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Semivariational numerical model of prelinear filtration with the special emphasis on nonlinear sources

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The initial–boundary value problem for quasi–linear parabolic equation with distribution coefficient modelling the nonlinear filtration is discussed. The presented results constitute extension of the earlier works of the authors concerning the filtration problem in the domains without sources to the case of filtration in the presence of sources.

1. Introduction

The purpose of the present paper is to present a new approach to the study of nonlinear filtration with point sources. Our approach is based on the new model of nonlinear filtration (*cf.* [17]), in which traditional constitutive formulae have been replaced by more adequate ones giving more accurate description of the filtration phenomenon in the range of low velocities. Moreover, we propose the new treatment of the point sources, i.e. sources having small geometrical size compared with the filtration domain.

The problem of filtration is old and has been treated by numerous authors, see e.g. works [1, 2, 20, 21, 23] and references therein.

The papers [1, 23] discuss general physical setting of the filtration problem. In works [17, 19] the new constitutive formula is proposed, the case when functions describing physical parameters of the system are not regular (e.g. are square summable) is also considered. The problem of point sources has not been discussed as yet.

The new treatment of point sources consists in modelling the sources operation by the use of Dirac measures. This leads to the equation with distribution coefficients. Equations of such kind have been used for modelling the vibrations of mechanical systems containing concentrated masses, see e.g. [8, 14], but their application to the study of filtration problems seems to be new.

The paper is organized as follows. The second paragraph contains discussion of various constitutive formulae used in the nonlinear filtration, and in the fourth paragraph the situations in which point sources appear are described. In the fourth one the theoretical result justifying approach based on new constitutive formula is presented. In the fifth paragraph we discuss mathematical justification of a model with point sources. The last three paragraphs contain the description of the numerical method for solving filtration equations in the presence of point sources, software for prelinear filtration problem, and the computational example.

2. Prelinear filtration — local modelling

The widely accepted constitutive filtration law, established by H. Darcy in 1858, has the linear form

$$v = k\, s, \tag{1}$$

where v denotes the filtration velocity, k is the (Darcy's) coefficient of permability, s stands for the hydraulic slope vector. In addition, it is assumed that $s = \operatorname{grad} h$, h being the piezometric height distribution. Darcy's formula remains valid for medium grained, moderately permeable soils, and in the middle range of hydraulic slopes.

Significant discrepancies begin to appear in the case of fine grained (more than 5% particles with diameters less than 5μ), cohesive or organic soils. In such a case the filtration velocity obeys exponential law in the lower range of slopes, $s \in [0, s_0]$ and then tends asymptotically to the straight line $v = M(s - s_0)$ relation.

The precise modelling of the phenomenon described above, the *prelinear filtration* phenomenon, plays the crucial role in the earth dam stability analysis as well as in the designing of the boggy terrain drainages.

There are some well known formulae describing prelinear filtration, such as threshold formula (*cf.* [10]), Hansbo power relationship (see Hansbo [7]) or the three parameter Swartzendruber's formula (see e.g. [20, 21])

$$v = M[s - s_0(1 - \exp(-\theta s/s_0))] \tag{2}$$

in which M is the "asymptotic Darcy coefficient", having the same dimension as Darcy coefficient, s_0 and θ are dimensionless constants: s_0 is the analogue of the threshold gradient and θ is the so called "nonlinearity index" (compare Fig. 1).

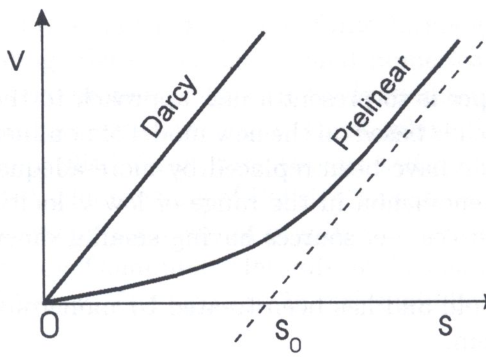

Fig. 1. Filtration velocity versus hydraulic slope

The latter seems to be very convenient since it permits to describe the filtration process in a wider range of slopes.

