Computer Assisted Mechanics and Engineering Sciences, 3: 97–107, 1996.

Inverse problem for the prelinear filtration of ground water

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(Received July 12, 1995)

A parameter identification problem for nonlinear parabolic equation describing the prelinear filtration phenomenon is considered. It is shown that the problem admits a non-empty solution set which is stable with respect to perturbations in the cost functional and the data. The numerical method for solving the inverse problem is given and the computational results are presented.

1. Introduction and engineering motivation

The inverse parameter problems usually consist in determining the unknown material parameters of an engineering object basing on observation of its behavior. The traditional areas of applications of inverse problems are: geological investigations (mineral deposit prospecting, recognition of underground water and oil resources) as well as defect detection.

The inverse analysis may be also effectively used in earthen dam monitoring. A control core wall, usually made of cohesive soil (clay or silt), is the most important part of each dam and decides on its tightness and stability. The precise knowledge of material parameter values in design phase and during exploitation enables us to forecast the filtration velocity field. Dangerous phenomena such as erosion and weakened zones or caverns may be further recognized.

Observations and experimental results (see [13, 8, 10, 15, 19]) prove that the 1-parameter Darcy formula $v = k \cdot s$ which establishes the linear dependence between the filtration velocity v and the hydraulic slope s does not suit well to the flow in fine-grained cohesive soils. The significant discrepancies usually appear when more than 5% of particles have the diameter less than 5.0μ. The behavior of filtration velocity in this case consists in more than linear growth in the initial range of slopes while for large slopes this velocity converges asymptotically to an affine function. The 3-parameter Swartzendruber formula (*cf.* [15])

$$v = M(s - s_0(1 - \exp(-\theta s/s_0))) \tag{1}$$

seems to be more flexible and adequate for the so-called *prelinear filtration* process. However, there are some formal difficulties if one tries to extend the Swartzendruber formula to the multidimensional flow preserving the initial and asymptotic features as well as the invariance with respect to the rotations of the coordinate system. One of the recent approaches (see [13, 14]) which takes into account the above needs is the following:

$$v_i(\mathbf{p}(t,x), Dh) = \varphi(\tilde{p}(t,x), |Dh|_L) \sum_{j=1}^{3} l_{ij}(t,x) \frac{\partial h}{\partial x_j}, \tag{2}$$

where $i = 1,2,3$, $\mathbf{p}(t,x) = (\{l_{ij}\},\tilde{p})(t,x)$, $\tilde{p} = (M, s_0, \theta)$, $|\eta|_L = \sum_{ij=1}^{3} l_{ij}\eta_i\eta_j$ for $\eta \in \mathbb{R}^3$ and

$$\varphi(\tilde{p}, s) = \begin{cases} M\left(1 - \frac{s_0}{s}\left(1 - \exp\left(-\frac{\theta s}{s_0}\right)\right)\right), & \text{if } E < s, \\ \left(\frac{M}{E^2}\left(s_0 - (s_0 + \theta E)\exp\left(-\frac{\theta E}{s_0}\right)\right)\right) s + M\left(1 - \frac{2s_0}{E} + \left(\frac{2s_0}{E} + \theta\right)\exp\left(-\frac{\theta E}{s_0}\right)\right), & \text{if } 0 \le s \le E. \end{cases}$$

Above $\{l_{ij}\}$ denotes the dimensionless anisotropy matrix, M has the similar meaning as the asymptotic Darcy coefficient and the same physical dimension, s_0 is an analogue of threshold gradient, θ is often called "index of nonlinearity" and the constant E has no physical meaning and depends only on the assumed accuracy. Moreover, we suppose that the inequality

$$\frac{M}{E^2}\left(s_0 - (s_0 + \theta E)\exp\left(-\frac{\theta E}{s_0}\right)\right) > 0 \tag{3}$$

holds in the whole filtration domain and for all time instances.

The significant sensitivity of the filtration velocity field with respect to changes of values of nonlinear parameters s_0 and θ was observed, in particular, in the earthen dam Wióry located on river Świślina in south-eastern Poland.

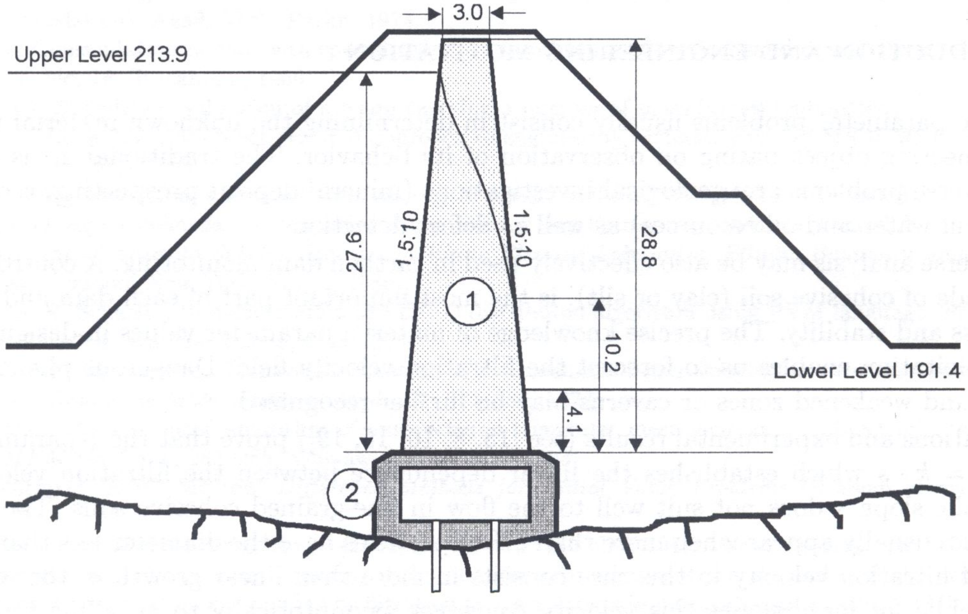

Fig. 1. Central cross-section through the dam "Wióry"

