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Numerical modelling of fibre composites with random-elastic components

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An original idea of the Stochastic Finite Element Method (SFEM) application in numerical modelling of random fluctuations of elastic properties of fibre composites components has been presented in this paper. The displacement and the stress random fields have been analysed for various contents of a fibre periodicity cell of such a composite, and for different coefficients of variation of the Young modulus of both phases.

1. Introduction

An elastic continuum, being for many years the centrefold of the classical theory of elasticity, has usually been treated as homogeneous. From the mathematical point of view this homogeneity is expressed by constant components of the constitutive tensor. Noticing molecular structure of matter, natural macroscopic heterogeneity (porous media), or randomness of material structure (defects and cracks), homogeneity assumption seems to be too idealistic. This occurs mainly in composite materials [12, 20, 25].

An idea of elastic properties randomization appeared in the beginning of the sixties [4, 6]. In the case of composite materials it has been often associated with searching of stochastic effective characteristics [1, 16, 27] or their deterministic [14, 24] and stochastic [2] upper and lower bounds. The applied methods were in that time purely mathematical because of the lack of adequate numerical methods.

Upon development of the Finite Element Method (FEM) the numerical simulation of deterministic behaviour of different kinds of composite structures has become possible. The comprehensive literature on this subject can be found in the reviewing position [10].

The following probabilistic methods have appeared simultaneously with the deterministic one: Monte-Carlo simulation [5, 30], Fisher theory of experiments [8], stratified sampling and Latin hypercube sampling [23]. Apart from the numerical methods connected with FEM, several other have appeared such as for example hierarchical equations of probabilistic moments derived by Dyson and Bethe-Salpeter [29], as well as methods connected with the existence of limit density (in the stochastic sense) of homogenized material [1]. The example of using the Monte-Carlo method for searching effective properties of a fibre composite together with proper numerical procedure has been presented in [16].

All the computer algorithms quoted above have, however, one fundamental defect of not regarding the influence of random elastic behaviour of a certain point belonging to one material on behaviour of another point of the same material. Expressed in a mathematical way, it means that elastic characteristics in different regions of the same material are uncorrelated random variables. This assumption, from the physical point of view, seems to be oversimplified.

Such a dependence is allowed for example by SFEM [15, 21, 23] developed from the late eighties. Its application in the mechanics of composites will continuously increase because of the possibility of modelling such phenomena as random fluctuations of periodicity conditions, random character of fibre-matrix boundary geometry or heterogeneity occurring on this boundary [18]. Due to the possibility of considering the correlation function of a random variable this method has already found its use in modelling of spatial structures undergoing degradation processes [17].

In the present paper, the qualitative and quantitative influence of randomness of composite component material elastic properties (with various fractions of such materials) on probabilistic displacement and stress fields is determined. In the numerical analysis (plane strain problem) a fibre composite under uniform tension in the plane orthogonal to the fibre direction is investigated. As it has been shown by computational tests presented in [26], these composites have the highest sensitivity to changes of elastic properties in this plane.

2. Mathematical model

Let us suppose that $Y \subset \Omega^2$ is a periodic random two-phase linear-elastic composite structure [16], where Ω is a periodicity cell of Y, Ω_1 is a fibre region and Ω_2 is a matrix region (Fig. 1).

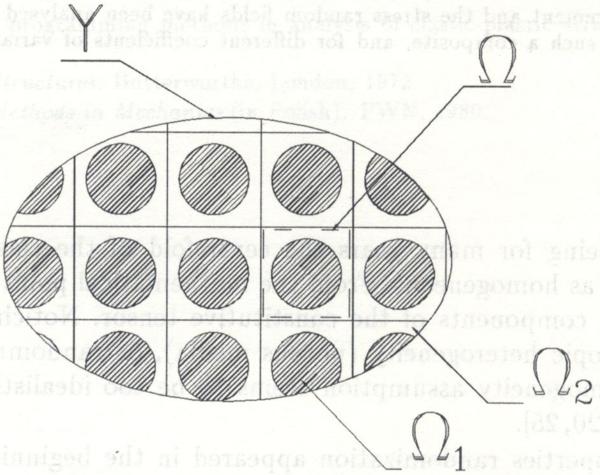

Fig. 1. Periodic composite structure Y

Next, we assume that Ω is a bounded coherent region uniplanar with the $x_3 = 0$ plane and having two perpendicular symmetry axes. Let Ω_1 and Ω_2 be disjoint coherent regions such that $\Omega = \Omega_1 \cup \Omega_2$ and let them contain linear-elastic isotropic homogenous media.

