Computer Assisted Mechanics and Engineering Sciences, 2: 231–248, 1995.

On avoiding of spurious mesh sensitivity in numerical analysis of plastic strain localization

Tomasz Łodygowski
Institute of Structural Engineering, Poznań University of Technology
ul. Piotrowo 5, 60-965 Poznań, Poland

(Received October 24, 1994)

The successful numerical analysis of plastic strain localization phenomena in ductile and brittle materials requires the fulfillment of several conditions in accordance with the formulation of the initial boundary value problem (IBVP). Sometimes it is necessary to use the regularization techniques which result in well-posedness of IBVPs. Then, there are several possibilities of introducing an internal length scale. If specific conditions are met, the results obtained in the numerical calculations are free of unexpected mesh sensitivity. In the paper the dynamical boundary values are studied. The rate-dependent regularized models of two materials are presented and used to solve practical engineering problems. The mathematical background which could be used to prove the well-posedness of IBVP as well as the physical arguments are discussed.

1. Introduction

The localization of deformations is a physical phenomenon observed for a wide range of materials such as ductile metals or polymers and brittle concrete, rock and soils. The deformations in specimens are localized into relatively narrow zones of intense straining. The results of numerous experiments expressed in the load-displacement space exhibit descending branches after the peak loads have been obtained. Some engineering materials, including metals, polymers, soils, concrete and rocks, are classified as so-called softening materials. These materials show the reduction of the load-carrying capacity together with the increasing localized deformations after the limit load has been reached.

The widths and the directions of localized zones, however, are not known a priori; they depend not only on material parameters but also upon the shape of the specimen and the initial and boundary conditions. In the phenomenological approach, which is used in the present paper and which is always used in continuum mechanics, a simple mapping of such experimental data into stress-strain relations provides a negative stiffness in the constitutive model. This fact can create considerable mathematical difficulties which, as a consequence, can result in unusual mesh sensitivity in the numerical calculations. There is no agreement on the question, whether strain softening is a true material property and can be used in stress-strain relations. In the FEM formulation, softening is attributed to the material and is incorporated into the constitutive equations. However, on the other hand, we know that the descending branch in the stress-deformation curve is the result of non-homogeneous deformation on the micro level which in fact is not visible from the level of continuum mechanics description. Therefore, special modelling techniques should be used to derive the proper material models with reliable parameters for strain softening.

For isothermal processes and rate-independent materials, this phenomenon has been analyzed as a material instability (see e.g. Rudnicki and Rice [43]). This kind of treatment, which does not include length-scale parameters, practically precludes post-localization analysis. Usually, the analysis of eigenvalues and eigenvectors of the acoustic tensor leads only to the formulation of localization criteria (the locations and directions of singular localization lines). This formulation also simplifies the observed physical fact that in any experiment the width of the intensive straining zone is always finite even for very fast processes (see e.g. Marchand and Duffy [27]).

In such a description when the critical point has been reached, the problem becomes ill-posed which prevents reliable analysis and as a result — reliable (useful) computations (see e.g. Perzyna [37] and Kibler et al. [18]). When post-critical analysis is to be continued after the plastic localization appears, the analysis of well-posedness of the initial boundary value problem (IBVP) plays a crucial role [18]. Then, if the conditions for well-posedness are fulfilled, one can continue the numerical calculations while avoiding the well-known problem of observed strong mesh sensitivity which is the result of changing the type of the operator of the governing system of incremental equations.

In order to regularize the system of partial differential equations (PDE) which govern the incremental process in post-localization states, one has to introduce a length-scale parameter a priori leading to the specification of the width of the localization. A comprehensive study has been presented by Sluys [44]. The length-scale parameter can be introduced into the formulation in different ways to obtain modelling of the softening continuum; for example, by using non-local theories (Bazant, Pan and Pijaudier-Cabot [3]), by the addition of higher-order (second order) strain terms (Belytschko [4]), by introducing a gradient model (de Borst [7, 8]), by the inclusion of micro-polar effects (de Borst [10], Steinmann and Willam [46]) or by using rate-dependent formulation (Perzyna [37], Needleman [29], Lengnick et al. [21], Łodygowski et al. [24]).

Finally, regularization is also possible by introducing the width of the localization zone (length-scale parameter) explicitly into the formulation on the level of approximation (finite elements) as was shown by Pietruszczak and Mróz [41], Belytschko et al. [5], Ortiz et al. [31] or Łodygowski [23]. But in this case, in view of the experimental evidence which confirms that the width of localization strongly depends on initial and boundary conditions, one can expect serious difficulties with its explicit definition.

In our formulation we concentrate on rate-dependent, namely viscoplastic (Perzyna's type [34]) formulation which allows to carry out a successful analysis of post-critical plastic states and introduces the length-scale parameter via the viscosity mechanical relaxation time and the speed of elastic wave propagation.

The only drawback of the rate-dependent formulation seems to be the necessity of full dynamic analysis of the process under consideration. In the author's opinion, the rate-dependent model is physically well-founded for both ductile metals and soil like materials (Tavenas and coworkers [22, 48, 47]), especially those which are subject to dynamic load conditions, and thus has a variety of advantages in comparison with the other models. The use of such a property of the material like viscosity, particularly for fast mechanical processes, has a deep physical background and can be explained on the level of micro-mechanical experiments. For these reasons we will use viscoplasticity to describe the physical phenomenon and also as a tool of mathematical regularization.

In view of the achievements [21, 24, 18] in viscoplastic (rate-dependent) formulation, the problem remains well-posed at each interval of time, so the unique solution in numerical calculations can be obtained. Using a dynamic formulation for two-dimensional cases, both failure modes (I-mode and II-mode) can be performed and also because of the wave propagation phenomenon no artificial imperfections are necessary in computations to be superposed to activate the process of localization.

The aim of the presentation is to show the reduction of unexpected mesh dependence and also to stress the necessity of deeper understanding of the mathematical consequences that can have an effect on ill-posedness, particularly when one solves problems which do not satisfy Drucker's postulate, uses nonassociative plasticity or unilateral constraints.

2. ASSUMPTIONS

We will examine the IBVP in the entire range of its behaviour (also in post-critical states that accompany the plastic strain localization). In this case it is necessary to keep the governing operator unchanged during the entire process. In the analysis of wave propagation and in the investigation of instability phenomena for inelastic flow processes, a fundamental role is played by the so-called acoustic tensors. If all three eigenvalues of an acoustic tensor **Q** are real and positive, the set of equations is hyperbolic. When one neglects the inertia terms one uses the elliptic term to characterize the situation. The term used here is hyperbolic with respect to space-time variables and elliptic with respect to space variables only.

If zero is an eigenvalue of the acoustic tensor **Q**, then the discontinuity cannot propagate (wave velocity $c = 0$) and this is the case of stationary discontinuity. In the quasi-static case such a requirement is used as the strain localization condition (loss of ellipticity). The analysis of the acoustic tensors is crucial in assessing the well-posed IBVPs, but according to [18] it does not create sufficient condition.

The viscoplastic formulation, which is the focus of our consideration, guarantees the invariability of the governing operator [38], also in post-critical states, satisfying the necessary conditions of well-posedness of the formulation. The length-scale parameter introduced by viscoplasticity can be measured by the speed of elastic wave propagation $c = \sqrt{E/\varrho}$ where E is the elasticity modulus, ϱ is the density and by the mechanical relaxation time T_m. The other important reason for using this method of regularization of plastic flow processes comes from experimental evidence. In this, we obtain the confirmation that all the processes are extended in time and the localization zone never appears as a mathematically singular line.

In this presentation we restrict our attention to dynamical processes for a brittle constitutive model and for a ductile material model. The temperature effects are neglected in the formulations and we restrict the attention to the isothermal processes.

3. Motivation

The width and the directions of localized zones depend not only on material parameters but also upon shape of the specimen and the initial and boundary conditions. Two important qualitative results should here also be stressed, that are:

- the width of localization zone, though in some experiments very narrow, has always finite dimensions,
- the width and directions of the zones of concentrated strains significantly depend on the type of loading and its velocity. The localization patterns are usually completely different for quasi-static and for dynamic type of loadings.

In order to present the motivation based on the experimentally observed physical facts and to define the background of this presentation, several levels of the treatment analysis of the localization are presented in Box.

