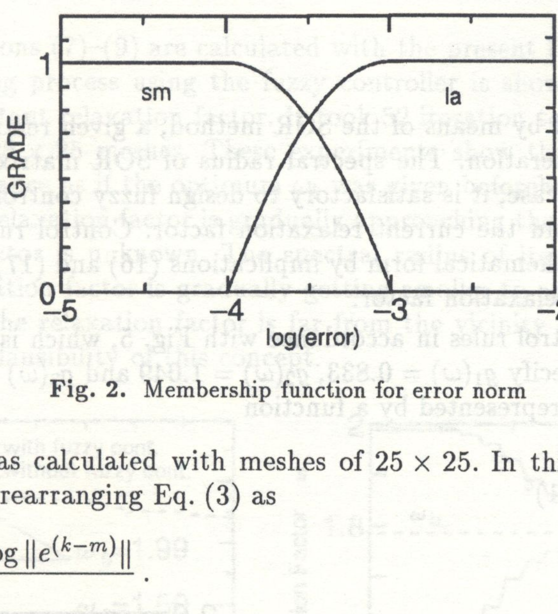

Fig. 2. Membership function for error norm

The model problem was calculated with meshes of 25×25. In this case, the average rate of convergence is defined by rearranging Eq. (3) as

$$R_k(G) = \frac{\log \|e^{(k)}\| - \log \|e^{(k-m)}\|}{m}. \tag{20}$$

The average rate of convergence can be represented in the linguistic form of "sl" (slow), "ms" (medium slow), "mf" (medium fast) and "fa" (fast) in Fig. 3, thus providing for reasonable definition of the fuzzy subsets. Figure 4 illustrates the relaxation factor in terms of the membership functions of "SM" (small), "MS" (medium small), "ML" (medium large), "LA" (large). In this model case, the optimum relaxation factor ω_b is determined to be 1.735 by the Eqs. (4), (12) and (13) according to ⑦. The optimum factor ω_b reduces error norm by 0.44 ($R_k(G) = 0.044$) per 10 iterations so that the solution converged only in 52 iterations. On the contrary, when $\omega_b = 1.2$, the error norm can be improved only by 0.11 ($R_k(G)$) per 10 iterations so that it takes 366 iterations to converge. The latter is about 7 times as many as the former. That is, the case when the average convergence rate is about 0.044 is defined fast in the linguistic form and the average convergence rate of 0.011 is slow. These are shown as fuzzy subsets of "fa" (fast), "sl" (slow) in Fig. 3, respectively, and it is proper to designate "ms" (medium-slow) and "mf" (medium-fast) between them.

The relaxation factor is used to the extent shown in ④. It takes many iterations to converge when $\omega \cong 1$ which is the same as in the Gauss-Seidel method. When $\omega \to 2$, reduction of the

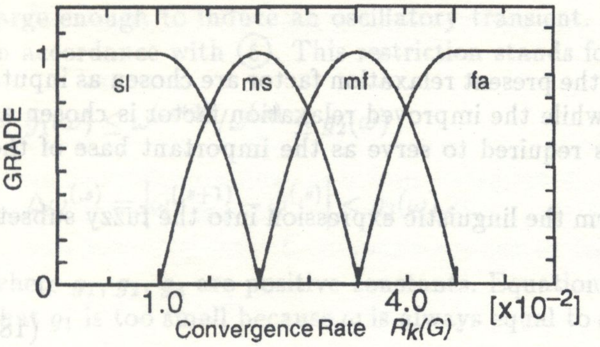

Fig. 3. Membership function for convergence rate

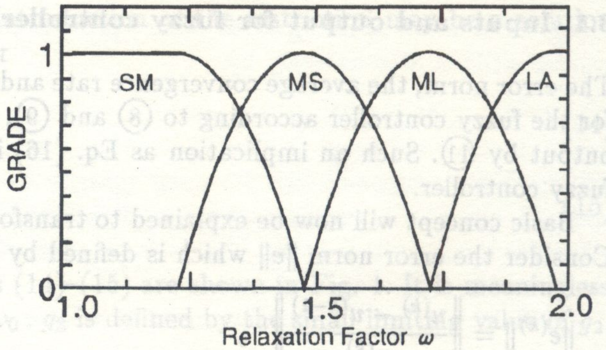

Fig. 4. Membership function for relaxation factor

error behaves oscillatory as shown in ⑤ to result in a large number of iteration as well. The point ⑦ indicates that the relaxation factor which minimizes $\rho(L_\omega)$ becomes close to 2 when the internal difference mesh gets smaller. Different relaxation factors are represented as fuzzy sets of "SM"(small), "MS" (medium small), "ML" (medium-large) and "LA" (large) in Fig. 4 respectively.