Let the Young modulus $e = e(\mathbf{x})$ be a Gaussian random variable $e(\mathbf{x}) = e(\mathbf{x};\omega)$, where $\mathbf{x} \in \Omega$; $\omega \in S$ and

$$E(e(\mathbf{x})) = \begin{bmatrix} E(e_1); & \mathbf{x} \in \Omega_1 \\ E(e_2); & \mathbf{x} \in \Omega_2 \end{bmatrix} \tag{1}$$

with the covariance matrix

$$\mathrm{Cov}(e_i, e_j) = \begin{bmatrix} \mathrm{Var}(e_1) & 0 \\ 0 & \mathrm{Var}(e_2) \end{bmatrix}, \tag{2}$$

where $\mathrm{Cov}(e_1, e_2) = 0$. This means uncorrelation of functions of random variables of elastic properties in Ω_1 and Ω_2 regions, which seems justified from the physical point of view.

The Poisson modulus is assumed to be a deterministic function so that:

$$\nu(\mathbf{x}) = \begin{bmatrix} \nu_1; & \mathbf{x} \in \Omega_1 \\ \nu_2; & \mathbf{x} \in \Omega_2 \end{bmatrix}. \tag{3}$$

Thus, let us define the random elasticity tensor field $C_{ijkl}(\mathbf{x};\omega)$ as follows:

$$C_{ijkl}(\mathbf{x};\omega) = e(\mathbf{x};\omega)\left[\delta_{ij}\delta_{kl}\frac{\nu(\mathbf{x})}{(1+\nu(\mathbf{x}))(1-2\nu(\mathbf{x}))} + (\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\frac{1}{2(1+\nu(\mathbf{x}))}\right], \tag{4}$$

where $i,j,k,l = 1,2$.

Problem formulation

Find displacement $u_i(\mathbf{x};\omega)$ and stress $\sigma_{ij}(\mathbf{x};\omega)$ random fields fulfilling the following boundary-differentiable equation system:

$$\sigma_{ij}(\mathbf{x};\omega) = C_{ijkl}(\mathbf{x};\omega)\,\varepsilon_{kl}(\mathbf{x};\omega), \tag{5}$$

$$\varepsilon_{ij}(\mathbf{x};\omega) = \frac{1}{2}\left(\frac{\partial u_i(\mathbf{x};\omega)}{\partial x_j} + \frac{\partial u_j(\mathbf{x};\omega)}{\partial x_i}\right), \tag{6}$$

$$\sigma_{ij,j} + \rho f_i = 0, \tag{7}$$

$$u_i(\mathbf{x};\omega) = \hat{u}_i(\mathbf{x};\omega); \qquad \mathbf{x} \in \partial\Omega_{\hat{u}}, \tag{8}$$

$$\sigma_{ij}(\mathbf{x};\omega) = \hat{\sigma}_{ij}(\mathbf{x};\omega); \qquad \mathbf{x} \in \partial\Omega_{\hat{\sigma}}, \tag{9}$$

where ρ and ρf_i are the material density and the body force per unit volume, respectively.

Denoting the random variable of our problem (Young modulus in this case) as a vector $\{b^r(\mathbf{x};\omega)\}$, its probability density as $g(b^r)$ and $g(b^r, b^s)$ respectively, where $r,s = 1,2,\ldots,R$, we can define the expected values of our variable as follows [31]:

$$E[b^r] = \int_{-\infty}^{+\infty} b^r g(b^r)\,\mathrm{d}b^r \tag{10}$$

and covariances

$$\mathrm{Cov}(b^r, b^s) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (b^r - E[b^r])\,(b^s - E[b^s])\,g(b^r, b^s)\,\mathrm{d}b^r\mathrm{d}b^s. \tag{11}$$