The central part of this dam is made in the form of a massive clay seal (see detail 1 in Figure 1) founded on the injection gallery made of concrete (see detail 2 in Figure 1). The clay wall is about 29.0 m high, 8.6 m thick at the basis and 3.0 m at the top. The upper water level reaches 27.6 m and the lower water level is 4.1 m above the basis. The asymptotic permeability was measured in laboratory and equals $M = 2.4 \cdot 10^{-6}$ cm/s in the whole area of protection wall. Strongly cohesive clay exhibites also the considerable nonlinear behavior, and the remaining parameters θ, s_0 were heuristically evaluated to be near 0.8, 2.5, respectively (we have also put $\{l_{ij}\} = \delta_{ij}$).

The simulation of stationary filtration in a wall cross-section was made using the finite element method in order to obtain the total leakage per unit width of the dam. Several iterations were made in order to establish the right position of the free water surface. The computations were carried out independently for different values of s_0 and θ, keeping the remaining parameters on their standard levels ($M = 2.4 \cdot 10^{-6}$, $s_0 = 0.8$, $\theta = 2.5$). The results exhibited a strong variability of computed leakage in both above cases (see Figure 2).

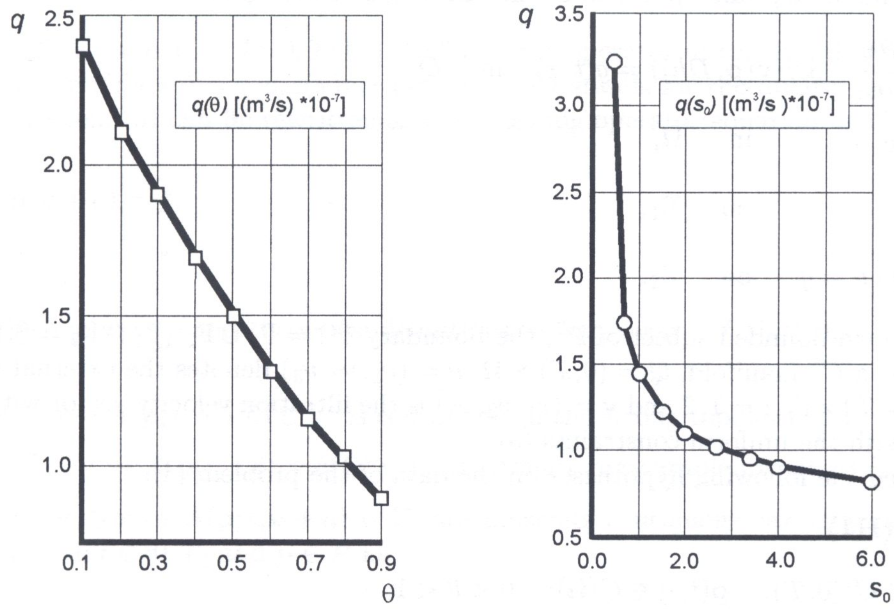

Fig. 2. Total leakage q per unit width versus parameters θ and s_0

These facts together with the earlier ones (see [13, 8, 10, 15, 19]) prove that the behavior of earthen dams is strongly sensitive with respect to the nonlinear filtration parameters, so their identification is indispensable for the stability monitoring. The inverse analysis is especially important in the initial period of exploitation in which all the physical parameters can dramatically change due to the consolidation process.

We would like to mention that another mathematical model (not used here) of the water flow through an earthen dam can be described by the elliptic variational inequality. The surface which separates the dry medium and the saturated one (called the free boundary) is not known a priori and it is one of the unknows of the problem. There exists an extensive literature on the free boundary problems, see [3, 9, 5] and [17], which contains a computer code of the program modelling a water filter.

The theoretical part of this paper describes the mathematical model for the filtration problem. We study a parameter–dependent nonlinear parabolic equation (see (4)) for the scalar piezometric pressure h which permits then to calculate the velocity field $\mathbf{v}$ by means of the formulae (2). We first formulate a result (Theorem 1) on the existence of the weak solutions to (4), their uniqueness and their continuous dependence (in appropriate spaces) on parameters and on the data. Furthermore, the weak solution to (4) can be found as the limit, taken in suitable space, of the sequence of solutions to the associated Galerkin equations (compare (7) below). Next, we consider the inverse problem formulated as the control one (see [1]) which consists in finding unknown parameters in the equation from known observations of the modelled process. We state a theorem on existence of solutions to the identification problem for the case of exact solutions as well as for the case of their Galerkin approximations and we show that the latter ones approximate the solution of the inverse problem (7) for exact solution provided the cost functional is continuous (see Section 3). Moreover, we give a result on the sensitivity of solution set for inverse problem with respect to perturbations in the cost functional and in the data (Theorem 3). Finally, in Section 5, we report on numerical results concerning an identification problem which is based on a real experiment.