3.2. Control rules

When the problem is solved by means of the SOR method, a given relaxation factor may serve as a means to minimize the iteration. The spectral radius of SOR matrix is always less than 1 for the model problem. In this case, it is satisfactory to design fuzzy controller for ω by observing the average convergence rate and the current relaxation factor. Control rules are shown in Table 1. This is described in the mathematical form by implications (16) and (17). That is to say, this table shows how to improve the relaxation factor.

Table 1 displays the control rules in accordance with Fig. 5, which is derived from a schematic presentation of Fig. 1 to specify $g_1(\omega) = 0.833$, $g_2(\omega) = 1.049$ and $g_3(\omega) = 0.074$. The fuzzy sets of VL, L, ..., U in Fig. 5 are represented by a function

$$\omega^{(,s+1)} = 1.00 + (\omega^{(,s)} - 1)^c \tag{21}$$

Table 1. Fuzzy control rules for Poisson's equation

		ω			
		SM	MS	ML	LA
$R_k(G)$	sl	VL	VL	VL	VS
	ms	VL	VL	L	S
	mf	L	L	U	U
	fa	U	U	U	U

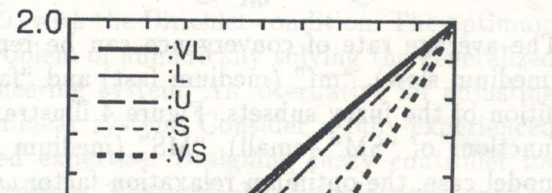

Fig. 5. Membership function for control rules

where

$$\begin{aligned}
\text{VL} = \text{very large} &\longrightarrow c = 0.80,\\
\text{L} = \text{large} &\longrightarrow c = 0.85,\\
\text{S} = \text{small} &\longrightarrow c = 1.50,\\
\text{VS} = \text{very small} &\longrightarrow c = 2.00,\\
\text{U} = \text{untouched} &\longrightarrow c = 1.00.
\end{aligned}$$

Here some examples are introduced as follows:

- implication (16)

  "If the average convergence rate is medium small (ms), and the current relaxation factor is small (SM), then the updated relaxation should be much larger (VL)."

- implication (17)

  "If the average convergence rate is medium fast (mf), and the current relaxation factor is medium large (ML), the updated relaxation factor should be kept unchanged (U)."

3.3. Examples

The model problem equations (7)–(9) are calculated with the present fuzzy controller to verify its plausibility. The converging process using the fuzzy controller is shown in Fig. 6 and compared against that with the constant relaxation factor. It took 52 iteration to converge for the optimum factor of $\omega_b = 1.785$ for 25×25 meshes. These experiments show that the fuzzy controller can produce the same performance as if the optimum ω_b was given beforehand.

Fig. 7 reveals that the relaxation factor is gradually approaching the optimum even if the initial guess of the relaxation factor is unknown. The spectral radius of iterative matrix ρ behaves as shown in Fig 8. The relaxation factor is gradually getting smaller to approach the minimum even when the initial guess of the relaxation factor is far from the vicinity of the optimum. This may clarify the mathematical plausibility of this concept.