Generally, the variational formulation equivalent to the specified above system of equations (5)–(9), obtained from the Hamiltonian theorem, leads to the following algebraic systems of equations [15, 21, 23]

$$\mathbf{K}^0\mathbf{q}^0 = \mathbf{Q}^0, \tag{12}$$

$$\mathbf{K}^0\mathbf{q}^{,r} = \mathbf{Q}^{,r} - \mathbf{K}^{,r}\mathbf{q}^0, \tag{13}$$

$$\mathbf{K}^0\mathbf{q}^{(2)} = \frac{1}{2}\left[\mathbf{Q}^{,rs} - 2\mathbf{K}^{,r}\mathbf{q}^{,s} - \mathbf{K}^{,rs}\mathbf{q}^0\right]\mathrm{Cov}(b^r, b^s), \tag{14}$$

where

$$\mathbf{q}^{(2)} = \frac{1}{2}\mathbf{q}^{,rs}\,\mathrm{Cov}(b^r, b^s) \tag{15}$$

and $1 \le r,s \le E$ (E — finite element number). The zero-th order functions from the FEM equations were denoted by $(\cdot)^0$ and $(\cdot)^{,r}$, $(\cdot)^{,s}$ denote the first and the second partial derivatives with respect to the random variables. $\mathbf{K}$, $\mathbf{q}$, $\mathbf{Q}$ denote global stiffness matrix, displacement and external load vectors, respectively. The stiffness matrix and its derivatives in two-dimensional elasticity problems are defined as follows:

$$K^0_{\alpha\beta} = \sum_{e=1}^{E} \int_{\Omega_e} C^0_{ijkl} B_{ij\alpha} B_{kl\beta}\, d\Omega\,, \tag{16}$$

$$K^{,r}_{\alpha\beta} = \sum_{e=1}^{E} \int_{\Omega_e} C^{,r}_{ijkl} B_{ij\alpha} B_{kl\beta}\, d\Omega\,, \tag{17}$$

$$K^{,rs}_{\alpha\beta} = \sum_{e=1}^{E} \int_{\Omega_e} C^{,rs}_{ijkl} B_{ij\alpha} B_{kl\beta}\, d\Omega\,. \tag{18}$$

Thus, in the algebraic equation systems (12)–(14) there are

- the first partial derivatives of the elasticity tensor

$$C^{,r}_{ijkl}(\mathbf{x};\omega) = \delta_{ij}\delta_{kl}\frac{\nu(\mathbf{x})}{(1+\nu(\mathbf{x}))(1-2\nu(\mathbf{x}))} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\frac{1}{2(1+\nu(\mathbf{x}))}\,, \tag{19}$$

- the second partial derivatives of the elasticity tensor

$$\forall_{r,s=1,\ldots,E} \left(C^{,rs}_{ijkl} = 0 \Leftrightarrow K^{,rs}_{\alpha\beta} = 0\right)\,, \tag{20}$$

- the first and the second partial derivatives of the external load vector

$$\forall_{r,s=1,\ldots,E} (Q^{,r}_{\alpha} = 0 \wedge Q^{,rs}_{\alpha} = 0)\,. \tag{21}$$

Finally we have to solve the following algebraic equation systems

$$\mathbf{K}^0\mathbf{q}^0 = \mathbf{Q}^0\,, \tag{22}$$

$$\mathbf{K}^0\mathbf{q}^{,r} = -\mathbf{K}^{,r}\mathbf{q}^0\,, \tag{23}$$

$$\mathbf{K}^0\mathbf{q}^{(2)} = -\mathbf{K}^{,r}\mathbf{q}^{,s}\,\mathrm{Cov}(b^r, b^s)\,. \tag{24}$$

In the above equations we compute successively $\mathbf{q}^0$ from (22), $\mathbf{q}^{,r}$ — from (23) and $\mathbf{q}^{,rs}$ — from (24) to finally determine the expected values of displacements

$$E[q_\beta] = q^0_\beta + \frac{1}{2}q^{,rs}_\beta\,\mathrm{Cov}(b^r, b^s) = q^0_\beta + q^{(2)}_\beta \tag{25}$$

and their covariances

$$\mathrm{Cov}(q^r_\alpha, q^s_\beta) = q^{,r}_\alpha q^{,s}_\alpha\,\mathrm{Cov}(b^r, b^s)\,. \tag{26}$$

The expected values of the stress tensor in the finite element e are given by

$$E[\sigma^{(e)}_{ij}] = C^{0\,(e)}_{ijkl} B^{(e)}_{kl\alpha} q^0_{\alpha\,(e)} + \frac{1}{2}\left[2C^{,r\,(e)}_{ijkl} q^{,s}_{\alpha\,(e)} + C^{0\,(e)}_{ijkl} q^{,rs}_{\alpha\,(e)}\right] B^{(e)}_{kl\alpha}\,\mathrm{Cov}(b^r, b^s)\,. \tag{27}$$

The assumed mathematical model allows global consideration of material heterogeneity by means of one variance value. In order to include local changes of individual material elastic properties, the covariance matrix should be specified as a local function. The existence of discontinuities or other weaknesses would be equivalent to maximum components of this matrix, and homogenous regions would correspond to minimum components.