2. Filtration model

The prelinear filtration process in the time interval $[0,T]$ through a body occupying a set Ω in $\mathbb{R}^3$ is described by the piezometric height distribution h which is the solution to the mixed Dirichlet/Neumann initial-boundary value problem for nonlinear parabolic equation:

$$\begin{cases} \alpha(t)e(x)\dfrac{\partial h}{\partial t} - \operatorname{div}(\mathbf{v}(\mathbf{p}, Dh)) = g(t,x) & \text{in} \quad Q, \\ h(0) = h_0 & \text{in} \quad \Omega, \\ h|_{\Sigma_1} = h_b & \text{on} \quad \Sigma_1, \\ \mathbf{v}(\mathbf{p}, Dh)\cdot\nu = q & \text{on} \quad \Sigma_2. \end{cases} \tag{4}$$

Here Ω is an open bounded subset of $\mathbb{R}^3$, the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ $(\Gamma_1 \cap \Gamma_2 = \emptyset, m(\Gamma_1) > 0)$ is supposed to be a C^1 manifold, $Q = (0,T)\times\Omega$, $\nu = (\nu_1, \nu_2, \nu_3)$ denotes the external normal vector to $\partial\Omega$, $\Sigma_i = (0,T)\times\Gamma_i, i = 1,2$ and $\mathbf{v} = (v_1, v_2, v_3)$ is the filtration velocity vector with components given by (2) with the uniform constraints (3).

We will need the following hypotheses on the data of the problem (4).

Hypothesis (H1)

$$\begin{cases} \mathbf{p}(\cdot, x) \in C^1(0,T), \quad \mathbf{p}(t,\cdot) \in C(\Omega); \quad 0 < \theta < 1; \\ c_0|\eta|_1 \le |\eta|_L \le c_1|\eta|_1 \quad \text{for} \quad \eta \in \mathbb{R}^3 \quad \text{with} \quad 0 < c_0 \le c_1, \quad (|\cdot|_1 \text{ is the Euclidean norm}); \end{cases}$$

Hypothesis (H2)

$$\begin{cases} \alpha \in C^1(0,T), \quad e \in C^1(\Omega); \quad 0 < \alpha_0 \le \alpha(t), \quad 0 < e_0 \le e(x); \\ \mathbf{d} = (g, h_0, h_b, q), \quad \mathbf{d} \in \mathcal{D}, \quad \text{where} \\ \mathcal{D} = C(0,T; L^\infty(\Omega)) \times L^2(\Omega) \times C^1(\Sigma_1) \times C(0,T; C^1(\Gamma_2)). \end{cases}$$

In order to state the results on the existence of the weak solutions to (4), their uniqueness and their continuous dependence on parameters and on the data, we can formulate (4) in the weak form:

$$\begin{cases} \langle e(x)\dfrac{\mathrm{d}\tilde{u}}{\mathrm{d}t}(t), w\rangle + b(\mathbf{p}; t, \tilde{u}(t) + \tilde{a}(t), \tilde{u}(t) + \tilde{a}(t), w) = \langle f(t), w\rangle, \quad \forall\, w \in V, \\ \tilde{u}(0) = h_0 - \tilde{a}(0) =: u_0, \end{cases} \tag{5}$$

where $V = \{w \in H^1(\Omega);\ w = 0 \text{ on } \Gamma_1\}$, $a \in C^1(\overline{Q})$ denotes the lifting of h_b i.e. $a|_{\Sigma_1} = h_b$; $\tilde{h}(t) = h(t,\cdot)$, $\tilde{h}(t) = \tilde{u}(t) + \tilde{a}(t)$, $\langle\cdot,\cdot\rangle$ is the duality pairing between V and its dual V' and for $z, v, w \in V$ we have put

$$b(\mathbf{p}; t, z, v, w) = \frac{1}{\alpha(t)} \int_\Omega \varphi\left(\tilde{p}(t,x), |Dz|_L\right) \sum_{i,j=1}^{3} l_{ij} \frac{\partial v}{\partial x_j}\frac{\partial w}{\partial x_i} \mathrm{d}x,$$

$$\langle f(t), w\rangle = \frac{1}{\alpha(t)} \int_{\Gamma_2} q(t,x) w(x)\, \mathrm{d}\sigma(x) + \frac{1}{\alpha(t)} \int_\Omega \left(g(t,x) - \alpha(t)e(x)\frac{\partial a}{\partial t}\right) w(x)\, \mathrm{d}x.$$

Next, we also introduce the families of operators $B(\mathbf{p}; t, \cdot, \cdot)\colon V \times V \to V'$, $A(\mathbf{p}; t, \cdot)\colon V \to V'$ and $\mathcal{A}(\mathbf{p})\colon \mathcal{V} \to \mathcal{V}'$ respectively given by the formulae

$$\langle B(\mathbf{p};t,z,v),w\rangle = b(\mathbf{p};t,z,v,w), \quad z,v,w\in V,$$

$$A(\mathbf{p};t,v) = B(\mathbf{p};t,v,v), \quad v\in V,\ t\in[0,T],$$

$$\mathcal{A}(\mathbf{p},\tilde{u})(t) = A(\mathbf{p};t,\tilde{u}(t)), \quad \tilde{u}\in\mathcal{V},\ t\in[0,T],$$

where $\mathcal{V}=L^2(0,T;V)$ and $\mathcal{V}'=L^2(0,T;V')$. Defining the operator of the distributional multiplication $\mathcal{E}:\mathcal{V}'\to\mathcal{V}'$ by $\langle\langle\mathcal{E}u,v\rangle\rangle=\langle\langle u,e(x)v\rangle\rangle$ (where $\langle\langle\cdot,\cdot\rangle\rangle$ stands for the duality pairing between $\mathcal{V}$ and $\mathcal{V}'$), the problem (5) can be rewritten as the following operator equation in $\mathcal{V}'$:

$$\begin{cases}\mathcal{E}\dfrac{\mathrm{d}\tilde{u}}{\mathrm{d}t}+\mathcal{A}(\mathbf{p};\tilde{u}+\tilde{a})=f,\\ \tilde{u}(0)=u_0.\end{cases}\tag{6}$$