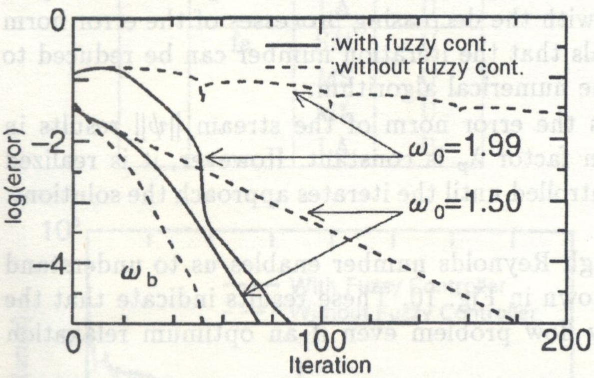

Fig. 6. Convergent process of Poisson equation

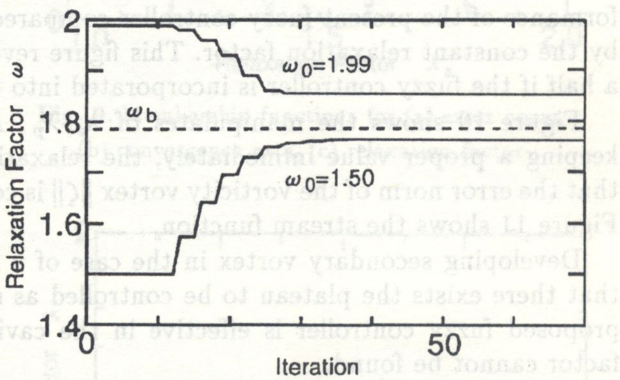

Fig. 7. Control results of relaxation factor

4. TWO-DIMENSIONAL CAVITY FLOW

4.1. Fuzzy controller

Consider a cavity flow which is described by the vorticity and stream function. The basic equations were transformed into the Leonard finite difference scheme and solved iteratively. These nonlinear

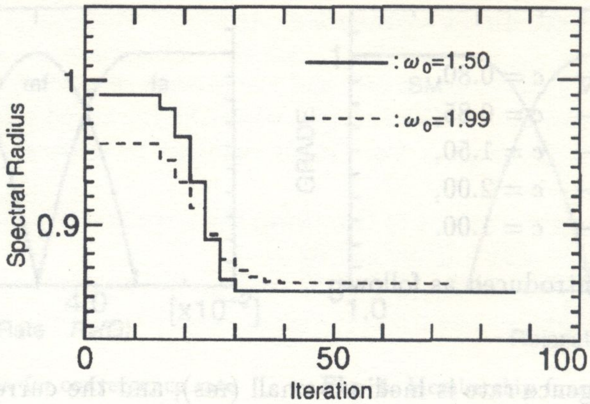

Fig. 8. Control results of spectral radius

equations may not give us the characteristics of their converging state by the iterative method. If the dynamic behavior of the iterates is formulated by the differential equation, the possible manipulation rule for relaxation could be obtained. The cavity flow problem suggests no information about an efficient iterative method, though.

This section attempts to develop the fuzzy controller choosing the cavity flow of $Re = 2000$ with 50×50 meshes. The incorporated rule is given in Table 2. The table says that inputs are the current error norm, the average rate convergence and the relaxation factor $\lambda_z^{(,s)}$, and output is the improved relaxation factor $\lambda_z^{(,s+1)}$ where VL, L, U, S, VS are defined in Eq. (21). These fuzzy subsets are shown in Fig. 9. For example, "If the error norm is large (la), the average rate convergence is medium-slow (ms), and relaxation factor is small (SM), then the relaxation factor should be much larger (VL)."

This implication appears as the fifth element from the top in Table 2.

4.2. Control performance

This section aims at studying the control performance. Figure 10 indicates the good control performance of the present fuzzy controller compared with the decreasing processes of the error norm by the constant relaxation factor. This figure reveals that the iteration number can be reduced to a half if the fuzzy controller is incorporated into the numerical algorithm.

Figure 10 shows the manipulates of λ_z, λ_p. As the error norm of the stream $\|\psi\|$ results in keeping a proper value immediately, the relaxation factor λ_p is constant. However, it is realized that the error norm of the vorticity vortex $\|\zeta\|$ is controlled until the iterates approach the solutions. Figure 11 shows the stream function.

Developing secondary vortex in the case of a high Reynolds number enables us to understand that there exists the plateau to be controlled as shown in Fig. 10. These results indicate that the proposed fuzzy controller is effective in the cavity flow problem even if an optimum relaxation factor cannot be found.

5. CONCLUSION

The present fuzzy controller for iterative methods results in generating a good performance in the two following numerical experiments. A Dirichlet problem of Poisson's equation, whose optimum relaxation factor can be found analytically, was solved by means of the odd-even SOR method. Two-dimensional cavity flow, whose relaxation factor cannot be specified, is solved by the Leonard scheme. Observing a change in the spectral radius of the iteration matrix during the convergent process, we achieved a large feasibility of the fuzzy controller for relaxation.