3. Numerical analysis and results

The purpose of the numerical analysis was investigation of elastic behaviour of a fibre composite when the Young modulus of composite components is a random variable. Moreover, the numerical simulations were carried out in order to find out, how various contents of fibre (with round section) in a periodicity cell, and random material properties of reinforcement and matrix, influence displacement and stress state in the cell.

A quarter of a fibre composite periodicity cell has been investigated in numerical analysis. Its discretization is shown in Fig. 2. This example has been already analysed, cf. [11, 16, 22] (computations of effective properties in deterministic and probabilistic case).

The numerical implementation allowing the computations has been done using 4-node rectangular plane element of POLSAP system [3, 15] written in FORTRAN 77 (Plane Strain/Stress and Membrane Element). The composite structure was subjected to uniform tension (100 kN/m) on the vertical cell boundary (60 finite elements with 176 degrees of freedom). On the remaining boundaries the vertical displacements were fixed and the analysis of the plain strain element with unit thickness has been ordered. Twelve numerical tests have been performed assuming fibre contents of 30, 40 and 50% and coefficients of variation specified in Table 1, have been calculated from the formula:

$$\alpha[b(\mathbf{x};\omega)] = \sqrt{\frac{\mathrm{Var}[b(\mathbf{x};\omega)]}{E^2[b(\mathbf{x};\omega)]}}\,. \tag{28}$$

Material properties of fibre and matrix were as follows: $E(e_1) = 84.0\,\mathrm{GPa}$, $\nu_1 = 0.22$, $E(e_2) = 4.0\,\mathrm{GPa}$, $\nu_2 = 0.34$.

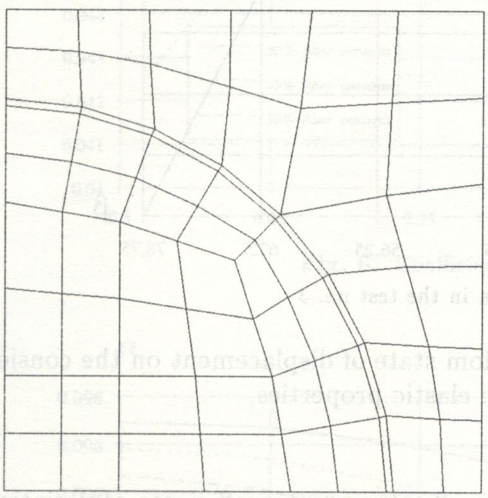

Fig. 2. Discretization of a periodicity cell quarter

Table 1. Coefficients of variation for different numerical tests

test number	$\alpha(e_1)$	$\alpha(e_2)$
1	0.10	0.10
2	0.10	0.05
3	0.05	0.10
4	0.05	0.05

In each case, the first two moments of displacement function were observed on the phase boundary and on the vertical edge subjected to tension. Regarding expected values of stress fields, location and maximum value of reduced stress have been examined.

Figures 3 and 4 show radial displacement coefficients of variation of points located on the fibre-matrix boundary as a function of β angle.

The results of the test no. 1 (Table 1) are presented in Fig. 3, and Fig. 4 shows the results of the test no. 3. Results of the remaining tests (no. 2 and 4) agree with them respectively. In both cases coefficients of variation for $\theta = 90°$ were omitted on the graphs because of their big values. For fibre contents equal to 50% they are approximately 1.5 times bigger than for $\theta = 0°$ (disproportion of data would give illegible picture).

