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Solution of 2D non-homogenous wave equation by using polywave functions

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The paper presents a specific technique of solving the non-homogenous wave equation with the use of Trefftz functions for the wave equation. The solution was presented as a sum of a general integral and a particular integral. The general integral was expressed in the form of a linear combination of Trefftz functions for the wave equation. In order to obtain the particular integral polywave functions were used. They were generated by using the inverse operator L^{-1} of the equation taking into consideration the Trefftz functions.

Keywords: polywave functions, Trefftz functions, wave polynomials, wave equation

1. Introduction

The solving functions methods for linear partial differential equations have been developing very quickly in recent years. The main idea of these methods is to find an approximate solution of the considered equation in terms of functions satisfying the equation (Trefftz functions) by fitting to given initial and boundary conditions properly. The method presented below belongs to them.

Solving a linear wave equation in regular areas leads to eigenproblems (see [1, 6, 16]). The solution of an eigenproblem is expressed by the linear combination of eigenfunctions. Work [16] presents solution for Laplace's eigenvalue problems, in paper [6] a solution of eigenvalue problems of polygonal membranes and plates is obtained with the use of Boundary Element Method (BEM). The question is if we can determine a general solution of the wave equation in an irregular area in the form of linear combination of base functions which satisfy a considered equation (wave polynomials).

Paper [3] presents two methods of determining one-dimensional wave polynomials for the wave equation in Cartesian coordinates. The first of them is based on a Taylor series expansion, the second uses the generating function and leads to recurrent formula for wave polynomials and their derivatives. As it turned out, both give the same formula for functions. An approximate solution of the considered equations is shown as a linear combination of wave polynomials. The unknown coefficients of this combination are sought by means of the minimization of a functional which fits the combination to given initial and boundary conditions in the L_2 norm. In papers [7, 8] the theory was extended on two- and three- dimensional cases.

The third method of determining the wave functions uses inverse operations. The inverse Laplace operator of harmonic functions is defined in [3, 14] for different co-ordinate systems. In papers [4, 5, 14] inverse operations were applied for determination of functions satisfying heat conduction equation not only in Cartesian co-ordinates. The paper [14] refers to heat conduction equation. It presents solutions of heat conduction problems in the form of heat functions given by inverse operations. Both direct and inverse problems are considered. Work [14] also contains a description of modified FEM with base functions (Trefftz functions for 2D heat conduction equation) given by inverse operations.

The papers [4, 5] present recurrent formulas for wave polynomials obtained by using inverse Laplace operator. In papers [9, 13] explicit formulas for these functions obtained from the Taylor series expansion can be found. It is worth mentioning that wave polynomials (see [4, 5, 9, 13]) are different from those obtained in [7, 8].

In paper [15] a numerical algorithm for solution of multi-dimensional wave equations is presented. The proposed method based on the Houbolt finite difference (FD) scheme, the method of particular solutions (MPS) and the Fundamental Solutions Method (MFS). It leads to transformation the wave equation into a Poisson-type equation with time-dependent loading.

Papers [9, 11, 12] are devoted to application FEM to wave equation. In FEM base functions (in general, see [2]) do not satisfy the given differential equations. Wave polynomials were used as base functions in FEM in paper [9]. The usage of Trefftz functions (wave functions) implies that the functional for FEM will have a different form than the traditional one. In [9] three kinds of modified FEM are compared (nodeless, continuous and discontinuous FEM). The problem of membrane vibrations was considered, moreover, two kinds of aforementioned functions as base functions were taken.

This work extends the wave polynomials method with the use of polywave function in non-homogeneous equation cases.

2. Solution of the 2D non-homogenous wave equation by the wave polynomials method

Take into account a non-homogenous wave equation

$$Lu(x,y,t) = Q(x,y,t), \qquad L = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) u. \tag{1}$$

It is commonly known that the general solution of this equation can be given by

$$u(x,y,t) = L^{-1}(0) + L^{-1}(Q) \tag{2}$$

where $L^{-1}(0)$ is the general solution of a homogenous equation and $L^{-1}(Q)$ is the particular solution of the non-homogenous one.

In the wave polynomials method a general solution is expressed by linear combination of the Trefftz functions for the wave equation $\nu_n(x,y,t)$

$$u \approx w(x,y,t) = \sum_{n=1}^{N} c_n \nu_n(x,y,t). \tag{3}$$

The unknown coefficients c_n are obtained by minimization of the functional describing adjustment of the approximate solution to given boundary and initial conditions, in the mean square sense.

The formula for determining the particular solution $L^{-1}(Q)$ is given in [6]. It is a result of applying operator L^{-1} to the Taylor series expansion of the function Q. In this work a different approach to calculate the particular solution is proposed, namely the expression of the particular solution in terms of polywave functions.

The presented method can easily be extended on other co-ordinate systems. Moreover, the solution given by (2) expressed in terms of wave and polywave functions may be used in modified FEM because these functions have no constrains as regards the shape of considered area.