The above mentioned formulae have been derived for one dimensional flows. Their major drawback lies in the fact that they cannot be simply extended to multidimensional cases without the loss of accuracy characteristic for the one dimensional prelinear filtration case, and preserving the invariance with respect to rotations of the coordinate system.

The first attempt to overcome the above difficulty has been made by Swartzendruber in [21], where the nonlinear part of (2) has been replaced by its Taylor expansion up to terms of second order

$$v = k(1 + D|s|)s, \quad k = M(1 - s_0\theta), \quad D = \frac{s_0\theta^2}{2(1 - s_0\theta)}. \tag{3}$$

$|s|$ denotes the norm of the slope vector, which is introduced in place of scalar value used in the one dimensional case. The above formula is valid only for small values of slopes, and it lacks the nice asymptotic behaviour exhibited by the one dimensional Swartzendruber law.

The first of the authors (see [17]) has proposed the new well conditioned constitutive formula of the real prelinear filtration remaining in agreement with both local and asymptotic behaviour, and being invariant with respect to rotations of coordinate system:

$$v_i(t,x,Dh) = \varphi(t,x,|Dh|)\frac{\partial h}{\partial x_i}, \quad i = 1,2,3, \tag{4}$$

with φ given by

$$\varphi(t,x,r) = \begin{cases} M\left(1 - \frac{s_0}{r}\left(1 - \exp\left(-\frac{\theta r}{s_0}\right)\right)\right), & \text{for} \quad r > E \\ \left[\frac{M}{E^2}\left(s_0 - (s_0 + \theta E)\exp\left(-\frac{\theta E}{s_0}\right)\right)\right] r & \\ \qquad + M\left(1 - \frac{2s_0}{E} + \left(\frac{2s_0}{E} + \theta\right)\exp\left(-\frac{\theta E}{s_0}\right)\right), & \text{for} \quad E \geq r \geq 0, \end{cases} \tag{5}$$

where $Dh = \operatorname{grad} h$, and $|Dh| = (Dh|Dh)^{1/2}$, $(\cdot|\cdot)$ stands for the usual scalar product in $\mathbb{R}^3$ (or generally, in $\mathbb{R}^n$).

The constant E appearing in (5) depends on the floating point arithmetic accuracy and on the features of the current soil pattern. Positive parameters M, s_0, θ, not necessarily constant, satisfy conditions below

$$0 < \theta < 1, \quad \frac{M}{E^2}\left(s_0 - (s_0 + \theta E)\exp\left(-\frac{\theta E}{s_0}\right)\right) > 0$$

resulting from the physical nature of the considered phenomenon and they are assumed to be enough regular in respect of arguments t, x, such that φ is C^1 in t, C^0 in x and C^1 in respect of the last argument.

The formula above has been described in details in [17], and thoroughly tested in the case of consolidated organic layers (see [25]), and in the case of filtration through the earth dams.

The local mass balance leads to the following initial–boundary value problem for the piezometric height distribution $h(x,t)$ governing the filtration process in the case of deformable skeleton, provided strains and displacements are small:

$$\beta p(t,x)\frac{\partial h}{\partial t} = \sum_{i=1}^{3}\frac{\partial}{\partial x_i}v_i(t,x,Dh) + Q(t,x) \qquad \text{for} \quad (t,x) \in (0,T] \times \Omega, \tag{6}$$

$$h(t,x) = h_b(t,x) \qquad \text{for} \quad (t,x) \in (0,T] \times \partial\Omega_1, \tag{7}$$

$$\sum_{i=1}^{3} n_i\, v_i(t,x,Dh) = q_b(t,x) \qquad \text{for} \quad (t,x) \in (0,T] \times \partial\Omega_2,$$

$$h(0,x) = h_0(x) \qquad \text{for} \quad x \in \Omega. \tag{8}$$

In the equations above Ω is the domain of filtration (e.g. it represents an earth dam protection screen), T is the time of filtration, $n = (n_1, n_2, n_3)$ denotes the unit vector, orthogonal to the part $\partial\Omega_2$ of the boundary $\partial\Omega$, $\beta > 0$ is a coefficient of the water compressibility, $Q = Q_1 + Q_2$, while functions Q_1, Q_2, q_b, p, h_b, h_0 represent the yield of sources, the volume strain velocity of the skeleton, the boundary flux, the soil porosity, the piezometric height on $\partial\Omega_1$, the initial height distribution, respectively.