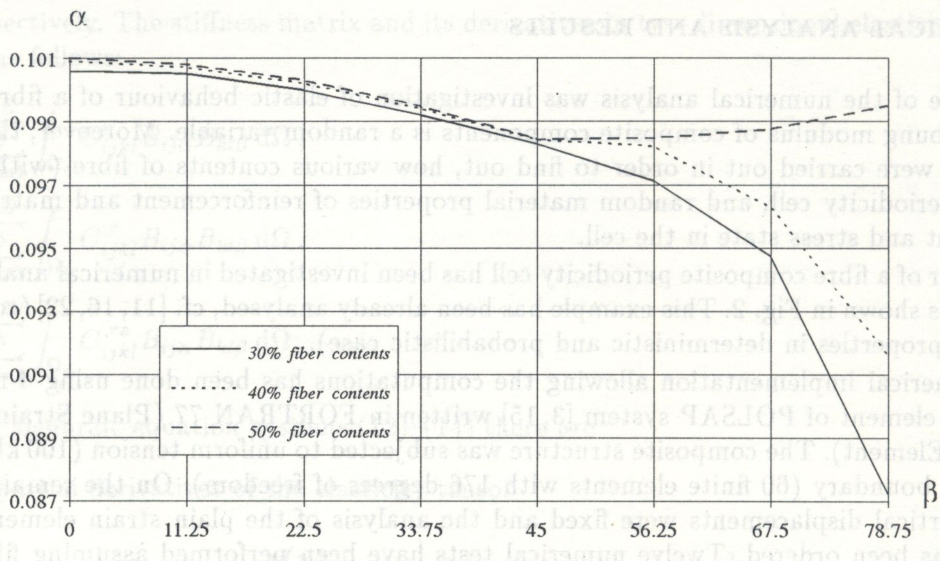

Fig. 3. Coefficients of variation in the test no. 1

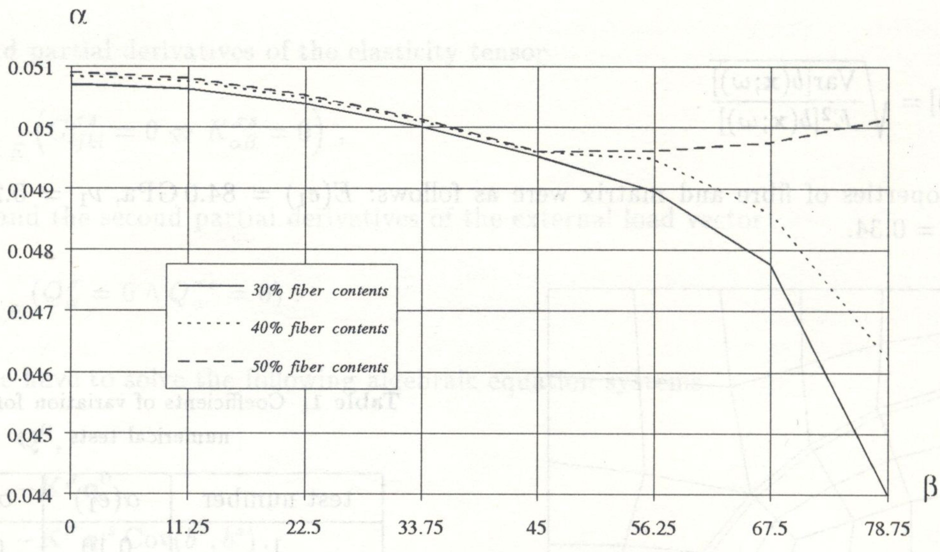

Fig. 4. Coefficients of variation in the test no. 3

On the basis of these figures we may state that random state of displacement on the considered boundary depends mainly on random character of fibre elastic properties,

$$\alpha[u(\mathbf{x})] \cong \alpha[e_1]; \qquad \mathbf{x} \in \partial\Omega_{1,2} . \tag{29}$$

The fibre contents in a periodicity cell influences also displacement coefficients of variation on $\partial\Omega_{1,2}$. This influence becomes evident in the range $0° \leq \theta \leq 45°$.

For the contents of 40% the decrease is not so sharp, and for 50% plane fraction the tendency is opposite: the coefficient increases up to about 1.5 times of the value obtained at $\theta = 0°$. Physically, it may be interpreted as increasing of random measure of uncertainty about displacements perpendicular to fibre boundary of the points belonging to its upper part with increase of this fibre radius.

Figures 5–8 show displacement variation coefficients of horizontal points belonging to the vertical, uniformly tensioned edge of a periodicity cell, obtained in the tests no. 1, 2, 3 and 4, respectively (Table 1). On the horizontal axes of these figures, real numbers in the decreasing order denote height on the vertical tensioned edge.

On the basis of the graphs shown in the above figures we may conclude that random character of matrix elastic properties mainly influences random displacement state of tensioned edge of the