Directly from the hypotheses and the above definitions we can verify (*cf.* [14]) that for every fixed $a\in C^1(\overline{Q})$,

(i) the map $\mathcal{P}\ni\mathbf{p}\mapsto A(\mathbf{p};t,u+\tilde{a})\in V'$ is continuous in the strong topology of V', for all $u\in V$,

(ii) the map $[0,T]\ni t\mapsto A(\mathbf{p};t,z(t)+\tilde{a}(t))\in V'$ is strongly measurable for all $z\in\mathcal{V}$

and the operators $V\ni v\mapsto A(\mathbf{p};t,v+\tilde{a})\in V'$ are Lipschitz continuous, strongly monotone and coercive, uniformly in $t\in[0,T]$ and $\mathbf{p}\in\mathcal{P}$ i.e.

(iii) $\|A(\mathbf{p};t,v+\tilde{a}(t))-A(\mathbf{p};t,w+\tilde{a}(t))\|_{V'}\le M\|v-w\|_V$,

(iv) $\langle A(\mathbf{p};t,v+\tilde{a}(t))-A(\mathbf{p};t,w+\tilde{a}(t)),v-w\rangle\ge m\|v-w\|_V^2$,

(v) $\langle A(\mathbf{p};t,v+\tilde{a}(t)),v\rangle\ge\rho(\|v\|_V)\|v\|_V$ and $\rho(s)\to\infty$ as $s\to\infty$,

for every $t\in[0,T]$, $v,w\in V$ with some $m,M>0$ independent of t and $\mathbf{p}$.

For the numerical purposes we can write the Galerkin system of the differential equations which corresponds to (6) and which has the form

$$\begin{cases}\mathcal{E}_n\dfrac{\mathrm{d}\tilde{u}_n}{\mathrm{d}t}+A_n(\mathbf{p};t,\tilde{u}_n(t)+(\tilde{a})_n(t))=f_n(t),\\ \tilde{u}_n(0)=(u_0)_n,\end{cases}\tag{7}$$

where the operators $\mathcal{E}_n, A_n, f_n$ denote (as usual in the Galerkin method) the restrictions of $\mathcal{E}, A, f$ to the finite–dimensional subspace V_n of V (possibly finite element type) and $(u)_n$ is the orthogonal projection of u on V_n. In practice, in order to solve the equations (7) numerically, we use the time discretization method based on the special kind of three-level finite difference scheme. As usually it consists in computing the approximate solution in the form

$$h_{n\tau}=u_{n\tau}+a_{n\tau},\tag{8}$$

where $u_{n\tau}\in U_{n\tau}=\{v:\Upsilon\to V_n\}$, $\Upsilon=\{\tau i;\ i=-1,0,1,\dots,k;\ k\tau=T\}$, $u_{n\tau}(i\tau)$ corresponds to the discrete value $\tilde{u}_n(i\tau)$ of the Galerkin solution of (7) for $i=1,\dots,k$, $u_{n\tau}(0)=(u_0)_n$, $u_{n\tau}(-\tau)=\overline{u}(-\tau)$, where $\overline{u}\in C^1(-t_0,0;V_n)$, $t_0>\tau$ is a special case of the initial condition which is necessary for three-level schemes. Moreover $a_{n\tau}(i\tau)=(\tilde{a})_n(i\tau)$ for $i=-1,0,1,\dots,k$. For more information about finite difference operator and the convergence of solutions as well as on the convergence of mixed finite element (Galerkin)/finite difference technique for this case, see [14].

The following theorem is obtained by using Faedo–Galerkin method and the theory of monotone operators, see Theorem 2 in [14] and Corollary 5.1 in [7].

Theorem 1. If hypotheses (H1), (H2) hold, then

(i) the problem (4) admits the weak solution $\tilde{h} = \tilde{h}(\mathbf{p}, \mathbf{d})$ corresponding to a parameter $\mathbf{p} \in \mathcal{P} \subset C(Q)$ and data $\mathbf{d} \in \mathcal{D}$, in the class $\mathcal{W} = \{v \in \mathcal{V} : v' \in \mathcal{V}'\} \subset C(0,T; L^2(\Omega))$;

(ii) $\tilde{h}$ can be written in the form $\tilde{h} = \tilde{u} + \tilde{a}$, where $\tilde{u} \in \mathcal{W}$ is the solution to the problem (6);

(iii) $\tilde{h}$ is unique in the sense that, if $a_1, a_2 \in C^1(\overline{Q})$ are two C^1 extensions of h_b and $\tilde{u}_1, \tilde{u}_2 \in \mathcal{W}$ are the corresponding solutions to (6), then $\tilde{h} = \tilde{u}_1 + \tilde{a}_1 = \tilde{u}_2 + \tilde{a}_2$;

(iv) $\tilde{h}$ can be obtained as the limit of its Galerkin approximations, that is, $\tilde{u}_n + \tilde{a} \to \tilde{u} + \tilde{a}$, as $n \to \infty$ weakly in $\mathcal{V}$ and strongly in $C(0,T; L^2(\Omega))$, where $\tilde{u}_n$ are the Galerkin approximations of the solution to (6);

(v) the mapping $(\mathbf{p}, \mathbf{d}) \mapsto \tilde{h}(\mathbf{p}, \mathbf{d})$ is continuous from $\mathcal{P} \times \mathcal{D}$ into $\mathcal{W}$. The analogous result holds for the Galerkin approximations of $\tilde{h}$.