To complete the figure how to treat the description of the localization phenomenon one has to add to this Box the level of numerical approximation on which we have chosen the finite element method and the level of comparison of the numerical results with experiments. If the numerical results are satisfactorily close to the experimental observations, one can stop the modelling process, otherwise the iteration starts on the level of mathematical description of the observed phenomena.

In the bold letters we have indicated the way which is adopted here and the approach the description of localization phenomenon. If we concentrate on the continuum mechanics basis, one can notice that the phenomenon of localization that starts from the micro level is not directly observable from this level of the description. It seems to be natural that we have to introduce the additional information that allows us to model more precisely the nucleation and growing of micro-defects which are macroscopically observable as a localization. Let us only stress that the thorough discussion on the level of mathematical formulation before we start any numerical approximation and computations in author's opinion is necessary. In this discussion one can predict

Box

Experimental Results
Mathematical Modelling of the Observed Physical Phenomena
Statistical Mechanics
Micro-level Description
Continuum Mechanics Description
Continuum Damage Mechanics
Fracture Mechanics
Plasticity with Softening
Smeared Crack Models
Non-local Theories
Higher Gradients Models
Embedded Models
Cosserat Models
Rate Dependent Models
others

some properties of the solutions and first of all answer the fundamental questions on their existence, uniqueness and stability.

The viscoplastic properties of ductile metals are well known and the model is well documented see e.g. the books of Lubliner [26], Perzyna [34, 36] or Cristescu and Suliciu [11]. For soil/rock like materials the reader can find the model parameters which describe their viscoplastic properties in the works by Adachi et al. [1, 2], Tavenas et al. [48, 47] and by Wood [51].

4. Mathematical background

The class of problems under consideration can be formulated on two levels: the so-called abstract Cauchy problem (ACP) and the problem including boundary conditions as an initial boundary value problem (IBVP). The main goal of this discussion is to show the way of proving the well-posedness of the IBVP (by this we understand the existence, uniqueness and stability of the solutions). In the author's opinion, particularly when the softening is taken into account, a thorough discussion on the level of mathematical (continuous) formulation is helpful and necessary before starting any computations. Then, on the level of numerical treatment, one can avoid the unexpected strong mesh sensitivity which is the result of ill-posedness and which is called primary mesh dependence (PMD) [21, 24, 18]. The important is that for this type of mesh dependence it is not possible to bound the errors, so the results obtained for this type of formulations are completely meaningless.

4.1. Abstract Cauchy problem

The inhomogeneous ACP can be formulated in the following form,

$$\begin{aligned} &\dot{\varphi}(t,\mathbf{x}) = \boldsymbol{\mathcal{A}}(t,\mathbf{x})\cdot\varphi(t,\mathbf{x}) + \mathbf{f}(t,\mathbf{x},\varphi) \qquad \text{for } t\in(\,0,T\,] \text{ and } \mathbf{x}\in\Omega \\ &\text{with the initial condition:} \quad \varphi(0,\mathbf{x}) = \varphi_0\,, \end{aligned} \tag{1}$$

where in our case $\varphi = (\mathbf{v},\mathbf{T},\mu)$. In the above, t represents time, $\mathbf{x}$ is the spatial variable that belongs to the domain Ω, by $\mathbf{v}$ we denote the velocity vector, $\mathbf{T}$ is the tensor of Cauchy stresses and μ is the only scalar internal state variable (e.g. relative density for brittle materials or isotropic hardening/softening yield limit for ductile materials). The existence, uniqueness and stability of ACP were studied by Pao [32], Pao and Vogt [33], Hughes et al. [14] and Perzyna [37, 38] in terms of nonlinear contraction and nonlinear negative contraction of semi-groups. The use of the method of semi-groups has the important advantage that it is possible to discuss the properties of the solution without the necessity of constructing it. If the operator $\mathbf{A} \otimes \mathbf{f}$ is the infinitesimal generator of a nonlinear negative contraction of a nonlinear semi-group $\{\mathbf{T}_t; t \geq 0\}$, then the existence, uniqueness and stability of the solution of ACP starting at $t = 0$ from φ_0 is ensured by the properties of semi-group.

The elastodynamics as a hyperbolic system, which is fundamental in the discussion of Cauchy problem, was also studied by Hughes and Marsden [15]. The solution of the IBVP,

$$\begin{aligned} &\varphi(t, \mathbf{x}) = \mathbf{T}_t\, \varphi_0(0, \mathbf{x}) \qquad \text{for } t \in (0, T] \text{ and } \mathbf{x} \in \Omega \\ &\text{with the initial condition:} \quad \varphi(0, \mathbf{x}) = \varphi_0\,, \end{aligned} \tag{2}$$

exists and the operator $\mathcal{A}$ is called well-posed if the domain of $\mathcal{A}$ operators is dense and its family is uniformly bounded; see [18].

4.2. Finite element approximation

In view of remarks of Richtmyer and Morton [42], the approximations obtained from FE equations can be thought of as represented by points in the solutions space. We have the sequence of points $\varphi^0, \varphi^1, \varphi^1, \ldots \varphi^n$ of which φ^n is supposed to approximate $\varphi(n\Delta t)$ where Δt is a small increment.

Based on the so called two-level formula [42], one can write the sequence of numerical results which are obtained for the consecutive time instants,

$$\varphi^{n+1} = \mathcal{C}(\Delta t)\varphi^n, \tag{3}$$

where the operator $\mathcal{C}$ depends on the series of finite increments of variables and the space variables themselves. Since $\frac{\varphi^{n+1} - \varphi^n}{\Delta t}$ is an approximation to the time derivative, the ratio $\frac{\mathcal{C}(\Delta t)\varphi - \varphi}{\Delta t}$ is an approximation to $\mathcal{A}\varphi$. One can say that the family $\mathcal{C}(\Delta t)$ of operators provides a consistent approximation of IVP if, for every $\varphi(t)$ in a class of admissible solutions with initial elements $\varphi_0(0)$, the operator is dense in the solutions space. It means that the difference between analytical and numerical solution tends to 0 if $\Delta t \to 0$ for $0 \leq t \leq T$. This also gives a chance to estimate the truncation error.

Now let us remind the Lax equivalence theorem, originally proposed for finite difference approximation which will be used as a basis in our further FE development.

Lax Theorem [42]

GIVEN A WELL POSED IVP AND ITS FE NUMERICAL APPROXIMATION THAT SATISFIES THE CONSISTENCY CONDITION, STABILITY IS THE NECESSARY AND SUFFICIENT CONDITION FOR CONVERGENCE.

The proof and detailed mathematical discussion can be found in [42].

4.3. Initial boundary value problem

In general, for arbitrary initial and boundary conditions, it is not possible to prove the stability of the solution without the necessity of constructing it. The perturbation method could be used [16], but unfortunately it can only give satisfactory results for a very restrictive class of IBVP (e.g. one-dimensional).

In order to transfer, at least partially, the proof of well-posedness on the level of numerical computations let us formulate the following corollary [18]:

Corollary

A WELL-POSEDNESS OF THE IBVP EXPRESSED IN A STRONG FORM ON THE LEVEL OF CONTINUOUS DESCRIPTION IS A NECESSARY AND SUFFICIENT CONDITION FOR THE WELL-POSEDNESS OF THE APPROPRIATE ALGEBRAIC PROBLEM. THE STABILITY OF THE NUMERICAL SOLUTION IS GUARANTEED BY ITS CONVERGENCE IN THE SENSE OF A DISCRETIZATION MEASURE.

Based on the above corollary, we will transfer the last part of the proof of well-posedness on the level of numerical treatment. If, for the different meshes introduced in computations, one can obtain the results convergent to each other in the whole range of the process, then the solution can be called stable. The well-posedness of the original IBVP is finally proved. This method was successfully applied in [18]. If one does not discuss the well-posedness on the level of mathematical (continuous) description and directly starts numerical calculations, ill-posed ACPs and as a consequence IBVPs can exhibit an unexpected mesh dependence PMD [21, 24, 18]. This spurious mesh dependence can be avoided by using a regularization technique (e.g. by introducing rate-dependence) on the level of formulation of the mathematical model and in consequence by recovery of well-posedness of an IBVP. As a result in numerical calculations we obtain the typical mesh dependence which comes from space discretization and for which the numerical errors can be estimated.