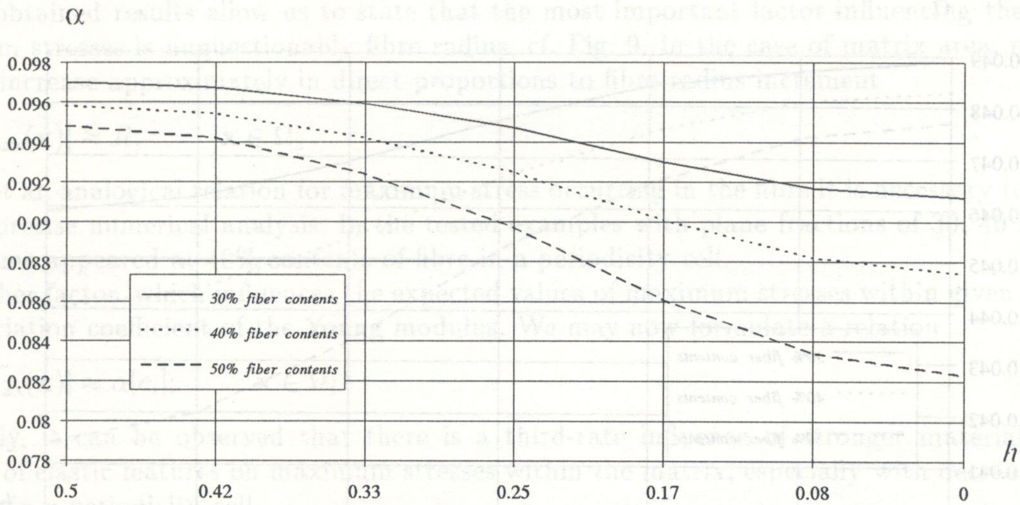

Fig. 5. Coefficients of variation in the test no. 1

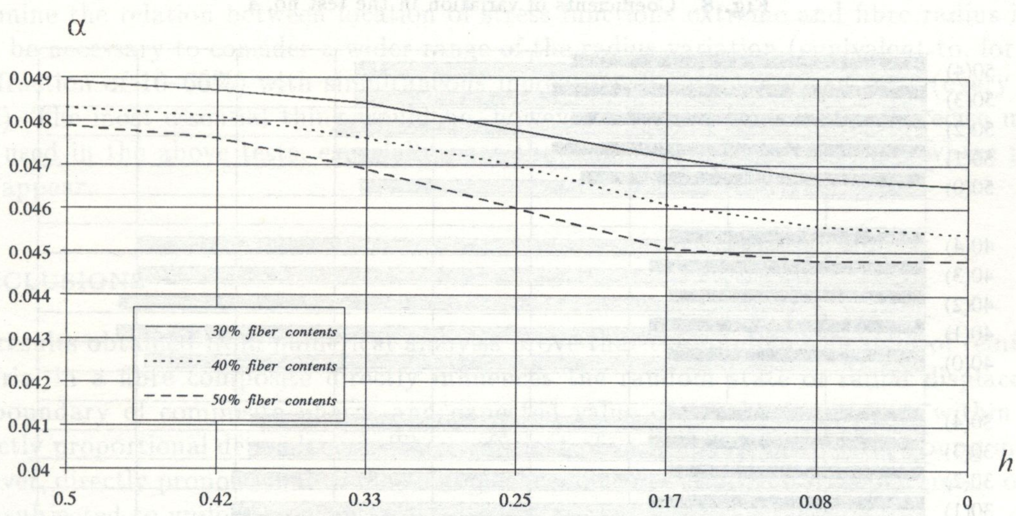

Fig. 6. Coefficients of variation in the test no. 2

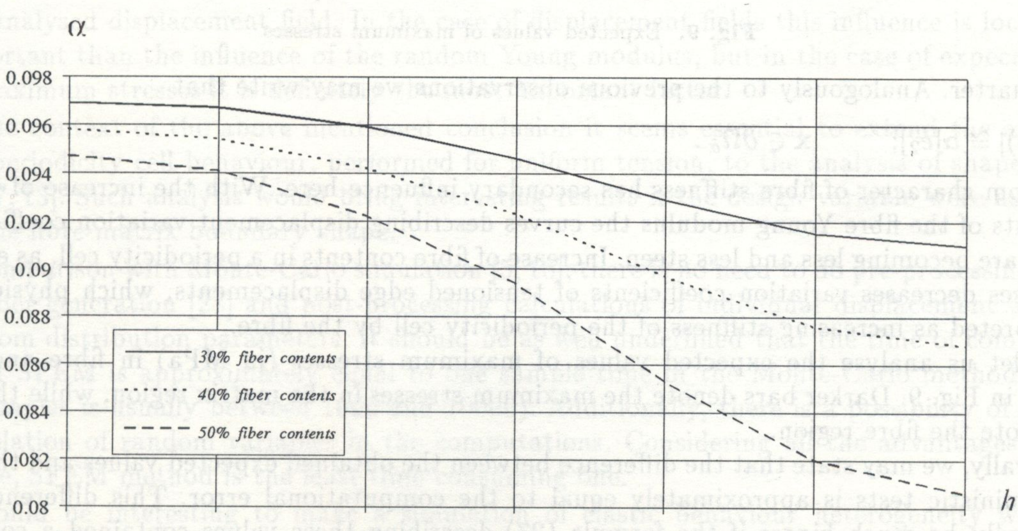

Fig. 7. Coefficients of variation in the test no. 3

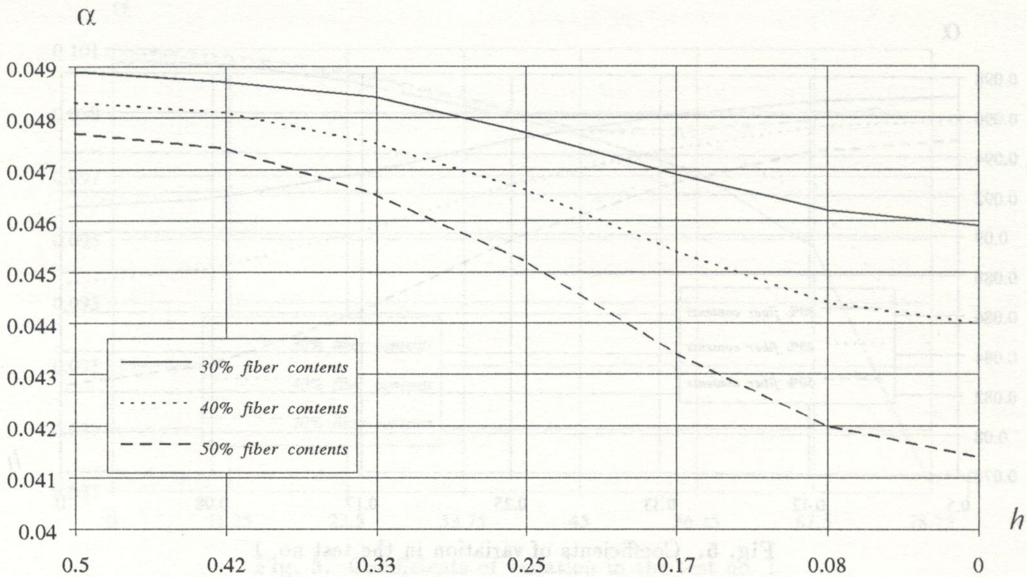

Fig. 8. Coefficients of variation in the test no. 4

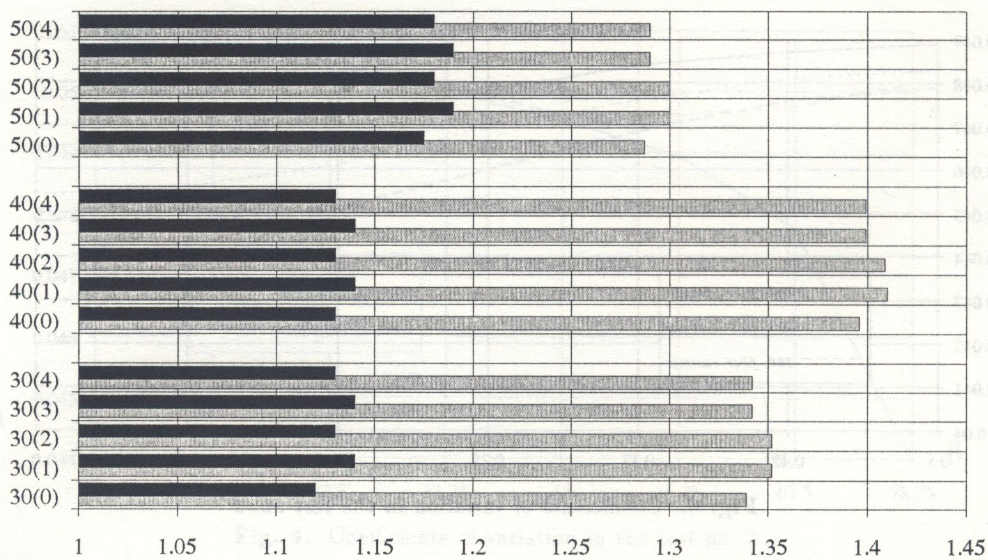

Fig. 9. Expected values of maximum stresses

tested quarter. Analogously to the previous observations we may write that

$$\alpha[u(\mathbf{x})] \cong \alpha[e_2]; \qquad \mathbf{x} \in \partial\Omega_{\hat{\sigma}}\,. \tag{30}$$

Random character of fibre stiffness has secondary influence here. With the increase of variation coefficients of the fibre Young modulus the curves describing displacement variation coefficients on the edge are becoming less and less steep. Increase of fibre contents in a periodicity cell, as expected, in all cases decreases variation coefficients of tensioned edge displacements, which physically can be interpreted as increasing stiffness of the periodicity cell by the fibre.