3. Identification problem

We are interested in the identification problem for (4), namely from a given information on the piezometric height distribution h, we would like to find the parameter vector $\mathbf{p}$. We formulate the inverse problem as an optimal control one: given an observator operation $\mathcal{C}$ from the space $\mathcal{W}$ of solutions of (4) to a space of observations $\mathcal{Z}$ and functionals $\mathcal{F}$ on $\mathcal{Z}$ and $\mathcal{G}$ on $\mathcal{P}$, find an element $\mathbf{p}^* \in \mathcal{P}$ which solves the minimization problem

$$\inf_{\mathbf{p} \in \mathcal{P}} \{\mathcal{F}(\mathcal{C}h(\mathbf{p}, \mathbf{d})) + \mathcal{G}(\mathbf{p})\}. \tag{9}$$

In order to get the existence result for the inverse problem (9), we apply the direct method of the calculus of variations. We admit the following

Hypothesis (H3)

$$\begin{cases} (a) \quad \mathcal{C}: \mathcal{W} \to \mathcal{Z} \ \text{ is a continuous operator (w.r. to the strong topologies);} \\ (b) \quad \mathcal{F}: \mathcal{Z} \to \mathbb{R} \cup \{+\infty\} \ \text{ is l.s.c. on } \mathcal{Z}; \\ (c) \quad \mathcal{G}: \mathcal{P} \to \mathbb{R} \cup \{+\infty\} \ \text{ is l.s.c. on } \mathcal{P}. \end{cases}$$

Theorem 2. Under hypotheses (H1), (H2), (H3), the inverse problem (9) admits a solution. Analogous conclusion holds for the case where the exact solutions to (4) are replaced by their Galerkin approximations.

Proof. Let us fix $\mathbf{d} \in \mathcal{D}$ and let m denote the infimum in (9). Let $\{\mathbf{p}_i\}_{i \geq 1} \subset \mathcal{P}$ be a minimizing sequence for (9) i.e. $m = \lim_{i \to +\infty} \left\{\mathcal{F}(\mathcal{C}h(\mathbf{p}_i, \mathbf{d})) + \mathcal{G}(\mathbf{p}_i)\right\}$. Using the fact that the set of parameters $\mathcal{P}$ is a compact subset of $C(Q)$, we find a subsequence of $\{\mathbf{p}_i\}$ (denoted in the same way) and an element $\mathbf{p}^* \in \mathcal{P}$ such that $\mathbf{p}_i \to \mathbf{p}^*$ in $\mathcal{P}$, as $i \to \infty$. From Theorem 1(v) and (H3)(a), it follows that $\mathcal{C}h(\mathbf{p}_i, \mathbf{d}) \to \mathcal{C}h(\mathbf{p}^*, \mathbf{d})$ in $\mathcal{Z}$, as $i \to +\infty$. Hence, by (H3)(b)(c), we have $\mathcal{F}(\mathcal{C}h(\mathbf{p}^*, \mathbf{d})) \leq \liminf_{i \to +\infty} \mathcal{F}(\mathcal{C}h(\mathbf{p}_i, \mathbf{d}))$, $\mathcal{G}(\mathbf{p}^*) \leq \liminf_{i \to +\infty} \mathcal{G}(\mathbf{p}_i)$, and finally

$$m \leq \mathcal{F}(\mathcal{C}h(\mathbf{p}^*, \mathbf{d})) + \mathcal{G}(\mathbf{p}^*) \leq \liminf_{i \to +\infty} (\mathcal{F}(\mathcal{C}h(\mathbf{p}_i, \mathbf{d})) + \mathcal{G}(\mathbf{p}_i)) = m.$$

□

To illustrate the formulation of our inverse problem we present the following examples:

(j) Let $\mathcal{Z} = L^2(Q) \simeq L^2(0,T;L^2(\Omega))$ and let $\mathcal{C}: \mathcal{W} \to \mathcal{Z}$ be the embedding operator. Let the functional $\mathcal{F}: \mathcal{Z} \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$\mathcal{F}(z) = \int_Q F(t,x,z(t,x))\,\mathrm{d}x\mathrm{d}t \tag{10}$$

with the integrand $F: Q \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ such that F is $\mathcal{L}([0,T] \times \mathbb{R}^3) \otimes \mathcal{B}(\mathbb{R})$ measurable (where $\mathcal{L}$ = Lebesgue and $\mathcal{B}$ = Borel $\sigma-$ algebra), $F(t,x,\cdot)$ is l.s.c. on $\mathbb{R}$ for a.e. $(t,x) \in Q$ and $-\beta(t,x) + \gamma|z|^2 \leq F(t,x,z)$ for a.e. (t,x) and for every $z \in \mathbb{R}$ with some $\beta \in L^1(Q)$, $\gamma \in \mathbb{R}$.

Under the above hypotheses on F, we know (see e.g. [6, Example 1.21]) that $\mathcal{F}$ satisfies (H3)(b) with the strong topology on $\mathcal{Z}$. For instance

$$\begin{cases} F(t,x,z) = |z(t,x) - z_d(t,x)|^2 \rho(t,x), \\ \mathcal{G}(\mathbf{p}) = ||\mathbf{p} - \mathbf{p}_d||_{C(Q)}, \end{cases} \tag{11}$$

where z_d, $\mathbf{p}_d$ are the desired elements and ρ is a nonnegative continuous (density) function. The weight distribution ρ enables us to concentrate on the arbitrary part of the process.