2.1. The general solution of homogenous 2D wave equation – wave polynomials generation

Let us consider a homogenous equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad (x,y) \in \Omega \subset R^2, \quad t \in (0,\infty). \tag{4}$$

The function $u = u(x,y,t)$ satisfying equation (4) is expanded into the Taylor series in a neighbourhood of an arbitrary point (x_0, y_0, t_0)

$$u(x,y,t) = u(x_0,y_0,t_0) + \sum_{i=1}^{\infty} \frac{d^n u(x_0,y_0,t_0)}{n!}. \tag{5}$$

The right hand side of the expansion is transformed so as to eliminate the derivatives $\frac{\partial^2 u}{\partial t^2}$ according to the wave equation. After grouping the same derivatives and extracting the Laplace operator Δu we obtain a form of expansion in which coefficients are functions satisfying wave equation, namely,

$$\begin{aligned}
u(x,y,t) &= u_0 \cdot 1 + \frac{\partial u}{\partial x}\bar{x} + \frac{\partial u}{\partial t}\bar{t} + \frac{\partial u}{\partial y}\bar{y} + \frac{\partial^2 u}{\partial x^2}\left(\frac{\bar{x}^2}{2!} - \frac{\bar{y}^2}{2!}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)\bar{t}\bar{x} \\
&\quad + \Delta u\left(\frac{\bar{t}^2}{2!} + \frac{\bar{y}^2}{2!}\right) + \frac{\partial^3 u}{\partial x^3}\left(\frac{\bar{x}^3}{3!} - \frac{\bar{x}\bar{y}^2}{2!}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right)\bar{t}\left(\frac{\bar{x}^2}{2!} - \frac{\bar{y}^2}{2!}\right) \\
&\quad + \frac{\partial}{\partial x}\Delta u\left(\frac{\bar{t}^2}{2!}\bar{x} + \frac{\bar{x}\bar{y}^2}{2!}\right) + \Delta\left(\frac{\partial u}{\partial t}\right)\left(\frac{\bar{t}^3}{3!}\bar{x} + \bar{t}\frac{\bar{y}^2}{2!}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\bar{x}\bar{y} + \frac{\partial^2 u}{\partial t \partial y}\bar{t}\bar{y} \\
&\quad + \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial y}\right)\left(\frac{\bar{x}^2\bar{y}}{2!} - \frac{\bar{y}^3}{3!}\right) + \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial t \partial y}\right)\bar{t}\bar{x}\bar{y} + \Delta\frac{\partial u}{\partial y}\left(\frac{\bar{t}^2\bar{y}}{2!} + \frac{\bar{y}^3}{3!}\right) + \cdots \\
&= \sum_{n=0}^{N}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\partial^{n-2k}}{\partial x^{n-2k}}(\Delta^k u)\left(\sum_{j=0}^{k}\frac{t^j}{j!}\Delta^{-k+j}F_{n-2k}(x,y)\right) \\
&\quad + \sum_{n=1}^{N}\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{\partial^{n-1-2k}}{\partial x^{n-1-2k}}\left(\Delta^k\frac{\partial u}{\partial y}\right)\left(\sum_{j=0}^{k}\frac{t^j}{j!}\Delta^{-k+j}G_{n-2k}(x,y)\right) + R_{N+1}
\end{aligned} \tag{6}$$

where $\bar{x} = x - x_0$, $\bar{y} = y - y_0$, $\bar{t} = t - t_0$, and F_n, G_n are the following harmonic functions

$$F_n(x,y) = \mathrm{Re}\left(\frac{(x + \mathrm{i}\,y)^n}{n!}\right) = \sum_{k=0,2,\ldots}^{n \geq k} (-1)^{k/2}\frac{x^{n-k}y^k}{(n-k)!k!}, \tag{7}$$

$$G_n(x,y) = \mathrm{Im}\left(\frac{(x + \mathrm{i}\,y)^n}{n!}\right) = \sum_{k=1,3,\ldots}^{n \geq k} (-1)^{k/2}\frac{x^{n-k}y^k}{(n-k)!k!}. \tag{8}$$

In this way two independent sequences of Trefftz functions (wave polynomials) are obtained. They are expressed by the inverse operation of harmonic functions. Obviously, it is possible to eliminate the other derivatives ($\frac{\partial^2 u}{\partial x^2}$ or $\frac{\partial^2 u}{\partial y^2}$), which leads to formulas for different sequences of wave polynomials. All possibilities are discussed in the work [13].

The inverse Laplace operator (Δ^{-1}) of harmonic functions includes monomials $\frac{x^m}{m!}\frac{y^k}{k!}$ and can be defined in the following manner [3]

$$\Delta^{-1}\left(\frac{x^m}{m!}\frac{y^k}{k!}\right) = \begin{cases} \dfrac{x^m}{m!}\dfrac{y^{k+2}}{(k+2)!} & \text{for } m = 0 \text{ or } 1, \text{ and } k \geq 0, \\[2ex] \dfrac{x^m}{m!}\dfrac{y^{k+2}}{(k+2)!} - \Delta^{-1}\left(\dfrac{x^{m-2}}{(m-2)!}\dfrac{y^{k+2}}{(k+2)!}\right) & \text{for } m \geq 2,\ k \geq 0. \end{cases} \tag{9}$$

However, harmonic functions $F_n(x,y)$, $G_n(x,y)$ are symmetrical with respect to variables x and y, whereas calculations of successive inverse operations in accordance with the formula (9) distinguish the variable y ($\Delta^{-1}(1) = \frac{y^2}{2!}$). By symmetry of Δ with respect to the variables x, y it is possible to define Δ^{-1}, which distinguishes the variable x. Both possibilities are shown in [14].

In consequence, the general solution of the homogenous equation can be expressed as a linear combination of wave polynomials.