4.4. Ductile material model

Assuming after Lee [20] the multiplicativity of the deformation gradient in the form $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$, we conclude that the total deformation rate $\mathbf{D}$ is simply a sum of its elastic $\mathbf{D}^e$ and plastic $\mathbf{D}^p$ parts. In Gurtin's notation [13], the evolution of the Cauchy stress tensor can be written in the form

$$\overset{\triangledown}{\mathbf{T}} = \mathcal{C}^e[\mathbf{D} - \mathbf{D}^p]\,, \tag{4}$$

where $\overset{\triangledown}{\mathbf{T}}$ is Jaumann rate of Cauchy stress and

$$\mathcal{C}^e = 2G\mathbf{I} + \left(K - \frac{2}{3}G\right)\mathbf{1}\otimes\mathbf{1}$$

is an elastic isotropic modulus. In the last formula G and K are the known shear and bulk moduli, respectively, while $\mathbf{I}$ and $\mathbf{1}$ denote fourth rank and second rank unit tensors, respectively.

The fundamental definition of inelastic part of deformation rate, takes usually the form

$$\mathbf{D}^p = F(\mathbf{S}, \Theta, \boldsymbol{\mu}), \tag{5}$$

where $\mathbf{S}$ is the deviatoric stress tensor, Θ is the temperature and $\boldsymbol{\mu}$ represents the vector of internal state variables which can consist of scalar, vector or tensor components. The variety of specific models can be analysed and/or constructed using the following generic set of relations:

- a flow rule of tensorial character,
- so called evolution equations describing the evolution of internal variables $\boldsymbol{\mu}$,
- a kinetic equation of a scalar type that relates stresses, inelastic parts of strain rate and temperatures.

We will restrict our attention to the isothermal processes and assume the *flow rule* in the form

$$\mathbf{D}^p = \Lambda\tilde{\mathbf{n}}\,, \tag{6}$$

where Λ denotes a scalar valued function and $\tilde{\mathbf{n}}$ represents a tensor of second rank, respectively. If we assume the flow rule in the form (6) and the definition of scalar function Λ as follows,

$$\Lambda = \begin{cases} \dot{\lambda} & : \quad \text{if} \quad \phi = 0 \quad \text{and} \quad \tilde{\mathbf{n}} : \mathcal{C} : \bar{\mathbf{D}} > 0\,, \\ 0 & : \quad \text{if} \quad \phi \le 0 \quad \text{or} \quad \phi = 0 \quad \text{and} \quad \tilde{\mathbf{n}} : \mathcal{C} : \bar{\mathbf{D}} \le 0\,, \end{cases} \tag{7}$$

where ϕ represents the yield condition, we arrive at the definition of rate-independent plasticity. Accepting $\tilde{\mathbf{n}} = \frac{\partial g}{\partial \mathbf{T}}$ we speak about the associative plasticity if $g = f$ or about nonassociative plasticity if $g \neq f$.

The parameter $\dot{\lambda}$ is derived from the consistency conditions. If we assume that Λ is an isothermal function of the type $\Lambda(\tilde{S}, \boldsymbol{\mu})$, we are starting to declare the rate-dependent plastic flow law which has to be supplemented by the next two relations which were pointed out above. It means that for rate-dependent flow law we arrive at

$$\mathbf{D}^p = \Lambda(\tilde{S}, \boldsymbol{\mu})\tilde{\mathbf{n}}. \tag{8}$$

Additionally, we have to define the evolution equations for the internal state variables. Among the variety of internal parameters $\boldsymbol{\mu}$ let us now restrict our attention on only two, namely the scalar value which describes the yield limit s, which is responsible for isotropic hardening/softening and tensorial value $\mathbf{B}$ which is called back stress (symmetric, traceless tensor) defining the kinematic hardening effect, both in stress units.

The *evolution of internal state variables* can then be proposed, for example, in the following form,

$$\dot{s} = r\,\Lambda\,, \tag{9}$$

$$\dot{\mathbf{B}} = H\,\mathbf{D}^p - \Lambda\,C\,\mathbf{B}\,, \tag{10}$$

where r is the hardening/softening parameter while H and C are parameters which are assumed to be functions of the variables $\boldsymbol{\mu}$.

A different function can be adopted to specify Λ defining the other type of rate-dependence. Let us for example assume

$$\mathbf{D}^p = \sqrt{\frac{3}{2}}\dot{\varepsilon}^p\tilde{\mathbf{n}} \tag{11}$$

and

$$\tilde{\mathbf{n}} = \sqrt{\frac{3}{2}}\frac{\mathbf{S}}{\tilde{S}}\,, \tag{12}$$

where $\mathbf{S} = \mathbf{T}' = \mathbf{T} - \frac{1}{3}\operatorname{tr}(\mathbf{T})\mathbf{1}$ is the deviatoric part of the Cauchy stress tensor and

$$\tilde{S} = \sqrt{\frac{3}{2}\mathbf{S} : \mathbf{S}} \tag{13}$$

is the equivalent stress.

Additionally, plastic equivalent strain rate $\dot{\varepsilon}^p = (\frac{2}{3}\mathbf{D}^p : \mathbf{D}^p)^{\frac{1}{2}}$ is prescribed as a function of current equivalent stress $\tilde{S}$ and state variables $\boldsymbol{\mu}$,

$$\dot{\varepsilon}^p = f(\tilde{S}, \boldsymbol{\mu}). \tag{14}$$

To complete the system of equations, it is necessary to add the evolution equations:

$$\dot{\boldsymbol{\mu}} = m(\tilde{S}, \boldsymbol{\mu}), \tag{15}$$

where

$$m(\tilde{S}, \boldsymbol{\mu}) = h(\boldsymbol{\mu})\,\dot{\varepsilon}^p \tag{16}$$

and $h(\boldsymbol{\mu})$ denotes the hardening/softening function.

The selection of functions $f(\tilde{S}, \boldsymbol{\mu})$ and $m(\tilde{S}, \boldsymbol{\mu})$ is based on phenomenological theories and should be strongly related to micro-mechanical observations and the experimental results obtained in physics of solids.

In further numerical consideration, let us restrict the class of functions $f(\tilde{S},\mu)$ in such a way that μ is characterized only by a scalar value s. If we accept the yield function in the form

$$f(\tilde{S},s)=\begin{cases}\eta\left(\frac{\tilde{S}}{s}-1.0\right)^n & \text{if } \tilde{S}\ge s\,,\\ 0 & \text{if } \tilde{S}<s\,,\end{cases} \tag{17}$$

we arrive at the particular case of viscoplasticity, originally introduced by Perzyna [34, 35, 36], where η is the viscosity parameter which is the reciprocal of the relaxation time of mechanical disturbances T_m, $(\eta=\frac{1}{T_m})$.

If we accept the yield function in the form:

$$f(\tilde{S},s)=\dot{\varepsilon}_0\left(\frac{\tilde{S}}{s}\right)^{\frac{1}{m}}, \tag{18}$$

we obtain the known viscoplastic power law. There also exist other possibilities of choosing functions $f(\tilde{S},s)$ which successfully serve in variety of particular cases.