(jj) Let $\mathcal{Z} = C(0,T;L^2(\Omega))$ and let $\mathcal{C}$ be the embedding operator. The performance index is given by the weighted least-squares functional $\mathcal{F}(z) = \sum_{i=1}^r \int_\Omega |z(t_i,x) - z_i^d(x)|^2 \rho_i(x)\,\mathrm{d}x$, where $0 < t_1 < t_2 < \ldots < t_r \leq T$ are points of measurements and $z_i^d \in L^2(\Omega)$ for $i = 1,\ldots,r$ are fixed targets. In particular (if $r = 1$, $t_1 = T$), it can be specified to $\mathcal{F}(z) = \int_\Omega |z(T,x) - z_d(T,x)|^2 \rho(x)\,\mathrm{d}x$.

(jjj) Let $\mathcal{Z} = L^2(\Sigma_2)$ and let $\mathcal{C}$ be the composition of the trace operator $\mathcal{W} \to L^2((0,T) \times \partial\Omega)$ and the restriction operator $L^2((0,T) \times \partial\Omega) \to L^2(\Sigma_2)$. As the functional $\mathcal{F}$ we can take $\mathcal{F}(z) = \int_{\Sigma_2} |z(t,x) - z_d(t,x)|^2 \tilde{\rho}(t,x)\,\mathrm{d}\sigma\mathrm{d}t$, where z_d represents the observations performed on the part of the boundary Σ_2.

In the case where the functional under infimum in (9) is continuous (i.e., for instance, if $\mathcal{F}$ and $\mathcal{G}$ are given by (10) and (11)), we can show (arguing as in [4, Theorem 3.1]) that the limit of any convergent subsequence of solutions of (9) with the Galerkin approximations for (4) is a solution of the inverse problem (9).

4. Sensitivity result

In practice, as a result of errors in modelling and observations we can meet perturbations in measured data and in the cost functionals. We present the theoretical result on the dependence of solutions to the inverse problem (9) on the data as well as on the cost functional. To this end, we consider the sequence, indexed by $k \in \mathbb{N} \cup \{\infty\}$ of the inverse problems

$$\inf_{\mathbf{p} \in \mathcal{P}} \{\mathcal{F}_k(\mathcal{C}h(\mathbf{p},\mathbf{d}_k)) + \mathcal{G}_k(\mathbf{p})\}, \tag{12}$$

where $h(\mathbf{p},\mathbf{d}_k)$ are solutions to (4) corresponding to the perturbed data $\mathbf{d}_k \in \mathcal{D}$ and $\mathcal{F}_k, \mathcal{G}_k$ are perturbed objective functionals. For every $k \in \mathbb{N} \cup \{\infty\}$, we denote by $\mathcal{S}_k$ the set of solutions to the inverse problem (12).

In order to formulate the next theorem, we need the notion of Γ-convergence and continuous convergence of functions. Given a sequence of functions from a metrizable topological space (X,τ) into $\overline{\mathbb{R}}$, we say that f_k sequentially Γ-converges to f and we write $f = \Gamma_{seq}(\tau - X)\liminf_k f_k$ if the following two statements hold:

(i) for every $x \in X$ and every $x_k \in X$, $x_k \xrightarrow{\tau} x$, we have $f(x) \leq \liminf_{k \to +\infty} f_k(x_k)$;

(ii) for every $x \in X$, there exists $x_k \xrightarrow{\tau} x$ such that $f(x) = \liminf_{k \to +\infty} f_k(x_k)$.

We say that f_k sequentially continuously converges to f and we write $f = C_{seq}(\tau - X)\lim_k f_k$, if for every $x \in X$ and for every $x_k \in X$, $x_k \xrightarrow{\tau} x$, we have $f(x) = \lim_{k\to+\infty} f_k(x_k)$. For a general definition of Γ–convergence in an arbitrary topological space, we refer to [2, 6].

Theorem 3. (The upper semicontinuity property of $\mathcal{S}_k$). If hypotheses (H1), (H2), (H3)(a) hold, $\mathcal{F}_k$, $\mathcal{G}_k$ satisfy (H3)(b)(c) uniformly with respect to $k \in \mathbb{N} \cup \{\infty\}$ and

$$\mathbf{d}_k \to \mathbf{d}_\infty \quad \text{in} \quad \mathcal{D},$$

$$\mathcal{F}_\infty = C_{seq}(\mathcal{Z}) \lim_k \mathcal{F}_k, \tag{13}$$

$$\mathcal{G}_\infty = \Gamma_{seq}(\mathcal{P}) \liminf_k \mathcal{G}_k, \tag{14}$$

(or $\mathcal{F}_\infty = \Gamma_{seq}(\mathcal{Z}) \liminf_k \mathcal{F}_k$ and $\mathcal{G}_\infty = C_{seq}(\mathcal{P}) \lim_k \mathcal{G}_k$ instead of (13) and (14)), then $\limsup_{k\to+\infty} \mathcal{S}_k \subseteq \mathcal{S}_\infty$, where $\limsup \mathcal{S}_k = \{\mathbf{p} \in \mathcal{P} : \mathbf{p} = \lim \mathbf{p}_{k_n}, \mathbf{p}_{k_n} \in \mathcal{S}_{k_n}, k_1 < k_2 < \ldots < k_n < \ldots\}$ stands for the sequential Kuratowski upper limit of the sets $\mathcal{S}_k$.

We say that the solution set for the inverse problem (9) is stable with respect to uncertain parameters if the thesis of Theorem 3 is satisfied. For a proof of Theorem 3, we refer to [11, 12]. We remark only that Theorem 2 implies that $\mathcal{S}_k$ are nonempty subsets of $\mathcal{P}$ for every k.