2.2. Particular solution for 2D non-homogenous wave equation given by polywave functions

Determining the particular solution for the non-homogenous wave equation by polywave functions is similar to generating the particular solution for the heat conduction equation by polyheat functions [14]. We start from the considered equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + Q(x,y,t), \quad (x,y) \in \Omega \subset R^2, \quad t \in (0,\infty). \tag{10}$$

Applying operator $L = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) u$ to (10) as many times as needed to get the right side of equation equal zero ($(M+1)$-times) leads to the following formula

$$L^{M+1} u(x,y,t) = 0. \tag{11}$$

The solution of (11) can be expressed as a linear combination of wave polynomials, namely,

$$L^M\big(Lu(x,y,t)\big) = L\big(L^M u(x,y,t)\big) = L\big(W(x,y,t)\big) = 0, \tag{12}$$

hence

$$W(x,y,t) = \sum_{n=0}^{\infty} a_n \nu_n(x,y,t) = \Theta_0(x,y,t) \tag{13}$$

and

$$L^M u(x,y,t) = \Theta_0(x,y,t). \tag{14}$$

Carrying out integration of the equation (14) with operator L^{-1} M-times we obtain

$$Lu(x,y,t) = \tilde{Q}(x,y,t) = \sum_{j=0}^{M} L^{-j} \Theta_{M-1-j}(x,y,t). \tag{15}$$

It denotes that the source function $Q(x,y,t)$ can be approximated by the following formula

$$\sum_{j=0}^{M} \beta_j L^{-j} \Theta_{M-1-j}(x,y,t). \tag{16}$$

So the function $Q(x,y,t)$ as well as the particular solution of the equation (10) can be given by linear combination of polywave functions (polywave polynomials). The unknown coefficient β_j can be derived from the problem interpolation or by fitting linear combination to the function $Q(x,y,t)$ in the mean square sense.

Table 1. Wave and polywave polynomials

Wave polynomials	Polywave function of order m, $L^{-m}(H) = L^{-1}\big(L^{-(m-1)}(H)\big)$
$\sum_{k=0}^{\mathrm{Floor}\left[\frac{n}{2}\right]}\left(\sum_{j=0}^{k}\frac{t^j}{j!}\Delta^{-k+j}F_{n-2k}(x,y)\right)$	$\sum_{k=0}^{\mathrm{Floor}\left[\frac{n}{2}\right]}\left(\sum_{j=0}^{k}L^{-m}\left(\frac{t^j}{j!}\Delta^{-k+j}F_{n-2k}(x,y)\right)\right)$
$\sum_{k=0}^{\mathrm{Floor}\left[\frac{n}{2}\right]}\left(\sum_{j=0}^{k}\frac{t^j}{j!}\Delta^{-k+j}G_{n-2k}(x,y)\right)$	$\sum_{k=0}^{\mathrm{Floor}\left[\frac{n}{2}\right]}\left(\sum_{j=0}^{k}L^{-m}\left(\frac{t^j}{j!}\Delta^{-k+j}G_{n-2k}(x,y)\right)\right)$

Formulas on polywave functions are based on an inverse operator L^{-1} proposed in [8], which should also be applied to wave functions (see Table 1). L^{-1} is defined (17) with accuracy to integration constant, which may be wave polynomial:

$$\begin{aligned}
&L^{-1}(x^k y^l t^m) = \\
&\quad\frac{1}{3(k+2)(k+1)}\Big(-x^{k+2}y^l t^m + m(m-1)L^{-1}\big(x^{k+2}y^l t^{m-2}\big) - l(l-1)L^{-1}\big(x^{k+2}y^{l-2}t^m\big)\Big) \\
&\quad+\frac{1}{3(l+2)(l+1)}\Big(-x^k y^{l+2} t^m + m(m-1)L^{-1}\big(x^k y^{l+2} t^{m-2}\big) - k(k-1)L^{-1}\big(x^{k-2}y^{l+2}t^m\big)\Big) \\
&\quad+\frac{1}{3(m+2)(m+1)}\Big(-x^k y^l t^{m+2} + k(k-1)L^{-1}\big(x^{k-2}y^l t^{m+2}\big) - l(l-1)L^{-1}\big(x^k y^{l-2}t^{m+2}\big)\Big).
\end{aligned} \tag{17}$$

For example polywave functions of the first and second order obtained from harmonic function $H(x,y,t) = 1$ are as follows

$$\begin{aligned}
&L^{-1}(H(x,y,t) = -\frac{t^2}{2!} - \frac{x^2}{2!} - \frac{y^2}{2!}, \\
&L^{-2}(H(x,y,t) = \frac{x^4}{4!} + \frac{y^4}{4!} + \frac{3t^4}{4!} + \frac{2x^2y^2}{2!2!} + \frac{2x^2t^2}{2!2!} + \frac{2y^2t^2}{2!2!}.
\end{aligned}$$

3. Description of the problem solution

Let us consider the direct problem described by the following mathematical model: the equation of motion

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + Q(x,y,t), \quad (x,y)\in\Omega, \quad t\in(0,\infty), \tag{18}$$

with the given boundary conditions

$$u(x,y,t)|_{\partial\Omega_i} = u_{\partial\Omega_i}(x,y,t)_{f_i}, \tag{19}$$

and the initial conditions

$$u(x,y,0) = f(x,y), \quad \frac{\partial u}{\partial t}(x,y,0) = u_0(x,y). \tag{20}$$