Following the discussion in [49, 50], we introduce for convenience rotational-neutralized form of our constitutive model. The so called bar form of Cauchy stress tensor $\bar{\mathbf{T}}$ is now expressed as

$$\bar{\mathbf{T}}=\mathbf{Q}^T\mathbf{T}\mathbf{Q}\,, \tag{19}$$

where the rotation tensor $\mathbf{Q}(t)$ is introduced here as the solution of the following initial value problem,

$$\begin{aligned}\dot{\mathbf{Q}}(t)\mathbf{Q}^T(t) &= \mathbf{W}(t) \quad \text{for} \quad t_n\le t\le t_{n+1}\,,\\ \mathbf{Q}(t_n) &= \mathbf{I}\,,\end{aligned} \tag{20}$$

in which the spin $\mathbf{W}(t)$ is defined as the non-symmetric part of the velocity gradient $\mathbf{L}$. This simplifies significantly the equation that describes the rate of Cauchy stresses to the form

$$\dot{\bar{\mathbf{T}}}=\mathbf{Q}^T\overset{\nabla}{\mathbf{T}}\mathbf{Q}=\mathcal{C}[\bar{\mathbf{D}}-\bar{\mathbf{D}}^p] \tag{21}$$

(also see Nagtegaal [28]). Using this bar formulation, we obtain the following system of equations that, together with Eq. (21), constitutes our rate-dependent model:

$$\bar{\mathbf{D}}^p = \sqrt{\frac{3}{2}}\dot{\varepsilon}^p\bar{\mathbf{n}}(\bar{\mathbf{S}},\tilde{S})\,, \tag{22}$$

$$\bar{\mathbf{n}}(\bar{\mathbf{S}},\tilde{S}) = \sqrt{\frac{3}{2}}\frac{\bar{\mathbf{S}}}{\tilde{S}}\,, \tag{23}$$

$$\tilde{S} = \sqrt{\frac{3}{2}\bar{\mathbf{S}}:\bar{\mathbf{S}}}\,, \tag{24}$$

$$\dot{\varepsilon}^p = f(\tilde{S},s)\,. \tag{25}$$

The evolution equation for the scalar value s is

$$\dot{s}=m(\tilde{S},s)=h(s)\,\dot{\varepsilon}^p. \tag{26}$$

Then, for different choices of functions $f(\tilde{S},s)$, we can specify the Perzyna's type viscoplasticity (17) or creep model (18). The integration of so prepared system (bar formulation) is computationally much more efficient.

The details of time integration procedure that was accepted on the level of integration point in finite element computations were presented by Łodygowski et al. [24]. The numerical calculations were performed using a general purpose finite element program ABAQUS. The environment of the program was supplemented with own constitutive relations introduced via user subroutine called UMAT.

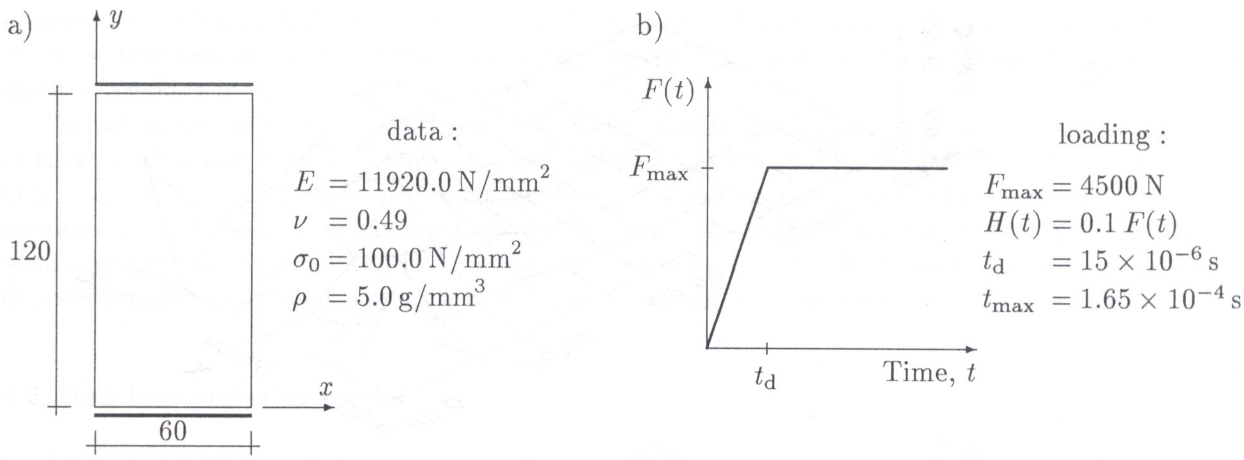

Fig. 1. Definition of the boundary value problem; a) geometry of the specimen, b) loading history

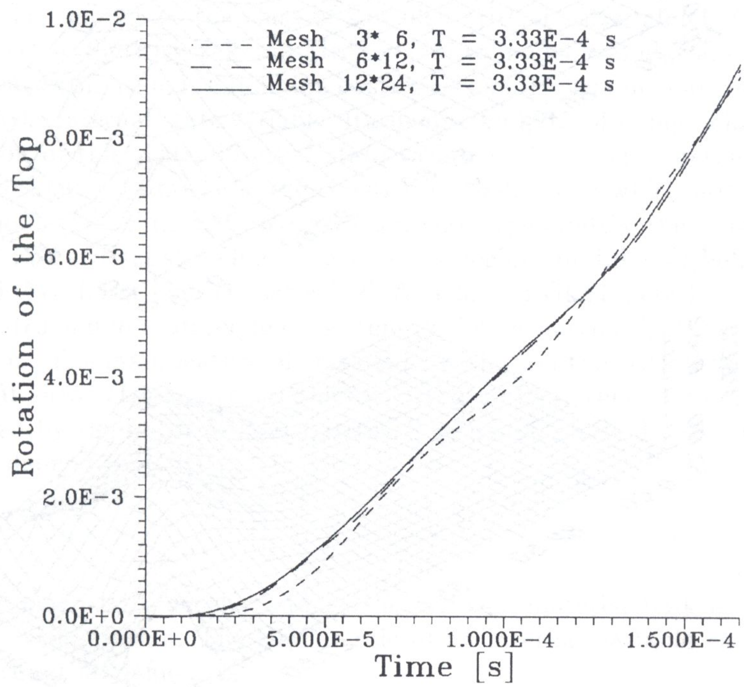

Fig. 2. Rotations of top side vs. time for three meshes

4.4.1. Numerical results

The specimen under consideration is 60 mm wide and 120 mm high. The problem is treated as a plane strain case. The above constitutive model is used and it shows the good convergence of results for different meshes (avoidance of the PMD) on both global ($P-\delta$) and local (here we have accepted plastic equivalent strains as a local measure) levels.

The material data used in the computations are as follows: $E = 11920\,\mathrm{N/mm^2}$, $\nu = 0.49$, $\sigma_0 = 100.0\,\mathrm{N/mm^2}$, $n = 1.0$, $\rho = 5.0\,\mathrm{g/cm^3}$. The hardening function $h(s)$ is assumed to be constant in the investigated example. The relaxation time used in calculations is $T_m = 10^{-6} - 10^{-4}\,\mathrm{s}$.

In the example we assume the dynamic loading at the top in the vertical direction by the force $F(t)$ and the horizontal one $H(t) = 0.1F(t)$. The force increases linearly until the value $F(15\times10^{-6}) = 0.75\,\sigma_0 A = 4500\,\mathrm{N}$ is recalled and then remains constant (see Fig. 1). The total time of the analysis is $t_m = 1.65\times10^{-4}\,\mathrm{s}$. It is assumed that the top side of the specimen remains rigid during the whole process. In Fig. 2 the plots of angles of rotation against time for different