Remark. The mixed problem (6)–(8) constitutes the part of the initial–boundary value problem of Biot's type concerning consolidating of soil (see Veruit [24]). Observe that in the case when the volume strain velocity is known, the Biot's system reduces to the problem considered here.

Note that v_i is nondifferentiable at $Dh = 0$, so the C^2-regularity of h cannot be expected, which implies that the solutions to (6)–(8) should be understood in the weak sense. Correspondingly, derivatives of h appearing in (4) are also understood in the distributional sense.

3. Nonlinear sources in consolidated organic layers

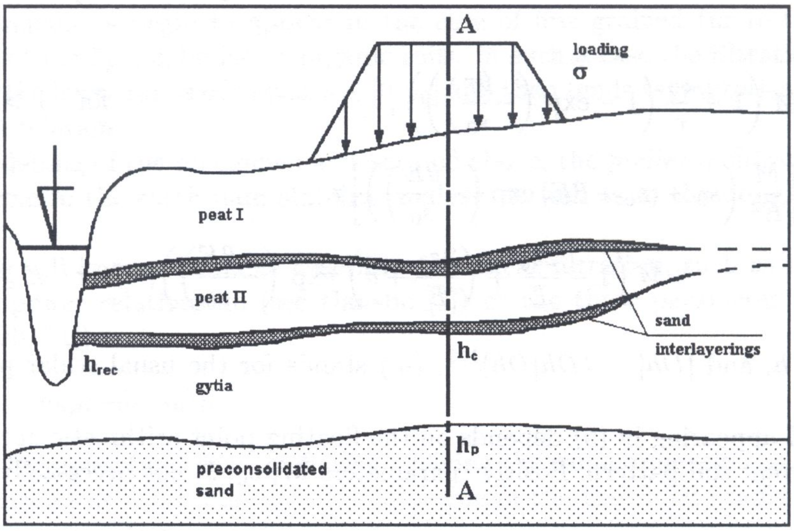

Fig. 2. Cross-section through the consolidated organic layers

The drainage flow through river banks composed of organic sediments horizontally interlayered with thin sand strata as depicted in Fig. 2, presents an important example of the prelinear filtration in the presence of sources.

Each interlayering, being much more permeable than the surrounding organic soil, forms a two-dimensional conductor transmitting water to the recipient. Consider the situation, when the part of organic layers is loaded from above by the massive structure, e.g. the flood protective wall. The movement of the water is stimulated by the skeleton deformation (consolidation), the artesian pressure in the interlayering, and the stable sand strata.

In a simplified model containing single sand interlayering, as presented in Fig. 2, the percolation of water in the cross-section **A–A** can be considered as a one-dimensional filtration problem through the domain being the segment $[0,l]$ of the real line in the presence of the negative source situated at the point $x_0 \in [0,l]$. In this case we set $\Omega =]0,l[$, $h(0,t) = r_0(t)$, $h(l,t) = r_l(t)$ for $t \in [0,T]$, $h(x,0) = h_0(x)$ for $x \in \Omega$. In addition, we assume compatibility conditions: $h_0(0) = r_0(0)$, $h_0(l) = r_l(0)$.

Since the cross-section of the sand interlayering is small in comparison with the length of the considered cross-section, it can be assumed that its action on the system is concentrated at a single point, and thus is modelled by the Dirac measure: $q(h)\delta_{x_0}$, where $\delta_{x_0} = \delta(x - x_0)$ denotes the Dirac distribution concentrated at x_0. The expression $q(h)$ describes the discharge intensity.

We will denote by h_c the piezometric pressure under the center of the loaded area (cross-section **A–A**) and by h_{rec} the piezometric height in recipient. If the transmission flow belongs to the laminar range ($h_c < h_{lam}$) then its discharge intensity g is assumed to be proportional to $h_c - h_{rec}$. In the case of turbulent flow ($h_c > h_{lam}$), q grows like $\sqrt{h_c - h_{rec}}$. Thus, the formula taking into account both cases takes the form

$$q(h_c) = \begin{cases} 0 & \text{for } h_c < h_{rec}, \\ a_1(h_c - h_{rec}) & \text{for } h_{rec} \le h_c < h_{lam}, \\ a_2\sqrt{h_c - h_{rec}} & \text{for } h_{lam} \le h_c \le h_{max}, \\ a_2\sqrt{h_{max} - h_{rec}} & \text{for } h_{max} < h_c. \end{cases} \tag{9}$$

The positive constants a_1, a_2, describe the hydraulic characteristics of the interlayering and are chosen in such a way, that q is continuous for all $h_c \in \mathbb{R}$.