In order to solve the problem approximately, we determine the solution successively in k-th time interval $k\Delta t \le t \le (k+1)\Delta t$, $k = 0,1,2,\ldots$, as a sum of two linear combinations. The first one consists of wave polynomials $v_i(x,y,t)$, the second one of polywave functions $p_j(x,y,t)$

$$\tilde{u}^k(x,y,t) = \sum_{i=1}^{M}\alpha_i^k v_i(x,y,t) + \sum_{j=1}^{N}\beta_j^k p_j(x,y,t). \tag{21}$$

The unknown coefficients α_i^k, β_j^k are determined in a similar way. We minimize the proper functional I^k, J^k, which describes the adjustment of the approximation in mean square sense, to the pre-set initial and boundary conditions and also to the known function $Q(x,y,t)$ in the whole domain of space-time continuum,

$$\begin{aligned} I^k = & \iint\limits_{\Omega} \Big[\tilde{u}^k(x,y,(k-1)\Delta t) - \tilde{u}^{k-1}(x,y,(k-1)\Delta t)\Big]^2 d\Omega \\ & + \iint\limits_{\Omega} \left[\frac{\partial \tilde{u}^k}{\partial t}(x,y,(k-1)\Delta t) - \frac{\partial \tilde{u}^{k-1}}{\partial t}(x,y,(k-1)\Delta t)\right]^2 d\Omega \\ & + \sum_i \int\limits_{(k-1)\Delta t}^{k\Delta t} \int\limits_{\partial\Omega_i} \left[\tilde{u}^k\Big|_{\partial\Omega_i} - u_{\partial\Omega_i}(x,y,\tau)_{f_i}\right]^2 d\Omega_i d\tau \end{aligned} \tag{22}$$

where $\tilde{u}^0(x,y,t) = u(x,y,0) = f(x,y)$, and

$$J^k = \iint\limits_{\Omega} \int\limits_{(k-1)\Delta t}^{k\Delta t} \left[\frac{\partial \tilde{u}^k}{\partial t^2}(x,y,\tau) - \frac{\partial \tilde{u}^k}{\partial x^2}(x,y,\tau) - \frac{\partial \tilde{u}^k}{\partial y^2}(x,y,\tau) - Q(x,y,t)\right]^2 d\tau\, d\Omega. \tag{23}$$

The approximate solution includes functions which have no restrictions with respect to the shape of considered area. Also the type of given initial and boundary conditions (continuous or not) have no influence on the approximate solution. The difference is only in the form of functionals I^k, J^k – some of the integrations have to be properly replaced by sums. In the case of more complicated areas the solution given by (2) expressed in terms of wave and polywave functions can be used in modified FEM as well. The method can be also used to solve both direct and boundary inverse problems. Moreover, the main idea of the construction of the approximate solution is the same in other co-ordinate systems, one should only use different functions, but they do not have to be polynomials.

4. Numerical examples

The effectiveness of the presented method is tested on a numerical example described by the following relations: the governing equation

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + Q(x,y,t), \quad (x,y) \in (-1,1)\times(-1,1), \quad t \in (0,\infty), \tag{24}$$

the boundary and initial conditions

$$U(0,y,t) = U(1,y,t) = U(x,0,t) = U(x,1,t) = 0, \tag{25}$$

$$U(x,y,0) = x(1-x)y(1-y), \quad \frac{\partial U}{\partial t}(x,y,0) = 0, \tag{26}$$

and three kinds of source functions $Q(x,y,t)$.

Example 1

We assume that the source is a periodic function, dependent only on time, namely,

$$Q(x,y,t) = \frac{1}{10}\sin(2\pi\, t). \tag{27}$$

In this case the exact solution is given as [10]

$$U(x,y,t)=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{0.4\sin(k\pi x)\sin(l\pi y)}{\sqrt{(k\pi)^2+(l\pi)^2}} \cdot\frac{4\sin\left(\frac{k\pi}{2}\right)^2\sin\left(\frac{l\pi}{2}\right)^2\left(\sqrt{k^2+l^2}\sin(2\pi t)-2\sin\left(\sqrt{k^2+l^2}\pi t\right)\right)}{kl\left(k^2+l^2-4\right)\pi^3}. \tag{28}$$

The solution $U(x,y,t)$ is approximated according to the equation (21)

$$U(x,y,t)=\tilde{u}^k(x,y,t)=\sum_{i=1}^{M}\alpha_i^k v_i(x,y,t)+\sum_{j=1}^{N}\beta_j^k p_j(x,y,t), \tag{29}$$

and then compared with the analytical solution. We obtained a good approximation in the whole first time interval $(0,\Delta t)$ both for the source and for the solution. For calculation we assume that $\Delta t=1$, because of the length of the source period.