Table 2. Fuzzy control rules for cavity flow problem

input			output
$\|\|e\|\|$	$R_k(G)$	$\lambda_z^{(,s)}$	$\lambda_z^{(,s+1)}$
la	sl	SM	VL
		MS	VL
		ML	VS
		LA	VS
	ms	SM	VL
		MS	L
		ML	S
		LA	VS
	mf	SM	L
		MS	U
		ML	U
		LA	S
	fa	SM	U
		MS	U
		ML	U
		LA	U
sm	sl	SM	L
		MS	U
		ML	U
		LA	S
	ms	SM	L
		MS	U
		ML	U
		LA	S
	mf	SM	U
		MS	U
		ML	U
		LA	U
	fa	SM	U
		MS	U
		ML	U
		LA	U

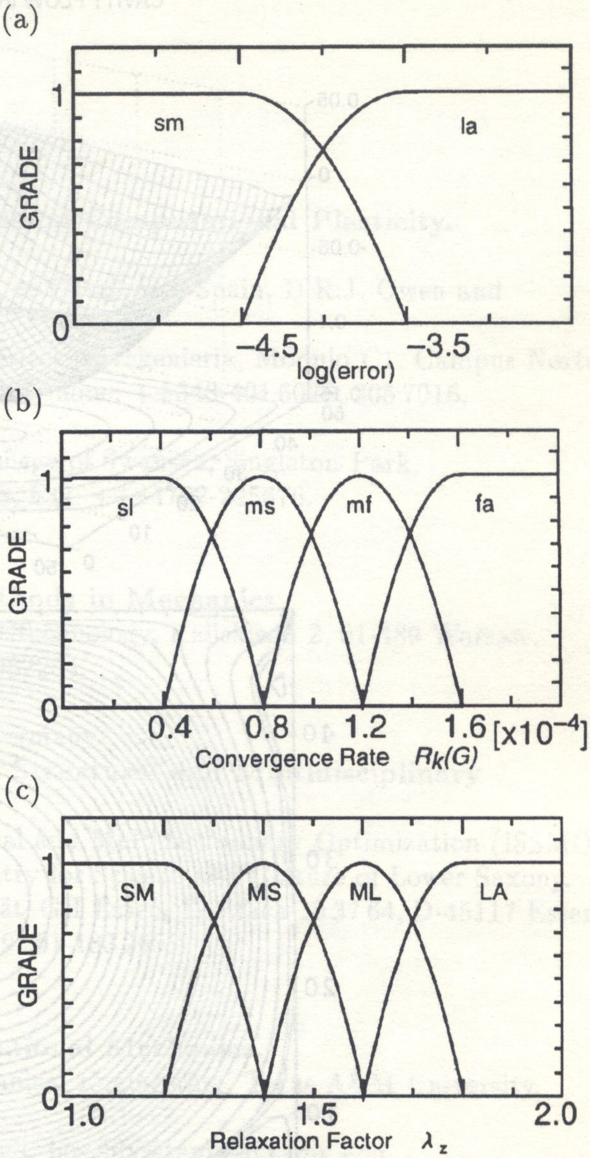

Fig. 9. Membership functions for (a) error norm, (b) convergence rate, (c) relaxation factor