The lack of discharge for h_c less than h_{rec} is justified on physical grounds, while the extension of q over h_{max} is only formal. The h_{max} is the maximal piezometric pressure, which can appear in the loading part of the strata as a result of summing the hydrostatic pressure, artesian pressure h_p and the upper loading σ.

4. Mathematical justification of the variational model

Let $L^2(\Omega)$ denote, as usual, the space of square summable functions defined on Ω and let $H^1(\Omega)$ be the Sobolev space of order 1 over Ω: $H^1(\Omega) = \{u : u, D_i u \in L^2(\Omega), i = 1,2,3\}$ ($D_i u$ stands for the distributional derivative of u with respect to x_i). Denote by $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{H^1(\Omega)}$ the norms in $L^2(\Omega)$ and $H^1(\Omega)$, respectively. Finally let $V = \{u \in H^1(\Omega) : \gamma_1 u = 0\}$, where $\gamma_1 u$ is the trace of u on $\partial\Omega_1$ (γ_1 is an extension of the restriction operator $u \mapsto u|_{\partial\Omega_1}$ on $H^1(\Omega)$ (for details consult, e.g. [6])).

Function $h : [0,T] \times \overline{\Omega} \to \mathbb{R}$ is said to be a *weak solution* to (6)–(8), if for any $w \in V$ it satisfies identicaly the equation (the variational form of (6)–(8))

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega p(t,x)w(x)\mathrm{d}x &= -\int_\Omega \varphi(t,x,|Dh(t,x)|)\,(Dh(t,x)|Dw(x))\,\mathrm{d}x \\ &\quad + \int_\Omega (Q(t,x))\,w(x)\,\mathrm{d}x + \int_{\partial\Omega_2} q_b(t,x)w(x)\,\mathrm{d}\sigma, \\ \gamma_1 h(t,\cdot) = h_0, \quad t \in (0,T], &\quad h(0,\cdot) = h_0. \end{aligned}\right\} \tag{10}$$

It is assumed that the functions appearing as coefficients in (10) are regular enough for the formula above to make sense.

Theorem below, proved in [19], justifies the well-posedness of the proposed variational model of prelinear filtration.

Theorem 1. *Suppose* (4) *and* (5) *hold. Let* $p(t,x) = \alpha(t)e(x)$, ($\alpha(t) > 0$, $0 < e_0 \le e(x) \le e_0$), *let* Q, q_b, e, h_0, $\alpha'(t)$ *be continuous in their domains and let* Ω *have sufficiently regular (e.g. of class* C^1*) boundary. If* h_b *is continuous, then* (6)–(8) *has the unique weak solution depending continuously on initial data.*

5. Nonlinear sources

The problem with nonlinear sources differs from the classical one by the presence in (6), in addition to $Q(t,x)$, the terms containing distributional coefficients, i.e. the expression $\sum_{k=1}^{r} q_k(h)\delta(x - x_k)$ representing the yield of point sources located at points $x_k \in \Omega$.

For the sake of simplicity we will consider the system with one negative source and without boundary supplies ($q_b(t,0) = q_b(t,l) \equiv 0$). From the proof below it will be clear that our considerations remain valid also in the general case.

In addition to definitions given in paragraph 4, let us introduce the following notations.

For a Banach space B, denote by $\|\cdot\|_B$ its norm, by B' its dual and by $\langle g, \xi\rangle$ ($g \in B'$, $\xi \in B$) the duality between B and B'.

$C^k(S;B)$ denotes the space of functions defined on S with values in B, k times continuously differentiable. $\|\cdot\|_{C^0(S;B)}$ denotes the supremum norm in $C^0(S;B)$, i.e. the norm of the uniform convergence on S.