Figure 1 shows the approximations of the source function in the middle of the area obtained for different number of polywave functions of the second order. In particular $N=121$ means that these

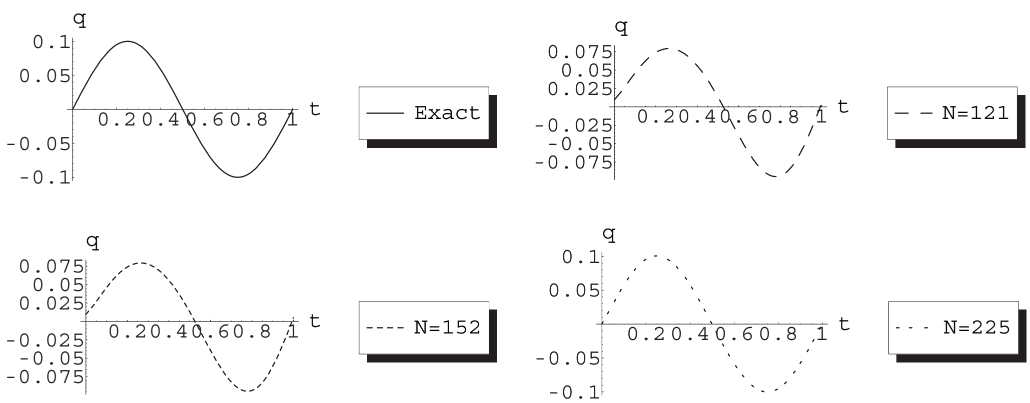


Fig. 1. The exact source and approximate ones at the point $x=y=0.5$ for N polywave functions of the second order, for the length of time interval $\Delta t=1$

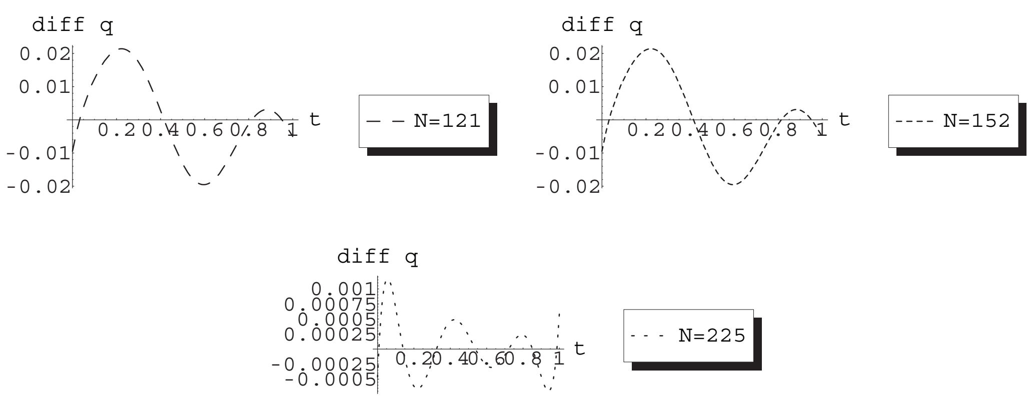


Fig. 2. The difference between exact and an approximate source at the point $x=y=0.5$ for N polywave functions of the second order, for the length of time interval $\Delta t=1$

polywave functions are generated from wave polynomials of the degree 10, however a satisfying result is not obtained until $N = 225$, so wave polynomials of degree 15 were used. Figure 2 shows the difference between the exact and the approximate source respectively.

To improve the result polywave functions of higher order were used. Figure 3 shows the comparison of the exact and approximate source obtained in the middle of the membrane for wave polynomials of the fifth order. In this case number $N = 180$ means that these polywave functions are generated from wave polynomials of the degree 6. Figure 4 shows the difference between the exact and the approximate source respectively. It is easy to notice that polywave functions of higher order give better results, although increasing their number caused the extension of time calculation while not improving the results significantly. The differences between the exact and the approximate source for 245 and 320 polywave functions are in similar range (Fig. 4).

The approximation of the problem solution (29) requires wave polynomials as well. Figure 5 shows the exact solution, an approximate one and the difference between them at the end of the first time interval. As could be expected, the highest difference is in the middle area.

For some approximation the following relative errors in the L_2 norm

$$\mathrm{error}_{q(x,y,t)} = \sqrt{\frac{\int_0^{\Delta t} \big(\mathrm{Approx}(0.5, 0.5, t) - \mathrm{Exact}(0.5, 0.5, t)\big)^2 dt}{\int_0^{\Delta t} \big(\mathrm{Exact}(0.5, 0.5, t)\big)^2 dt}} \cdot 100\%, \tag{30}$$

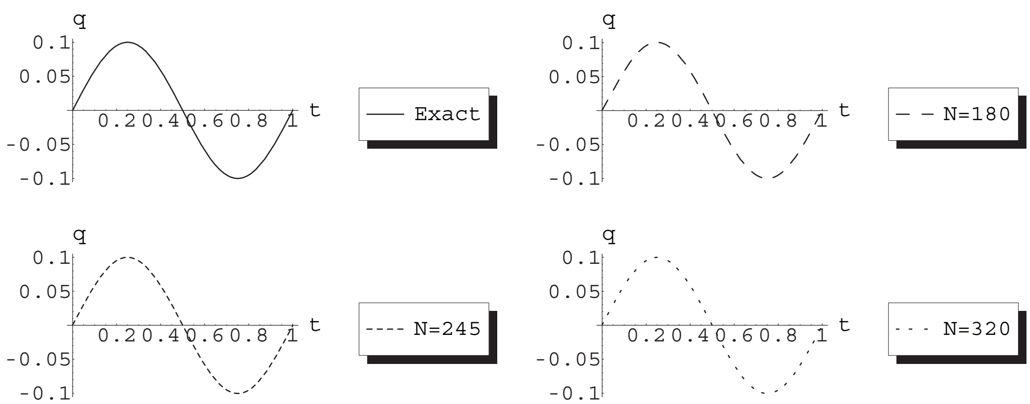


Fig. 3. The exact source and an approximate ones at the point $x = y = 0.5$ for N polywave functions of the fifth order, for the length of time interval $\Delta t = 1$