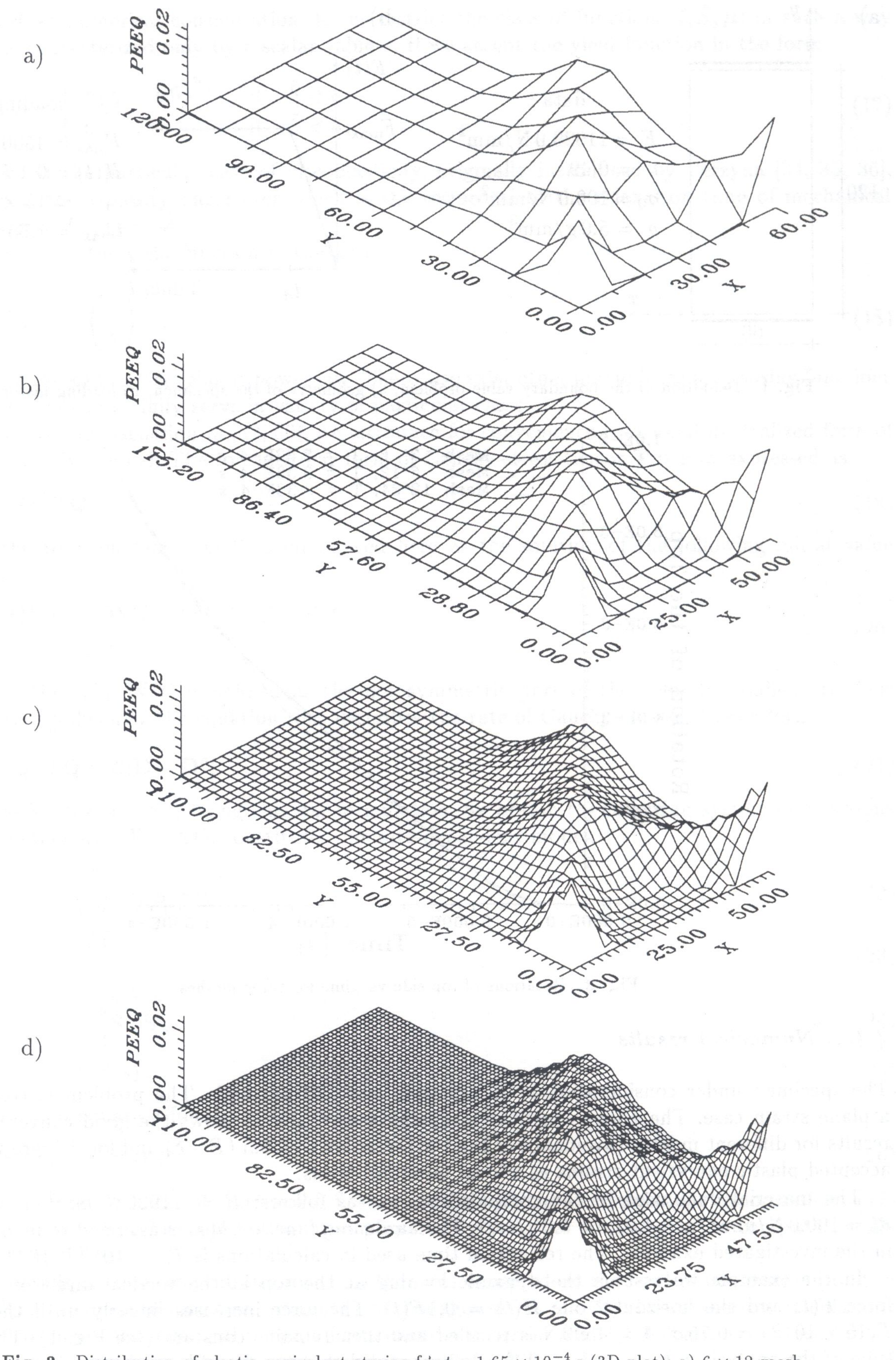

Fig. 3. Distribution of plastic equivalent strains for $t = 1.65 \times 10^{-4}$ s (3D plot); a) 6×12 mesh, b) 12×24 mesh, c) 24×48 mesh, d) 48×96 mesh

meshes are depicted. This figure reflects the behaviour of the specimen on the global level of the analysis. One can observe that the results for finer meshes converge to each other, they are stable and the primary mesh sensitivity is avoided.

The set of results presented in Fig. 3 shows the distribution of plastic equivalent strains PEEQ at time t_m obtained in computations for four different meshes ($6\times 12, 12\times 24, 24\times 48$ and 48×96). One can conclude that the results are qualitatively and quantitatively close to each other and spurious mesh dependence is avoided also on the local level of the behaviour of the specimen. Both results presented on global and local levels can be used as a numerical proof of avoidance of mesh dependence and simultaneously as a numerical confirmation of well-posedness of the IBVP.

4.5. Soil like material model

In the second example let us use the model of material which is soil-type and naturally refers to the dynamical processes. Let us summarize some basic assumptions of this elastic-plastic model of soil-like material with the yield condition which depends upon the hydrostatic pressure and porosity. This material belongs to the class of so called critical state models (see Wood [51]) and was originally adopted by Pietruszczak and Mróz in [40] to model granular and rock materials. Let us assume that the yield condition is a function of stresses and irreversible part of porosity or density variation η, which is the internal state variable. Hardening and/or softening behaviour is introduced through the evolution of this scalar internal state variable. The model presented here intends to simulate, contrary to known Cam-clays, the constitutive behaviour which does not restrict to the purely cohesionless materials. Similarly to other Cam-clay-type models, the yielding depends on the hydrostatic pressure and critical state line separates two regions of different behaviour (for detailed discussion see Wood [51], Loret [25], Drescher [12], Adachi and Oka [1] etc.) On the so called "dry" side the material dilates and in a consequence softens, while on the so called "wet" side it hardens as a result of compaction. The main feature of this model is that on the critical state line the material can yield at constant shear stress with no volume changes. The other important property is that the yield is influenced by the mean principal stress. The relative bulk density, which is introduced as scalar variable is defined by

$$\eta = \frac{\rho}{\rho_0}\,, \tag{27}$$

where ρ denotes the mean bulk density and ρ_0 the intrinsic material bulk density at a reference unloaded in general configuration. The similar role of an internal variable can also be played by the porosity, which could be defined as

$$\beta = \frac{V_v}{V_t}\,, \tag{28}$$

where V_v and V_t denote the void and the total volume of the representative specimen, respectively. Also, η can be expressed as

$$\eta = \frac{V_m}{V_t} = \frac{V_t - V_v}{V_t} = 1 - \beta\,, \tag{29}$$

where additionally V_m denotes the material volume. The change of the internal state variable η is given by

$$\dot{\eta} = \frac{\dot{\rho}}{\rho_0}\,. \tag{30}$$

From the continuity condition

$$\dot{\rho} + \rho\,\mathrm{div}(\mathbf{v}) = 0\,, \tag{31}$$

where $\mathbf{v}$ denotes the velocity, we arrive at

$$\dot{\rho} = -\rho\,\mathrm{tr}(\bar{\mathbf{D}}) \tag{32}$$

and after the decomposition of $\bar{\mathbf{D}}$ into elastic and plastic parts we obtain

$$\dot{\eta} = \underbrace{-\eta\,\mathrm{tr}(\mathbf{D}^e)}_{\dot{\eta}^e} - \underbrace{\eta\,\mathrm{tr}(\mathbf{D}^p)}_{\dot{\eta}^p} . \tag{33}$$

The second term on the right hand side of (33), $\dot{\eta}^p$, is the rate of change of porosity. Finally, the set of equations for the material model expressed in transformed form with respect to rotation group $\mathbf{Q}$ includes also the evolution equation for the Cauchy stresses,

$$\dot{\bar{\mathbf{T}}} = \mathbf{C} : (\bar{\mathbf{D}} - \bar{\mathbf{D}}^p) , \tag{34}$$

where

$$\mathbf{C} = 2G\mathbf{I} + \left(K - \frac{2}{3}G\right) \mathbf{1} \otimes \mathbf{1}$$

is the elastic material tensor. $\bar{\mathbf{D}}^p$ is the tensor of the rate of plastic deformation which for the rate-independent model appears as

$$\bar{\mathbf{D}}^p = \langle\dot{\lambda}\rangle\,\bar{\mathbf{N}}, \qquad \bar{\mathbf{N}} = \frac{\partial f}{\partial \bar{\mathbf{T}}} \tag{35}$$

and where

$$\langle\dot{\lambda}\rangle = \begin{cases} \dot{\lambda} & \text{if} \quad f = 0 \quad \text{and} \quad \bar{\mathbf{N}} : \mathbf{C} : \bar{\mathbf{D}} > 0 , \\ 0 & \text{if} \quad f \neq 0 \quad \text{or} \quad f = 0 \quad \text{and} \quad \bar{\mathbf{N}} : \mathbf{C} : \bar{\mathbf{D}} \leq 0 . \end{cases}$$

f is the yield function which depends, similarly as in critical state line type models, upon deviatoric stresses, mean pressure and also on the introduced internal state variable η and its evolution (33). The yield function is assumed in the form

$$f = \hat{f}(S, p, c) = (p - c)^2 + \frac{1}{2}\left(\frac{S}{d}\right)^2 - \left(\frac{\mu_c c}{d}\right)^2 \leq 0 \tag{36}$$

where the following denotations are used:

$$S = \sqrt{\bar{\mathbf{S}} : \bar{\mathbf{S}}} , \qquad \bar{\mathbf{S}} = \mathrm{dev}\,(\bar{\mathbf{T}}) , \qquad p = -\frac{1}{3}\,\mathrm{tr}\,(\bar{\mathbf{T}}) . \tag{37}$$

The geometrical interpretation of this yield function in the (S, p) space and its evolution is discussed in many papers, see e.g. Drescher [12]. Following the last paper we have adopted the material function c in the form

$$c = \hat{c}\,(\eta^p) = \alpha\,(\eta_0 + \eta^p) - c_0 \tag{38}$$

as well as constant parameters α, η_0, c_0 in numerical calculations. The needed system of equations that describes the constitutive relations consists of Eqs. (33–38). Different values of parameters c, d, μ_c define in $(S, -p)$ space different yield surfaces for which, contrary to the well documented Cam-clay models, the carrying of small tension stresses is also possible. The detailed discussion of the model can be found in the paper of Pietruszczak and Mróz [40] while for the other similar types of models see Wood [51], and for the practical applications the reader is referred to the works of Adachi and Oka [1, 2]. In the rate-independent formulation one can expect the problems with mathematical posedness and as a consequence in numerical computations with significant mesh sensitivity when the yielding enforce the shrinking of the yield surface (softening), see discussion in [39] and also the study of de Borst [9], Sluys [45].