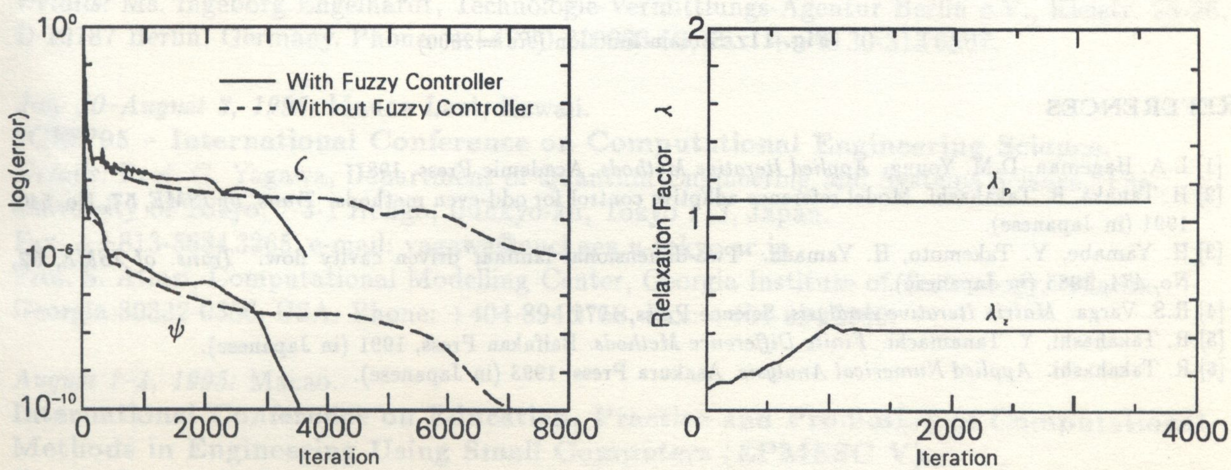

Fig. 10. Left: convergent process of cavity flow problem; right: control rules of relaxation factor

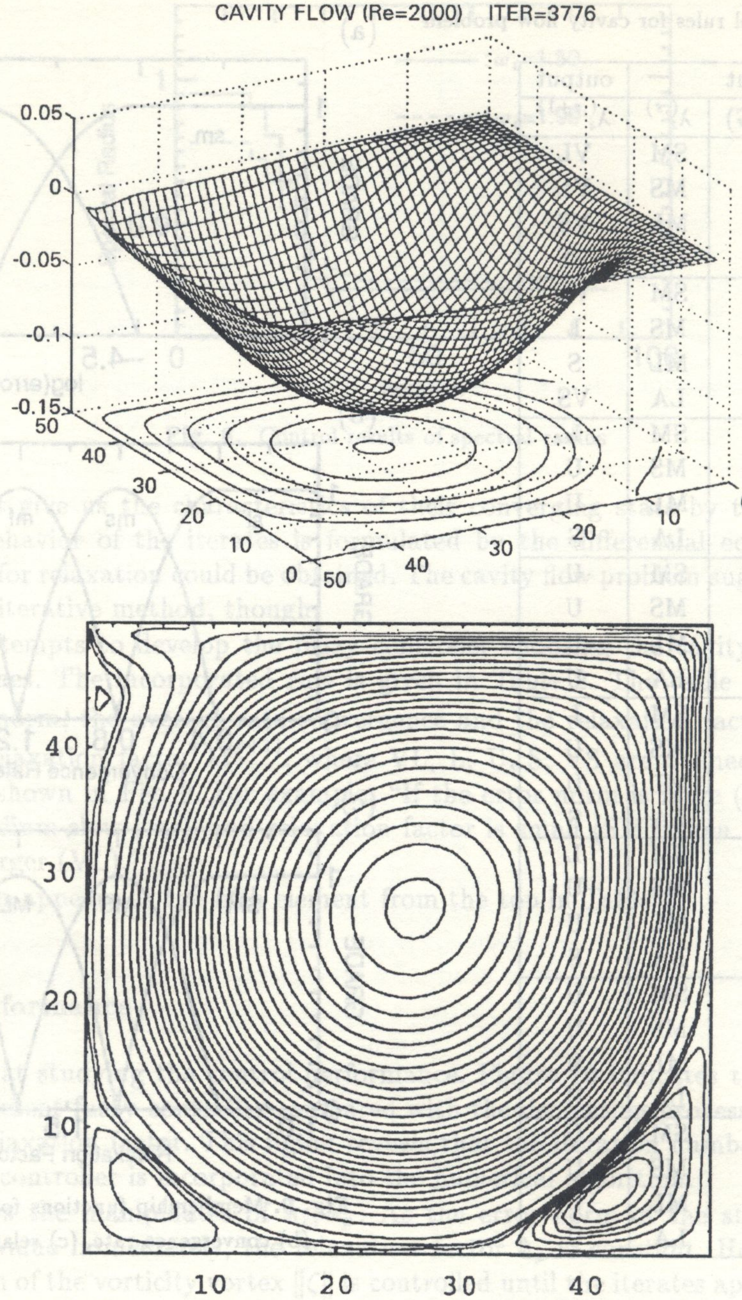

Fig. 11. Stream function (Re = 2000)

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