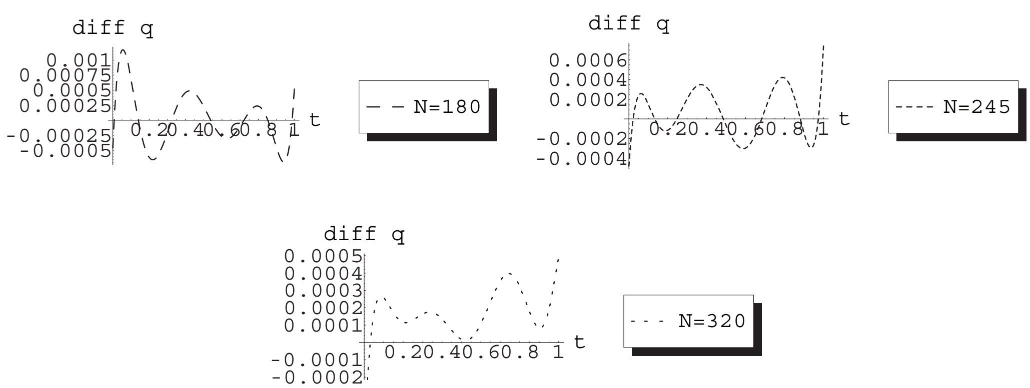


Fig. 4. The difference between exact and an approximate source at the point $x = y = 0.5$ for N polywave functions of the fifth order, for the length of time interval $\Delta t = 1$

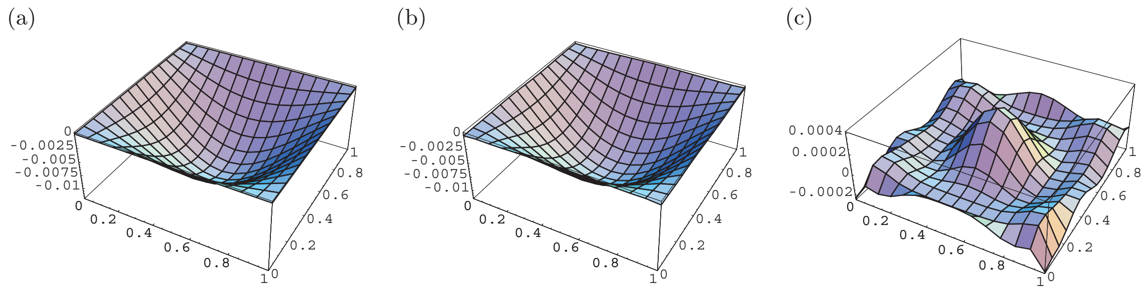


Fig. 5. The solution $U(x, y, t)$: exact (a), approximate (b), and the difference (c), at the point $x = y = 0.5$ for 152 wave and 180 polywave functions of the fifth order at time $t = 1$

$$\mathrm{error}_u = \sqrt{\frac{\int_0^{\Delta t} \left(\mathrm{Approx}(0.5, 0.5, t) - \mathrm{Exact}(0.5, 0.5, t)\right)^2 dt}{\int_0^{\Delta t} \left(\mathrm{Exact}(0.5, 0.5, t)\right)^2 dt}} \cdot 100\% \tag{31}$$

were calculated. These errors are presented in Table 2. It is worth mentioning that in all cases a smaller error for source is clearly seen.

Table 2. Relative errors for the source function and for the solution obtained with the use of 152 wave polynomials and different number of polywave functions

Number of polywave functions of the fifth order	$\mathrm{error}_q(x, y, t)$	$\mathrm{error}_u(x, y, t)$
180	0.6396%	5.5074%
245	0.0512%	4.4926%
320	0.0197%	4.4716%
784	4.9216%	over 30%

The comparison accuracy of an approximate and the exact solution in consecutive time intervals is shown on Figs. 6-8. In Figs. 6 and 7 one can see a big conformity between the approximate and exact solutions. Figure 8 shows extrapolation in time of the solution calculated in different time intervals. It is interesting that the approximate solution can be extrapolated in time backwards with a very good accuracy, but in the next time interval the extrapolation in time forward loses accuracy fast. In last case only the approximation from the first time interval gives a satisfying result.