Considering the case $\Omega =]0,l[$, observe that $V = H_0^1(\Omega) = \{u \in H^1(\Omega) : u(0) = u(l) = 0\}$. Note that for $\Omega \subset \mathbb{R}$ functions $u(0)$ and $u(l)$ are well defined, see *Remark 1* below. The norm $\|\cdot\|_{V}$. is defined by $\|u\|_V = \|u'\|_{L^2(\Omega)}$. Note that $\|u\|_V$ is equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$:

$$c_1\|u\|_{H^1(\Omega)} \le \|u\|_V \le \|u\|_{H^1(\Omega)}, \tag{11}$$

where $c_1 = (1+4l^2/\pi^2)^{-1/2}$ ([12, Th. 2.3.4]). For $g, \xi \in L^2(\Omega)$ the duality $\langle g, \xi\rangle$ reduces to the usual scalar product $(g,\xi) = (g,\xi)_{L^2(\Omega)}$ in $L^2(\Omega)$. Recall that for $v, w \in C^0(]0,l[;\mathbb{R})$ and Dirac measure δ concentrated at $x_0 \in]0,l[$, $\langle v\delta, w\rangle = v(x_0)w(x_0)$.

Let $X = L^2(0,T;V)$, so its dual is $X' = L^2(0,T;V')$ and $\langle\langle f,u\rangle\rangle = \int_0^T \langle f(s), u(s)\rangle\,\mathrm{d}s$ is the scalar product of $f \in X'$ and $u \in X$. The $\frac{\mathrm{d}u}{\mathrm{d}t} = u'$ denotes the time derivative of u in the sense of scalar distributions on $[0,T]$. Define $W = \{u : u \in X, u' \in X'\}$. For $u, v \in W$ the formula of the integration by parts holds ([6, Ch. IV, Th.1.17]).

Now (10) can be stated as follows:
find a function $h \in L^2(0,T;H^1(\Omega))$ satisfying for all $w \in V$ the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(eh, w) = -b(t,h,h,w) + f(t,w) + \langle q(h)\delta, w\rangle \quad \text{for } t \in]0,T], \tag{12}$$

the boundary conditions $h(t,0) = r_0(t)$, $h(t,l) = r_l(t)$ $(t \in (0,T])$ and the initial condition $h(0,x) = h_0(x)$, $x \in]0,l[$. The mappings $b : [0,T] \times (H^1(\Omega))^3 \to \mathbb{R}$, $f \in L^2(0,T;V')$, are given by expressions

$$b(t,u,v,z) = \frac{1}{\beta\,\alpha(t)} \int_0^l \varphi\left(t,x,\left|\frac{\partial u}{\partial x}\right|\right) \frac{\partial v}{\partial x}\frac{\partial z}{\partial x}\,\mathrm{d}x, \tag{13}$$

$$\langle f(t), z\rangle = f(t,z) = \frac{1}{\beta\,\alpha(t)} \int_0^l Q(t,x)z(x)\,\mathrm{d}x. \tag{14}$$

It will be convenient to formulate the problem above as the initial value problem for the operator equation. For this sake, introduce operators $B : [0,T] \times (H^1(\Omega))^2 \to V'$, $G : H^1(\Omega) \to V'$, and $C : V' \to V'$:

$$\langle Cr, u\rangle = \langle er, u\rangle = \langle r, eu\rangle, \qquad u \in V, \quad r \in V', \tag{15}$$

$$\langle B(t,u,v), w\rangle = b(t,u,v,w) \qquad \text{for} \quad w \in H^1(\Omega), \tag{16}$$

$$\langle G(u), w\rangle = \langle q(u)\delta, w\rangle, \qquad w \in H^1(\Omega) \tag{17}$$

and set $B(t,h) = B(t,h,h)$.

Remark 1. For $\Omega =]0,l[$ the Sobolev embedding theorem implies that $H^1(\Omega) \subset C^0([0,l];\mathbb{R})$, from which it follows that G is well defined. Moreover the inclusion map is continuous:

$$\|u\|_{C^0([0,l];\mathbb{R})} \leq c_0 \|u\|_{H^1(\Omega)}, \quad u \in H^1(\Omega) \tag{18}$$

with $c_0 = (\tanh l)^{-1/2}$ (see [12, Remark 2.2.1]).