The convergences (13) and (14) hold, for instance, for sequences $\mathcal{F}_k(z) = ||(z - z_d^k)\rho||^2_{\mathcal{Z}}$, $\mathcal{G}_k(\mathbf{p}) = ||(\mathbf{p} - \mathbf{p}_d^k)\rho||^2_{\mathcal{P}}$, where $\{z_d^k\}$ and $\{\mathbf{p}_d^k\}$ are sequences of perturbed measurements (observations) performed in $\mathcal{Z}$ and perturbed preferable parameters, respectively. If $z_d^k \to z_d$ in $\mathcal{Z}$ and $\mathbf{p}_d^k \to \mathbf{p}_d$ in $\mathcal{P}$, as $k \to +\infty$, then (13) and (14) hold with $\mathcal{F}_\infty(z) = ||(z - z_d)\rho||^2_{\mathcal{Z}}$ and $\mathcal{G}_\infty(\mathbf{p}) = ||(\mathbf{p} - \mathbf{p}_d)\rho||^2_{\mathcal{P}}$. For three possible choices of the space of observations $\mathcal{Z}$, see examples (j)–(jjj) of Section 3.

5. Numerical implementation

As it was stated in [7], one of the methods of solving the problem (9) in an iterative way is to study the finite-dimensional approximation obtained by the finite element/finite difference technique (see (8) in Section 2 for detailed description):

$$\inf_{\mathbf{p}_{n\mathcal{T}} \in \mathcal{P}_{n\mathcal{T}}} \{\mathcal{F}_{n\mathcal{T}}(\mathcal{C}h_{n\mathcal{T}}(\mathbf{p}_{n\mathcal{T}}, \mathbf{d})) + \mathcal{G}(\mathbf{p}_{n\mathcal{T}})\}.$$

In this approach, the identification problem consists in finding the parameter vector $\mathbf{p}_{n\mathcal{T}}$ under the assumption that $\mathbf{p}_{n\mathcal{T}}$ is constant on some clusters of elements and the time intervals of the solution. It is adequate to physical observations that the components $p^i_{n\mathcal{T}}$ of $\mathbf{p}_{n\mathcal{T}}$ satisfy the constraints

$$0 < p^{iL}_{n\mathcal{T}} \le p^i_{n\mathcal{T}} \le p^{iU}_{n\mathcal{T}},$$

where $p^{iL}_{n\mathcal{T}}$, $p^{iU}_{n\mathcal{T}}$ are known lower and upper bounds, respectively.

We describe a computational example that concerns the problem of identification of the parameter $\mathbf{p}$. All computations are based on real measurements (observations) which were made at the experimental range of The Warsaw Agricultural University near Białośliwie (see [18]). At the same time, we were able to compare our numerical results with the values of the parameter $\mathbf{p}$ provided by laboratory methods (tests of soil patterns).

We consider one-dimensional vertical groundwater flow under a prismatic embankment (see Figure 3) founded on two organic strata peat and gytia. The organic soil is subjacent by a stable, preconsolidated and well permeable sand layer. The groundwater flow was caused by artesian pressure (measured in the bottom sand layer) and by deformations of organic layers caused by embankment weight. During the experiment the embankment was heightened and this was the main reason of intensification of the vertical flow of water. The area of the experiment was equipped

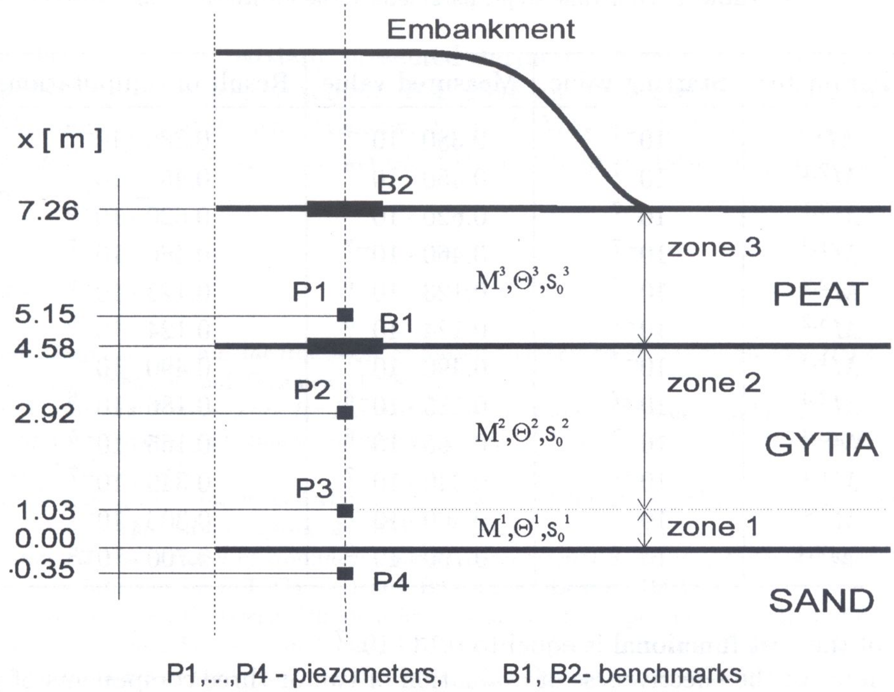

Fig. 3. Cross-section through consolidating layers

with several piezometers and benchmarks. Measurements of pressure in these piezometers as well as benchmarks and piezometer displacements were used for calculating the volume strain velocity.

For the numerical simulations we have used the data from the piezometers P1–P4 and the benchmarks B1, B2 which were situated as it is shown in Figure 3. In this way the filtration area was divided into three zones: the first one between the bottom of the gytia layer and the piezometer P3, the second one between the piezometer P3 and the back of the gytia layer, and the third one which occupied all the saturated parts of the peat.