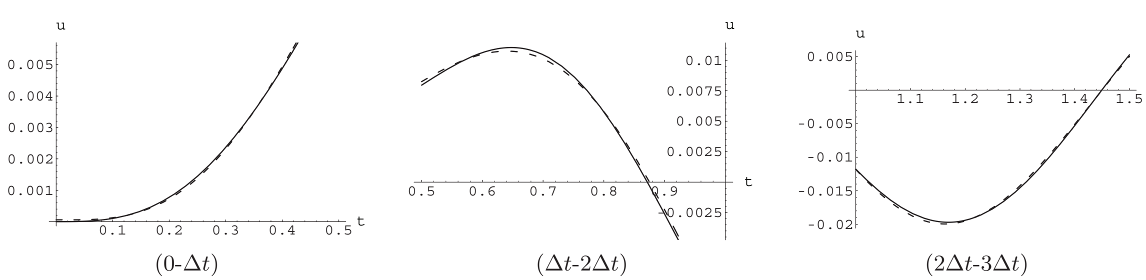
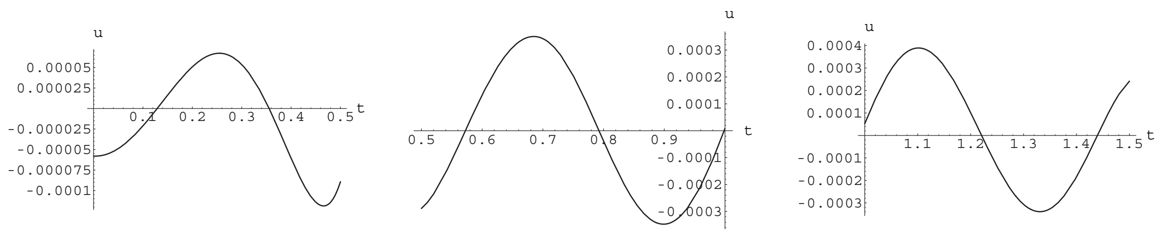
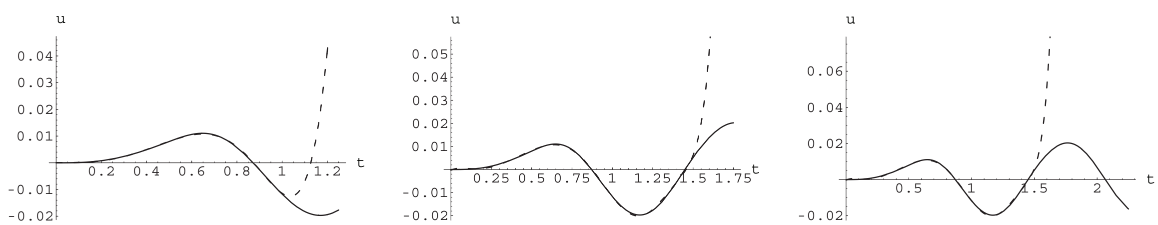


Fig. 6. The solutions: exact (continuous line) and approximate (dashed line) at the point $x = y = 0.5$ for the length of time interval $\Delta t = 0.5$



(0-Δt) (Δt-2Δt) (2Δt-3Δt)

Fig. 7. The difference between the exact and approximate solutions at the point $x = y = 0.5$ for the length of time interval $\Delta t = 0.5$



(0-Δt) (Δt-2Δt) (2Δt-3Δt)

Fig. 8. Extrapolation in time: the exact (continuous line) and approximate (dashed line) solutions calculated in indicated time intervals

Example 2

In the second test example a quickly expiring source is assumed

$$Q(x,y,t) = (x - x^2 + y - xy + 2t^2xy + x^2y - 2t^2x^2y - y^2 + xy^2 - 2t^2xy^2 - y^2 + xy^2 - 2t^2x^2y^2)e^{-t^2}. \quad (32)$$

For this case the exact solution is given as

$$U(x,y,t) = x(1-x)y(1-y)e^{-t^2}. \quad (33)$$

Similarly as in the previous case the approximate solution is constructed according to Eq. (21) for different number of wave and polywave polynomials. All results presented below are obtained for $\Delta t = 1$. Figure 9 shows the approximations of the source function in the middle of the area obtained for different number polywave functions of the fifth order (a), and the difference between the exact and the approximate sources (b).

Figure 10 shows the exact solution, an approximation by 152 wave polynomials and 180 polywave functions, and the difference between them in the midpoint. It is obvious that the approximation source and solution are characterized by high accuracy as well.

(a) (b)

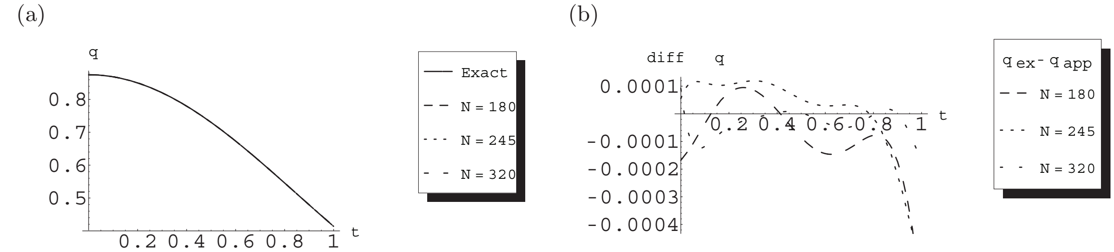


Fig. 9. The exact and approximate sources (a) and the difference between the exact and approximate sources (b) at the point $x = y = 0.5$ for N polywave functions of the fifth order, for the length of time interval $\Delta t = 1$

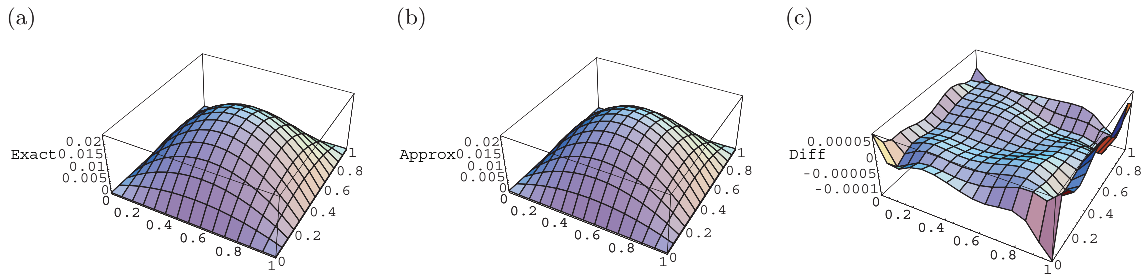


Fig. 10. The solution $U(x, y, t)$: exact (a), approximate (b), and the difference (c), at the point $x = y = 0.5$ for 152 wave and 180 polywave functions of the fifth order at time $t = 1$

For some approximations the relative errors in the L_2 norm calculated according to formulae (30) and (31) are presented in Table 3. It is worth mentioning that compared to the previous example, here there is no significant difference in the range of error for the source and the solution.