The change of variable $h(t,x) = a(t,x) + u(t,x)$ ($a(t,x) = r_0(t) + x(r_l(t) - r_0(t))/l$) transforms the problem (12), $h(t,0) = r_0(t)$, $h(t,l) = r_l(t)$, $h(0,x) = h_0(x)$ into an equivalent problem with homogeneous boundary conditions:
Given $f \in L^2(0,T;V')$, find u satisfying

$$C\frac{\mathrm{d}u}{\mathrm{d}t} + B(t,u+a(t)) = G(u+a(t)) + f_1(t), \tag{19}$$

$$u(0) = h_1, \quad u \in W, \tag{20}$$

where $a(t) = a(t,\cdot)$, $f_1(t) = f(t) - C(\partial/\partial t)a(t)$ and $h_1 = h_0 - a(0)$.

Note that $W \subset C^0([0,T];L^2(\Omega))$, hence the first condition in (20) is meaningful.

It can be proved (*cf.* [19]) that (4) and conditions imposed on $\alpha(t)$ and $e(x)$ in Theorem 1 imply that the operator B is Lipschitz continuous, strongly monotone and coercive, the operator G is bounded, Lipschitz continuous and monotone, i.e.:

$$\|B(t,u+a(t)) - B(t,v+a(t))\|_{V'} \leq M_1\|u-v\|_V, \tag{i}$$

$$\langle B(t,u+a(t)) - B(t,v+a(t)), u-v\rangle \geq m\|u-v\|_V^2, \tag{ii}$$

$$\langle B(t, u+a(t)), u\rangle \geq \xi(\|u\|_V)\,\|u\|_V, \quad \xi(s) \to \infty \quad \text{as} \quad s \to \infty. \tag{iii}$$

$$\|G(t, u+a(t))\|_{V'} \leq a_2\sqrt{h_{max} - h_{rec}}, \tag{iv}$$

$$\|G(t, u+a(t)) - G(t, v+a(t))\|_{V'} \leq a_1\|u-v\|_V, \tag{v}$$

$$\langle G(t, u+a(t)) - G(t, v+a(t)), u-v\rangle \geq 0. \tag{vi}$$

all inequalities hold for any $u, v \in V$, uniformly in $t \in [0,T]$. Moreover

$$e_0\|u-v\|^2_{L^2(\Omega)} \leq \langle C(u-v), u-v\rangle \leq e^0\|u-v\|^2_{L^2(\Omega)} \quad \text{for} \quad u, v \in L^2(\Omega), \tag{vii}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Cu, u\rangle = 2\langle Cu', u\rangle \quad \text{for} \quad u \in W. \tag{viii}$$

Now, we are in a position to state and prove the main result of the paper.

Theorem 2. *Assume conditions of Theorem 1. Let $G = q_b \equiv 0$. Let c_0, c_1 be defined as above. If constants m, a_1 satisfy $m\,c_1 \geq c_0\,a_1$, then for any $h_0 \in L^2(\Omega)$ and $r_0, r_l \in C^1([0,T];\mathbb{R})$ problem (6), (7), (8) with $Q(t,x)$ replaced by $Q(t,x) + q(h)\delta$ has exactly one weak solution h,*

$$h \in L^2(0,T; H^1(\Omega)) \cap C^0([0,T]; L^2(\Omega)) \tag{21}$$

depending continuously on initial data.

Proof. It is clear that in order to show the theorem it suffices to prove, that the problem (19), (20) has the unique solution u depending continuously on the initial data. The proof is based on the classical Faedo–Galerkin method combined with monotonicity and compactness techniques (compare [8, Ch.VI], [11, Ch.II]). As it will be seen such an approach is acceptable in case of the distribution operator equation considered here.