In the numerical algorithm we have supposed that the cost functional is of the form

$$\mathcal{F}(\mathbf{p}) = \sum_{j=1}^{N} \sum_{i=1}^{3} (h_d^{ij} - h^{ij}(\mathbf{p}))^2,$$

where h_d^{ij} and $h^{ij}(\mathbf{p})$, $i = 1, 2, 3$, $j = 1, \ldots, N$ are, respectively, measured and computed values of the piezometric height, $\mathcal{G} \equiv 0$ and $N = 1, \ldots, 4$ denotes the number of time steps. So we look for the parameter $\tilde{p} = (M, s_0, \theta)$ which is assumed to be constant in each of the three zones mentioned above and in each time step. These conditions lead to a nonlinear optimization problem with $9N$ parameters $(M^{ij}, s_0^{ij}, \theta^{ij})$, which are to be identified.

The results obtained from scalar implementation of standard nonlinear programming methods applied in MINUIT package (from CERN, Geneva) exhibit that there are problems with effective computation of $\mathbf{p}$. The results strongly depend on the starting points, there are local minima, sometimes there is no convergence at all. For these methods the satisfactory results were gained in the identification of one component of $\mathbf{p}$ (the other being fixed). The mentioned methods can be used effectively for many time steps (for instance in a long period of the consolidation process in dams). In the case where the components of $\mathbf{p}$ do not change rapidly (for s_0 and θ this agrees with observations), their initial values (obtained from measurements during construction of the object) give a good starting point for the computations.

The example of identification of the parameter M is shown in Table 1.

Table 1. Four time steps, parameter to be identified — M

Parameter	Starting value	Measured value	Result of computations
$M^{1,1}$	10^{-7}	$0.380 \cdot 10^{-7}$	$0.380 \cdot 10^{-7}$
$M^{2,1}$	10^{-7}	$0.450 \cdot 10^{-7}$	$0.450 \cdot 10^{-7}$
$M^{3,1}$	10^{-7}	$0.620 \cdot 10^{-6}$	$0.620 \cdot 10^{-6}$
$M^{1,2}$	10^{-7}	$0.460 \cdot 10^{-7}$	$0.460 \cdot 10^{-7}$
$M^{2,2}$	10^{-7}	$0.123 \cdot 10^{-6}$	$0.123 \cdot 10^{-6}$
$M^{3,2}$	10^{-7}	$0.124 \cdot 10^{-5}$	$0.124 \cdot 10^{-5}$
$M^{1,3}$	10^{-7}	$0.490 \cdot 10^{-7}$	$0.490 \cdot 10^{-7}$
$M^{2,3}$	10^{-7}	$0.185 \cdot 10^{-6}$	$0.186 \cdot 10^{-6}$
$M^{3,3}$	10^{-7}	$0.165 \cdot 10^{-5}$	$0.165 \cdot 10^{-5}$
$M^{1,4}$	10^{-7}	$0.340 \cdot 10^{-7}$	$0.340 \cdot 10^{-7}$
$M^{2,4}$	10^{-7}	$0.300 \cdot 10^{-7}$	$0.300 \cdot 10^{-7}$
$M^{3,4}$	10^{-7}	$0.700 \cdot 10^{-6}$	$0.700 \cdot 10^{-6}$

Final value of the cost functional is equal to $0.13 \cdot 10^{-7}$.

In order to improve the effectiveness of evaluation of two or three components of $\mathbf{p}$ the authors propose to apply parallel and distributed computations in algorithms with hierarchic decomposition of the initial problem and several parallel processes starting with different initial points. The algorithms are described in [16].

An example of the so–called hierarchic optimization for identification of parameters M and θ is presented in Table 2.

Table 2. One time step, parameters to be identified — M and θ

Parameter	Starting value	Measured value	Result of computations
$M^{1,1}$	$0.5 \cdot 10^{-7}$	$0.38 \cdot 10^{-7}$	$0.38004 \cdot 10^{-7}$
$M^{2,1}$	$0.5 \cdot 10^{-7}$	$0.45 \cdot 10^{-7}$	$0.44996 \cdot 10^{-7}$
$M^{3,1}$	$0.5 \cdot 10^{-7}$	$0.62 \cdot 10^{-6}$	$0.62001 \cdot 10^{-6}$
$\theta^{1,1}$	10^{-7}	0.86	0.86010
$\theta^{2,1}$	10^{-7}	0.86	0.85995
$\theta^{3,1}$	10^{-7}	0.80	0.80000

Final value of the cost functional is equal to $2.71 \cdot 10^{-6}$.

6. Conclusions

The behavior of earthen dams strongly depends on the nonlinear filtration process in its seal. The identification of the hydraulic parameters and their updating is essential for the stability monitoring. The inverse analysis is especially important in the initial period of exploitation in which all physical parameters can change due to the consolidation.

The problem of identification of parameters in the nonlinear filtration model (which takes into account a new constitutive formula (2)) is solvable and its solutions are stable with respect to the uncertain parameters (see the remark following Theorem 3).

The inverse problems for the prelinear filtration can be effectively solved by the hierarchic optimization combined with FEM/FDM schemes (*cf.* Section 5 and [16]).

ACKNOWLEDGEMENTS

The results of the paper were partially presented during the Second International Symposium on Inverse Problems, Paris, France, November 2–4, 1994 and the Twelfth Conference on Computer Methods in Mechanics, Zegrze near Warsaw, Poland, May 9–13, 1995. We would like to thank graduate students P. Gołuch and R. Olewiński for their help in data collection and the referee for valuable suggestions.

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