For this purpose the viscoplastic regularization is proposed to overcome the ill-posedness and assure invariability of governing operator type during the whole process even in the post-critical states.

The rate-dependent material model expressed in a transformed form with respect to the rotation group $\mathbf{Q}$ can be performed in the following manner,

$$\dot{\bar{\mathbf{T}}} = \mathbf{C} : (\bar{\mathbf{D}} - \bar{\mathbf{D}}^{vp}), \tag{39}$$

where $\bar{\mathbf{D}}^{vp}$ is the viscoplastic rate of deformation tensor and is assumed following [34] as

$$\bar{\mathbf{D}}^{vp} = \varphi\langle\phi(S, p, \eta^p)\rangle \frac{\partial \phi}{\partial \bar{\mathbf{T}}}. \tag{40}$$

In the above, φ denotes the viscosity and the associative type of plasticity is taken into account, ϕ is empirical overstress function and $\langle\cdot\rangle$ denotes the MacCauley bracket which is understood as

$$\langle \phi(F) \rangle = \begin{cases} \Phi(F) & \text{if} \quad F > 0, \\ 0 & \text{if} \quad F \leq 0. \end{cases} \tag{41}$$

The viscosity φ is sometimes denoted by $\varphi = \frac{1}{T_m}$ where T_m is relaxation time for mechanical disturbances. The evolution equation for the irreversible part of the porosity takes the form

$$\dot{\eta}^{vp} = -\eta \operatorname{tr}(\bar{\mathbf{D}}^{vp}). \tag{42}$$

The important feature that is derived from the viscoplastic formulation of the problem for numerical calculations is the existence, uniqueness and well-posedness of the Cauchy problem. The discussion of this problem after the condition specified by Kato [17] and Hughes, Kato and Marsden [14] is presented by Perzyna in [37, 39] and recently also discussed on the level of the so called abstract Cauchy problem (ACP) and on the level of well-posedness of initial boundary value problem (IBVP) by Kibler et al. [18].

Since in numerical calculations we restrict our attention to the 2-D plane strain problems, let us now specify the constitutive equations for this case. The matrix $\mathbf{C}$ takes now the form

$$\mathbf{C} = 2G\mathbf{I} + (k - G)\,\mathbf{1} \otimes \mathbf{1} \tag{43}$$

where $k = K + \frac{1}{3}G$ and the function ϕ is expressed similarly to Eq. (36):

$$\phi = (p - c)^2 + \frac{1}{2}\left(\frac{S}{d}\right)^2 - \left(\frac{\mu_c c}{d}\right)^2 \tag{44}$$

with the material function c given by (38). The viscoplastic part of the rate of deformation is now obtained from

$$\bar{\mathbf{D}}^{vp} = \varphi\langle\phi()\rangle \left[\frac{S}{d^2}\bar{\mathbf{n}} - (p - c)\mathbf{1}\right] \tag{45}$$

where $\bar{\mathbf{n}} = \bar{\mathbf{S}}/S$.

The idea of using the rate-dependent plasticity model for soil-like material is sometimes argued. But, in particular for dynamic processes, the use of rate-dependent models exhibiting viscous effects is reasonable and fully justified. The experimental results describing the properties of clays confirm this statement (see for example Leroueil et al. [22] or Tavenas et al. [47, 48]).

The details of the numerical strategy which has been accepted in numerical integration of the constitutive relations are presented by Lengnick et al. [21].

4.5.1. Numerical results

Let us consider the pillar problem whose geometry is presented in Fig. 4 and let us assume the frictionless interaction with the rigid plates on the top and bottom sides of the specimen. Let us assume also the displacement controlled process in such a way that all nodes of the top side move down with equal displacements.

The results of numerical studies for rate-independent model which confirm the expected mesh sensitivity are shown in [21]. This effect of mesh dependence was documented on both global and local level of the analysis and for static and dynamic cases of loading. Different meshes used in the calculations create extremely different behaviour in post-critical states (after the peak load has been reached) and also differently predict the maximum value of forces that the structure can carry.

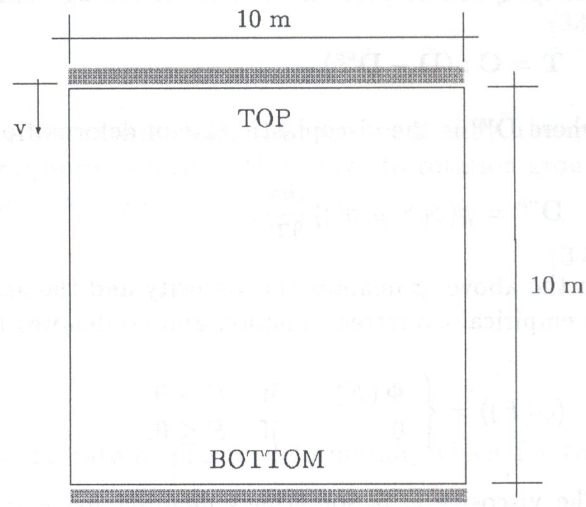

Fig. 4. Two-dimensional plane strain pillar problem

Let us now assume that the specimen is loaded dynamically. The top side moves with the constant velocity $v = 3.125\,\mathrm{m/s}$ such that after $t = 0.16\,\mathrm{s}$ its displacement is $0.5\,\mathrm{m}$. In this case, contrary to the static one, the localization does not propagate symmetrically but it starts first to develop from the top side. One can also observe the influence of the elastic wave propagation which, in the time duration of the process, is able to run over the pillar, reflect from the bottom and come back to the top.

Now we will examine the pillar problem using regularized model. The main goal of this part of presentation of numerical results is to show the effect of minimization and avoiding the strong primary mesh sensitivity. Only the dynamical calculations are taken into consideration. The me-

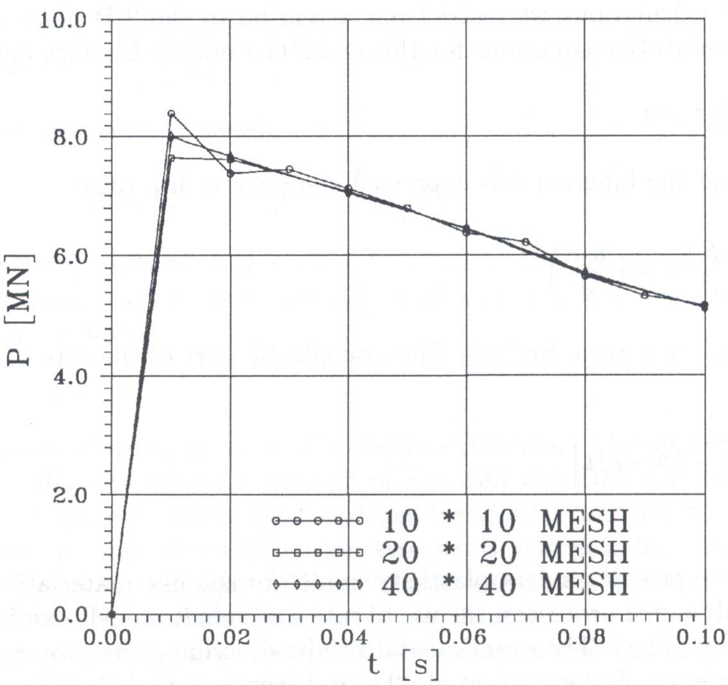

Fig. 5. The effect of viscoplastic regularization in the dynamic case, total force vs. time (velocity driven problem)