Denote by $\{w_n\}$ a countable basis of V (e.g. the finite element basis) and denote by V_n the subspace of V spanned on $w_1, w_2, \ldots, w_n$. Choose $\{u_{0n}\}$ such that $u_{0n} \in V_n$, $u_{0n} \to h_1$ in V as $n \to \infty$. Let $u_n(t) = \sum_{j=1}^n g_{jn}(t)\,w_j$, where $g_{jn}(t)$ are so defined that $u_n(t)$ is a solution of the initial value problem

$$(eu_n', w_i) = -b(t, u_n + a(t), u_n + a(t), w_i) + \langle f_1(t) + G(u_n + a(t)), w_i\rangle, \tag{22}$$

$$u_n(0) = u_{0n}. \tag{23}$$

Correspondingly, n-vector g_n with components $g_{jn}(t)$, $j = 1, \ldots, n$ solves the problem (the Galerkin approximation of (19))

$$C_n g_n' + B_n(t, g_n)g_n + r_n(t, g_n) = f_{1n}(t), \tag{24}$$

$$g_n(0) = d_n, \tag{25}$$

where C_n, $B_n(t, g_n)$ are $n\times n$-matrices with entries (ew_i, w_j), $b(t, u_n + a(t), w_j, w_i)$. The $f_{1n}(t)$, $r_n(t, g_n)$ are n-vectors with components $f_1(t, w_i) - \frac{\mathrm{d}}{\mathrm{d}t}(ea(t), w_i)$, $b(t, u_n + a(t), a(t), w_i) - \langle G(u_n + a(t)), w_i\rangle$. The components d_{jn} of d_n are defined by $\sum_{j=1}^n d_{jn}\,w_j = u_{0n}$).

Obviously, matrix C_n is invertible. The $B_n(t, g_n)$ and $r_n(t, g_n)$ are continuous in t and by (i), (iv), (v), they are bounded and Lipschitz continuous with respect to g_{in}, hence the initial value problem (24), (25) is uniquely solvable with solution defined for all $t \in [0,T]$ and the same holds true for (22), (23).

Applying to the formula

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle Cu_n(t), u_n(t)\rangle + \langle B(t, u_n(t) + a(t)), u_n(t)\rangle - \langle B(t, a(t)), u_n(t)\rangle$$
$$= \langle f_1(t) + G(u_n(t) + a(t)) - B(t, a(t)), u_n(t)\rangle.$$

inequalities $\langle f_1(t) - B(t,a(t)), u_n(t)\rangle \le c_2\|u_n\|_V$, $\langle G(u_n + a(t)), u_n\rangle \le a_2\sqrt{h_{max} - h_{rec}}\|u_n\|_V$ (c_2 is a suitable constant; the last inequality results from (iv)), and using (ii) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle Cu_n(t), u_n(t)\rangle + m\|u_n\|_V^2$$

$$\le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle Cu_n(t), u_n(t)\rangle + \langle B(t, u_n(t) + a(t)), u_n(t)\rangle - \langle B(t, a(t)), u_n(t)\rangle$$

$$\le (a_2\sqrt{h_{max} - h_{rec}} + c_2)\|u_n\|_V$$

from which, by (vii), using reasoning similar to this in [6, Ch.VI] we conclude that the sequence $\{u_n\}$ is bounded in X and in $C^0([0,T]; L^2(\Omega))$. This in turn implies, that the sequences $\{\langle\langle B(\cdot, u_n + a), u_n\rangle\rangle\}$ and $\{\langle\langle G(u_n + a), u_n\rangle\rangle\}$, are bounded, which by the monotonicity of B and G, implies that $\{B(\cdot, u_n + a)\}$ and $\{G(u_n + a)\}$ are bounded in X'. In consequence (*cf.* [6, Ch. VI, Lemma 1.4]) there is a subsequence $\{u_\mu\} \subset \{u_n\}$ such that

$$u_\mu \to u \qquad \text{weakly in } X, \tag{j}$$

$$u_\mu(T) \to z \qquad \text{weakly in } L^2(\Omega) \tag{jj}$$

$$B(\cdot, u_\mu + a(\cdot)) \to v \qquad \text{weakly in } X' \tag{jjj}$$

$$G(\cdot, u_\mu + a(\cdot)) \to w \qquad \text{weakly in } X' \tag{jv}$$

and the limits defined above satisfy

$$u \in X, \quad u(0) = h_1, \quad u(T) = z, \quad Cu' + v = f_1 + w. \tag{v}$$

By Remark 1, (j) and (jv) imply, that

$$\langle G(u_\mu + a(\cdot)), u_\mu\rangle \to \langle G(u + a(\cdot)), u\rangle = \langle w, u\rangle.$$