Now let us analyse the expected values of maximum stresses (in MPa) in fibre and matrix specified in Fig. 9. Darker bars denote the maximum stresses in the matrix region, while the lighter ones denote the fibre region.

Generally, we may state that the difference between the obtained expected values and the results of deterministic tests is approximately equal to the computational error. This difference would undoubtedly be much bigger if the formula (27) describing these values contained a component connected with elasticity tensor derivatives. The present version of the program includes only the first two components, which correspond with the expected values of displacement functions.

The obtained results allow us to state that the most important factor influencing the value of maximum stresses is unquestionably fibre radius, cf. Fig. 9. In the case of matrix area, maximum stresses increase approximately in direct proportions to fibre radius increment

$$E[\sigma_{\max}(\mathbf{x})] \approx R; \qquad \mathbf{x} \in \Omega_2 . \tag{31}$$

To get an analogical relation for maximum stress occurring in the fibre it is necessary to perform a more precise numerical analysis. In the tested examples with plane fractions of 30, 40 and 50%, an extreme appeared at 40% contents of fibre in a periodicity cell.

Another factor, which influences the expected values of maximum stresses within given material, is its variation coefficient of the Young modulus. We may now formulate a relation

$$E[\sigma_{\max}(\mathbf{x})] \approx \alpha[e_i]; \qquad \mathbf{x} \in \Omega_i . \tag{32}$$

Finally, it can be observed that there is a third-rate influence of stronger material random changes of elastic features on maximum stresses within the matrix, especially with decreasing fibre contents in a periodicity cell.

In the context of present numerical analysis of maximum stresses it should be added that, apart from changes in expected values of these stresses, a change of their locations was observed. In order to determine the relation between location of stress functions extreme and fibre radius increment it would be necessary to consider a wider range of the radius variation (equivalent to, for example, surface fraction of 10–60%) with simultaneous increasing of the number of tests (every 1–5% for example). The most essential thing would be, however, creating a much more precise mesh than the one used in the above tests, especially near the composite phase boundary, where maximum stresses appear.

4. Conclusions

1. The results obtained from numerical analysis prove that the random character of reinforcement material in a fibre composite directly influences the random state of radial displacements of the boundary of composite phases and expected value of maximum stresses within the fibre (directly proportional dependence). The coefficient of variation of the matrix Young modulus is, however, directly proportional to the analogical coefficient of horizontal displacement on vertical edge subjected to uniform tension.
2. It appeared that a very important factor in the presented tests is fibre radius parameter, which influences both changes of expected values of maximum stresses within fibre and matrix, and the analysed displacement field. In the case of displacement fields this influence is locally more important than the influence of the random Young modulus, but in the case of expected values of maximum stresses it is definitely the most important factor.
3. In the context of the above mentioned conclusion it seems essential to extend the analysis of the periodicity cell behaviour, performed for uniform tension, to the analysis of shape sensitivity [7, 13]. Such analysis would bring interesting results if the design variable were assumed to be the fibre-matrix boundary shape.
4. In comparison with Monte-Carlo simulation [5, 16], there is no need to do pre-processing random number generation [28] and post-processing calculations of individual displacement and stress random distribution parameters. It should be as well underlined that the time of computations using SFEM is approximately equal to one sample time in the Monte-Carlo method (number of samples is usually between 1000 and 10000). Additionally, there is a possibility of including correlation of random variables in the computations. Considering all the advantages specified above, SFEM method is the least time consuming one.
5. It would be interesting to make a simulation of elastic behaviour heterogeneity within one material (cracks or inclusions). Such simulation is possible when using for this purpose the presented numerical method and defining properly the covariance matrix [15]. Using for this purpose the deterministic FEM formulation seems to be more complicated.

6. From a numerical point of view it would be interesting to extend the presented stochastic model to the stochastic sensitivity problem, where the Young modulus would be treated as a design variable [15, 21]. It would enable, for example, to analyse fluctuations of periodicity conditions on the boundaries of an analysed cell.

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