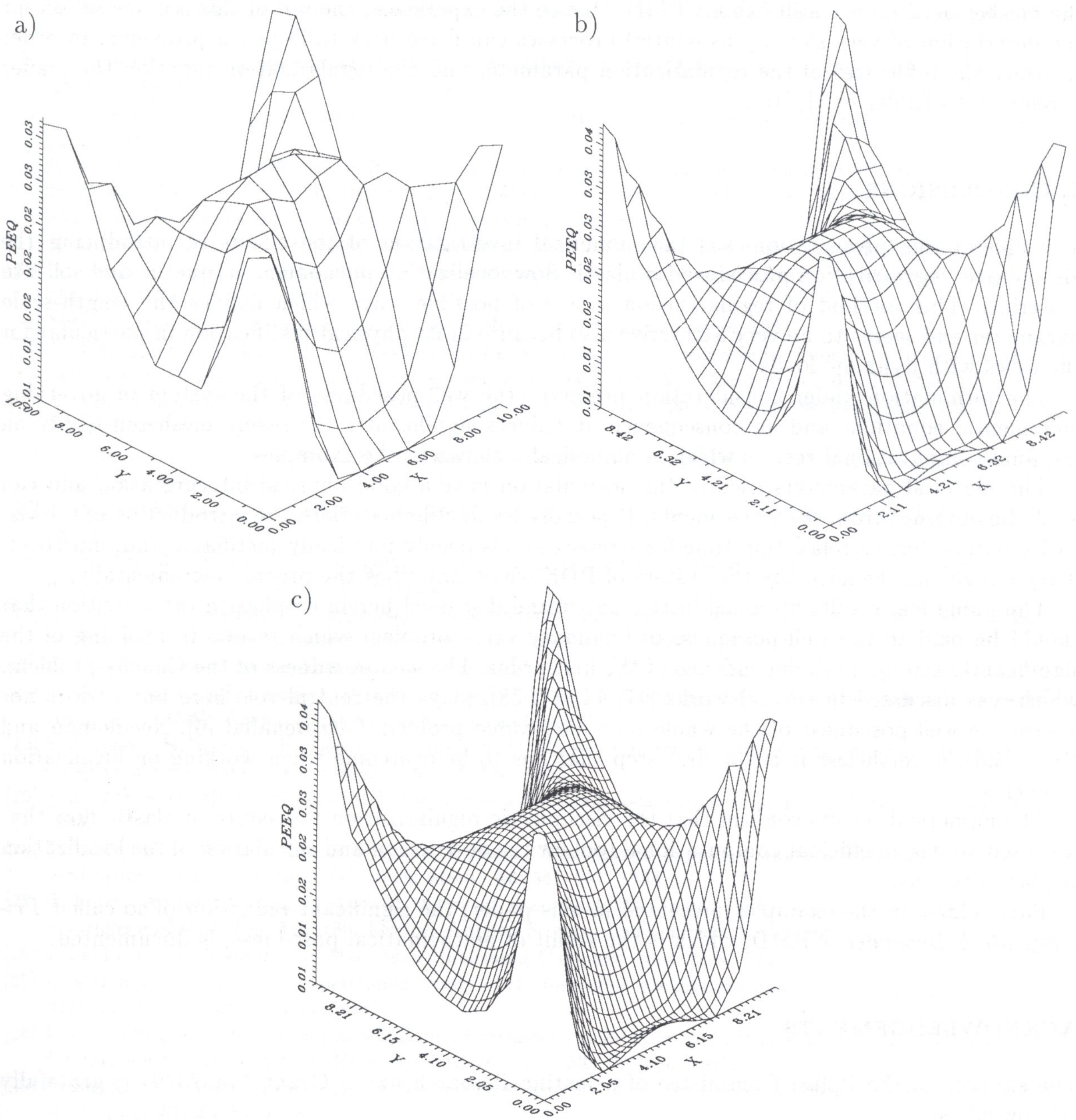

Fig. 6. Distribution of plastic equivalent strains PEEQ for three meshes in viscoplastic regularized cases; a) 10×10 mesh, b) 20×20 mesh, c) 40×40 mesh

chanical relaxation time which was used in calculations was of the order of $T_m = 10^{-3}$ s. In Fig. 5 results on the global level are presented. One can observe that the results for different meshes are close to each other also in post-critical states. In Fig. 6 we present the distribution of the equivalent plastic strains for three meshes: 10×10, 20×20 and 40×40 elements. The three-dimensional plots of these strains confirm the good agreement of both results for the specific time step $t = 0.16$ s. The results for the equivalent plastic strains are close both qualitatively and quantitatively. The differences obtained for 20×20 and 40×40 meshes do not exceed the value of 5%.

One can observe that the regularization introduced by viscoplasticity makes the localization zones not so sharp. The diffusion degree of these zones depends significantly on viscosity (relaxation time) used in the calculations.

For different values of velocities (initial conditions) one can expect different responses of the structure observed on both local and global level, however, for sufficiently large rates of deformations the meshes used do not still exhibit PMD. Due to the experience, the use of this regularization for the description of very slow (quasi-static) processes can drive back to ill-posed problems. In order to study the influences of the regularization parameter and the regularization function the reader is referred to Kibler et al. [19].

5. Conclusions

The aim of this presentation was the numerical investigation of the effects of introducing the viscoplastic regularization procedure for plastic flow localization phenomena in ductile and soil-like materials. This method of regularization is one of possible ways which defines the length-scale parameter and seems to be very attractive also because of its physical justification in particular for the cases with dynamic loads.

The formulation under consideration preserves the well-posedness of the system of governing incremental equations and, in consequence, it reduces significantly the severe mesh sensitivity on the final computational results which is numerically shown in the examples.

The material parameters used in this formulation have a good physical interpretation and can easily be obtained from the experiments. Especially for ductile materials, the introduction of the viscosity parameters (a relaxation time for stresses T_m) is deeply physically justifiable and, moreover, it regularizes mathematically the system of PDE which describes the process incrementally.

The numerical results obtained in this paper and discussed herein emphasize the attention that should be paid to the well-posedness of boundary value problem which results in avoiding of the significantly strong mesh dependence of the first order. The well-posedness of the Cauchy problem, which was discussed in several works [17, 14, 37, 38], plays the central role here but it does not assure the well-posedness of the whole boundary value problem (see Benallal [6], Needleman and Ortiz [30]); nevertheless it is the first step that has to be overcome when working on localization problems.

The numerical results confirm that the viscoplastic regularization procedure of plastic flow that was used here is an efficient computational tool for the description and calculation of the localization of plastic strains.

Particularly in the examples presented in this paper, the significant reduction of so called *Primary Mesh Dependence* PMD, which is the result of mathematical posedness, is documented.

Acknowledgements

The support of the Polish Committee of Scientific Research under Grant 1963/3/91 is gratefully acknowledged.

References

[1] T. Adachi and F. Oka. Constitutive equations for normally consolidated clay based on elasto-viscoplasticity. *Soils and Foundations*, **22**: 57–70, 1982.

[2] T. Adachi, F. Oka, M. Mimura. Mathematical structure of an overstress elasto-viscoplastic model for clay. *Soils and Foundations*, **27**: 31–42, 1987.

[3] Z. P. Bazant, J. Pan, G. Pijaudier-Cabot. Softening in reinforced concrete beams and frames. *J. Str. Engng. ASCE,*, **113**: 2333–2347, 1987.

[4] T. Belytschko, D. Lasry. A study of localization limiters for strain-softening in statics and in dynamics. *Comp. Struc.*, **33**: 707–715, 1989.

[5] T. Belytschko, X.-J. Wang, Z. P. Bazant, Y. Hyun. Transient solutions for one-dimensional problems with strain softening. *J. Appl. Mech.*, **54**: 513–518, September 1987.

[6] A. Benallal. Ill-posedness and localization in solid structures. In: D.R.J. Owen, E. Oñate, E. Hinton, eds., *COMPLAS III. Proc. Third Intern. Conference on Computational Plasticity, Fundamentals and Applications*, Barcelona, April 6–10, 1992.