Table 3. Relative errors for the source function and for the solution obtained with the use of 152 wave polynomials and different number of polywave functions

Number of polywave functions of the fifth order	$\text{error}_q(x, y, t)$	$\text{error}_u(x, y, t)$
180	0.0261%	0.0171%
245	0.0213%	0.0082%
320	0.0142%	0.0306%

Example 3

In the third test example the source is assumed to be a periodic function, dependent not only on time but also on space variables:

$$Q(x, y, t) = (2x - 2x^2 + 2y - xy + x^2y - 2y^2 + xy^2 - x^2y^2)\cos t. \tag{34}$$

In this case the exact solution has the following form

$$U(x, y, t) = x(1 - x)y(1 - y)\cos t. \tag{35}$$

In Fig. 11 the comparison of the exact and approximate sources in the first time interval is shown. For different number of polywave functions approximation of source is close to the exact one.

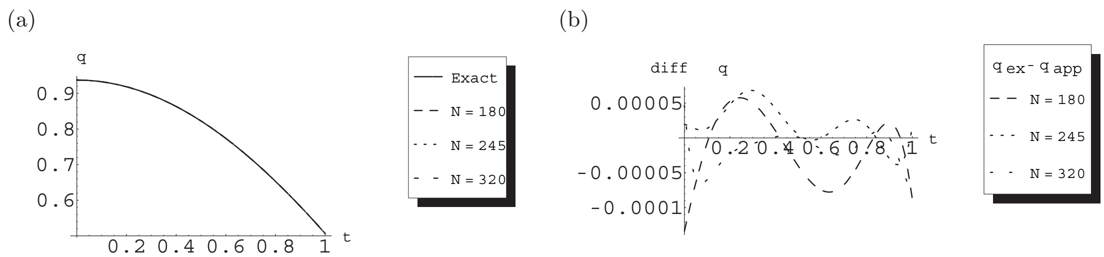


Fig. 11. The exact and approximate sources (a) and the difference between them (b) at the point $x = y = 0.5$ for N polywave functions of the fifth order, for the length of time interval $\Delta t = 1$

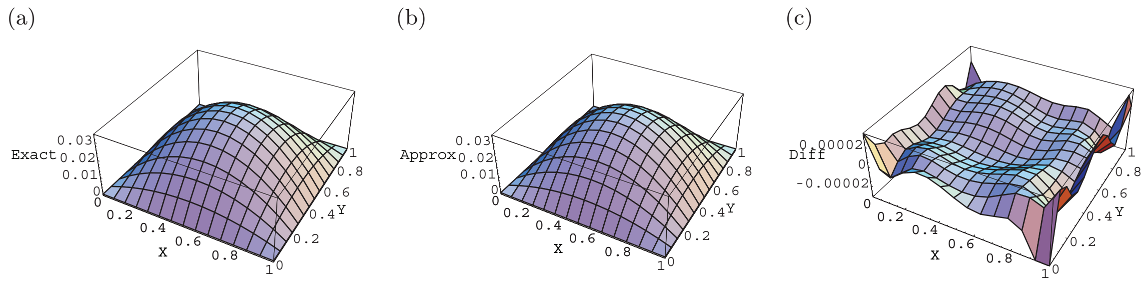


Fig. 12. The solution $U(x, y, t)$: exact (a), approximate (b), and the difference (c), at the point $x = y = 0.5$ for 152 wave and 180 polywave functions of the fifth order at time $t = 1$

The usefulness of the presented approach is confirmed by Fig. 12. We can see that the approximation of the exact solution in the midpoint of the area at the end of the first time interval is relatively accurate.

The relative errors in the L_2 norm were computed and put together in Table 4. The solution was approximated with the use of 152 wave polynomials and different numbers of polywave functions. As in the examples mentioned above, the computational results confirm a very good accuracy of the method.

Table 4. Relative errors for the source function and for the solution obtained with the use of 152 wave polynomials and different number of polywave functions

Number of polywave functions of the fifth order	$\text{error}_q(x, y, t)$	$\text{error}_u(x, y, t)$
180	0.0125%	0.0079%
245	0.0106%	0.0037%
320	0.0069%	0.0151%

5. Concluding remarks

The analysis of the results presented above leads to following conclusions.

The presented computational algorithm is simple and easy to implement in all co-ordinates systems. There is no difficulty in generating wave polynomials and polywave functions by using inverse operators.

The results obtained from numerical calculations confirmed a good accuracy of this method, although the number of functions in the approximation has not increased ad infinitum. The method is convergent, but a too big number of functions caused ill-conditioning of the matrix in the linear system of equations for the unknown coefficients of the approximation.

This approach can be applied to areas of more complicated shape by using presented wave and polywave functions as base functions in modified FEM.

The numerical calculations show that the method presented here seems to be a useful approach to solving some problems in thermoelasticity.

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