[7] R. de Borst. Simulation of strain localization: A reappraisal of the Cosserat continuum. *Eng. Comp.*, **8**: 317–332, 1991.
[8] R. de Borst. Gradient-dependent plasticity: Formulation and algorithmic aspects. *Int. J. Num. Meth. in Eng.*, **35**: 521–539, 1992.
[9] R. de Borst. Fundamental issues in finite element analyses of localization of deformation. *Eng. Comp.*, **10**: 99–121, 1993.
[10] R. de Borst, H.-B. Muehlhaus, J. Pamin, L. J. Sluys. Computational modelling of localization of deformation. In: D.R.J. Owen, E. Oñate, E. Hinton, eds., *COMPLAS III. Proc. Third Intern. Conference on Computational Plasticity, Fundamentals and Applications*, Barcelona, April 6–10, 1992.
[11] N. Cristescu, I. Suliciu. *Viscoplasticity.* Martinus Nijhoff Publishers, The Hague, Boston, London, 1982.
[12] A. Drescher. Zagadnienia doświadczalnej weryfikacji modelu ciała o wzmocnieniu gęstościowym. *Rozp. Inż.*, **3**: 351–387, 1972.
[13] M. E. Gurtin. *An Introduction to Continuum Mechanics.* Academic Press, 1981.
[14] T. J. R. Hughes, T. Kato, J. E. Marsden. Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Arch. Rat. Mech. Anal.*, **63**: 273–294, 1977.
[15] T. J. R. Hughes, J. E. Marsden. Classical elastodynamics as a linear symmetric hyperbolic system. *Journ. Elasticity*, **8**: 97–110, 1987.
[16] I.R. Ionescu, M. Sofonea. *Functional and numerical methods in viscoplasticity.* Oxford University Press, Oxford, New York, Tokyo, 1993.
[17] T. Kato. The Cauchy problem for quasi-linear symmetric hyperbolic systems. *Arch. Rational Mech. Anal.*, **58**: 181–205, 1975.
[18] K. Kibler, M. Lengnick, T. Łodygowski. Selected aspects of the well-posedness of the localized plastic flow processes. *CAMES* (submitted for publication).
[19] K. Kibler, M. Lengnick, T. Łodygowski. Viscoplasticity as a regularization method of plastic strain localization phenomena; mathematical aspects and numerical treatment. *ZAMM*, 1994.
[20] E.H. Lee. Elastic-plastic deformations at finite strains. *J. Appl. Mech.*, **36**, 1969.
[21] M. Lengnick, T. Łodygowski, P. Perzyna, E. Stein. On regularization of plastic flow localization in a soil material. *Num. Anal. Meth. Geomech. (submitted for publication)*, 1994.
[22] S. Leroueil, M. Kobbaj, F. Tavenas, R. Bouchard. Stress-strain-strain rate relation for the compressibility of sensitive natural clays. *Geotechnique*, **35**: 159–180, 1985.
[23] T. Łodygowski. Mesh independent beam elements for strain localization. *Comp. Meth. in Civil Eng.*, **3**: 9–24, 1993.
[24] T. Łodygowski, M. Lengnick, P. Perzyna, E. Stein. Viscoplastic numerical analysis of dynamic plastic strain localization for a ductile material. *Arch. Mech.*, **46**: 541–557, 1994.
[25] B. Loret. An introduction to classical theory of elastoplasticity. In: F. Darve, ed., *Geomaterials: constitutive equations and modelling*, 149–186. Elsevier Applied Science, London–New York, 1990.
[26] J. Lubliner. *Plasticity theory.* Macmillan Publishing Company, New York, 1990.
[27] A. Marchand, J. Duffy. An experimental study of the formation process of adiabatic shear bands in a structural steel. *J. Mech. Phys. of Solids*, 1987.
[28] J. C. Nagtegaal. On the implementation of inelastic constitutive equations with special reference to large deformation problems. *Comp. Meth. Appl. Mech. Eng.*, **33**: 469–486, 1982.
[29] A. Needleman. Material rate dependence and mesh sensitivity in localization problems. *Comp. Meth. in Appl. Mech. Eng.*, **67**: 69–85, 1988.
[30] A. Needleman, M. Ortiz. Effect of boundaries and interfaces on shear-band localization. *Int. J. Solids Struc.*, **28**: 859–877, 1991.
[31] M. Ortiz, Y. Leroy, A. Needleman. A finite element method for localized failure analysis. *Comp. Meth. in Appl. Mech. Eng.*, **61**, 1987.
[32] C. V. Pao. The existence and stability of the solutions of nonlinear operator differential equations. *Arch. Rat. Mech. Analysis*, **35**: 16–29, 1969.
[33] C. V. Pao, W. G. Vogt. On the stability of nonlinear operator differential equations, applications. *Arch. Rat. Mech. Analysis*, **35**: 30–46, 1969.
[34] P. Perzyna. The constitutive equations for rate-sensitive plastic materials. *Quart. Appl. Math.*, **20**: 321–332, 1963.
[35] P. Perzyna. Fundamental problems in viscoplasticity. In: C.-S. Yih, ed., *Advances in Applied Mechanics*, vol. **9**: 243–377. Academic Press, 1966.
[36] P. Perzyna. Thermodynamic theory of viscoplasticity. In *Advances in Applied Mechanics*, vol. **11**, 313–354. Academic Press, 1971.
[37] P. Perzyna. Constitutive equations of dynamic plasticity. In: D.R.J. Owen, E. Oñate, E. Hinton, eds., *COMPLAS III. Proc. Third Intern. Conference on Computational Plasticity, Fundamentals and Applications*, Barcelona, April 6–10, 1992.

[38] P. Perzyna. Instability phenomena and adiabatic shear band localization in thermoplastic flow processes. *Acta. Mech.*, **94**: 1–31, 1994.

[39] P. Perzyna, E. Stein, M. Lengnick, T. Łodygowski. Regularization of the plastic flow process in soil and metal media and the localization phenomena. In: B. Raniecki O. Bruhns, S. Kowalski, eds., *Proc. VI Polish–German Symposium on Inelastic behaviour of solids and structures, Czerniejewo, September 13–17, 1993*, p. 33, 1993.

[40] S. Pietruszczak, Z. Mróz. Numerical analysis of elastic–plastic compression of pillars accounting for material hardening and softening. *Int. J. Rock Mech. Min. Sci. Geomech.*, **17**: 199–207, 1980.

[41] S. Pietruszczak, Z. Mróz. Finite element analysis of deformation of strain–softening materials. *Int. J. Num. Meth. in Eng.*, **17**: 327–334, 1981.

[42] R. D. Richtmyer, K. W. Morton. *Difference Methods for Initial-Value Problems.* Interscience Publishers, New York, 2nd edition, 1967.

[43] J. W. Rudnicki, J. R. Rice. Conditions for the localization of deformations in pressure–sensitive dilatant materials. *J. Mech. Phys. of Solids*, **23**: 371–394, 1975.

[44] L. J. Sluys. *Wave propagation, localization and dispersion in softening solids.* Dissertation, Delft University of Technology, Department of Civil Engineering, 1992.

[45] L. J. Sluys, J. Block, R. de Borst. Wave propagation and localization in viscoplastic media. In: D.R.J. Owen, E. Oñate, E. Hinton, eds., *COMPLAS III. Proc. Third Intern. Conference on Computational Plasticity, Fundamentals and Applications*, Barcelona, April 6–10, 1992.

[46] P. Steinmann, K. Willam. Localization within the framework of micropolar elasto–plasticity. In *Advances in Continuum Mechanics.* Springer, 1991.

[47] F. Tavenas, S. Leroueil. The behaviour of embankments on clay foundations. *Canadian Geotechnical Journal*, **17**: 236–260, 1980.

[48] F. Tavenas, S. Leroueil, P. La Rochelle, M. Roy. Creep behaviour of an undisturbed lightly overconsolidated clay. *Canadian Geotechnical Journal*, **15**: 402–423, 1978.

[49] G. Weber, L. Anand. Finite deformation constitutive equations and a time integration procedure for isotropic, hyperelastic-viscoplastic solids. *Comp. Meth. in Appl. Mech. Eng.*, **79**: 173–202, 1990.

[50] G. Weber, A. M. Lush, A. Zavalangos, L. Anand. An objective time-integration procedure for isotropic rate-independent and rate-dependent elastic-plastic constitutive equations. *Int. J. of Plasticity*, **6**: 701–744, 1990.

[51] D. M. Wood. *Soil Behaviour and Critical State Soil Mechanics.* Technical report, Cambridge University Press, 1990.