Again from the monotonicity of B and the formula

$$\limsup\langle\langle B(\cdot, a(\cdot) + u_\mu), u_\mu\rangle\rangle$$

$$= \limsup\Big(\langle\langle f_1 + G(a(\cdot) + u_\mu), u_\mu\rangle\rangle + \tfrac{1}{2}(\langle Ca_\mu, a_\mu\rangle - \langle Cu_\mu(T), u_\mu(T)\rangle)\Big)$$

$$\le \langle\langle f_1 + w, u\rangle\rangle + \tfrac{1}{2}(\langle Ca, a\rangle - \langle Cz, z\rangle) = \langle\langle f_1 + w, u\rangle\rangle - \langle\langle Cu', u\rangle\rangle = \langle\langle v, u\rangle\rangle$$

we have (compare [6, Ch. III, Lemma 1.3, 11, Ch. II]) $v = B(\cdot, u + a(\cdot))$, so u satisfies (19). From conditions $u \in W$ and (v) it follows, that u satisfies also (20). Clearly, $h = a + u$ is the weak solution of (6), (7), (8). If h_1, h_2 are solutions to (19), (20), then by (ii), (v) and the inequality imposed on m, a_1, a_2, we get

$$\tfrac{1}{2}(\langle C(h_1(t) - h_2(t)), h_1(t) - h_2(t)\rangle - \langle C(h_1(0) - h_2(0)), h_1(0) - h_2(0)\rangle)$$

$$= \int_0^t \langle C(h_1(s) - h_2(s))', h_1(s) - h_2(s)\rangle\,\mathrm{d}s$$

$$= -\int_0^t \langle B(s, h_1) - B(s, h_2), h_1 - h_2\rangle\,\mathrm{d}s + \int_0^t \langle G(h_1) - G(h_2), h_1 - h_2\rangle\,\mathrm{d}s$$

$$\le \int_0^t (-m\,c_1 + c_0\,a_1)\|h_1(s) - h_2(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s \le 0,$$

which by (vii) implies finally

$$e_0\|h_1(t) - h_2(t)\|^2_{L^2(\Omega)} \le e^0\|h_1(0) - h_2(0)\|^2_{L^2(\Omega)},$$

proving the uniqueness and the continuous dependence of the solution on initial data.

6. APPROXIMATE SOLUTIONS

The Galerkin system (24), (25) solved numerically provides the approximate solution of the problem (6)–(8). The Galerkin equations will be solved using difference methods. For this sake write (24), (25) in the form

$$\left.\begin{aligned} C_n g_n' + A_n(t, g_n) &= f_{1n}(t), \\ g_n(0) &= d_n, \end{aligned}\right\} \tag{26}$$

where $A_n(t, g_n) = B_n(t, g_n) g_n + r_n(t, g_n)$ and C_n, $B_n(t, g_n)$, $r_n(t, g_n)$, f_{1n} has been already defined.

Denote by $Y_n = C^1([-\tau, T]; \mathbb{R}^n)$ the space of continuously differentiable vector valued functions defined on the interval $[-\tau, T]$ with the supremum norm. Fix $\tau > 0$ and let $S = \{\tau i : i = -1, 0, 1, \ldots, k;\ k\tau = T\}$ be the time–grid over $[-\tau, T]$. Denote by $g_\tau = g|_S$ the grid function corresponding to the function $g \in Y_n$ and set $g_\tau^i = g_\tau(i\tau)$. Let Y_{nh} denote the space of grid functions corresponding to Y_n. The $\|g_\tau\|_{k\tau} = max\{|g_\tau^i| : i = -1, 0, \ldots, k\}$ is the norm in Y_{nh}.

One of the possible formulations of the difference problem corresponding to (26) reads as follows (*cf.* [22]):

given the "initial guess" $\rho \in C^1([-t_0, 0]; \mathbb{R}^n)$, $\rho(0) = d_n$, find $g_\tau \in Y_{nh}$ satisfying the system of equations

$$R_\tau^{i+1} g_\tau = (f_{1n})_\tau^i, \quad i = 0, 1, \ldots, k, \quad g_\tau^{-1} = \rho(-\tau), \quad g_\tau^0 = d_n, \tag{27}$$

where $R_\tau : Y_{nh} \to Y_{nh}$ is the three–level linearized difference approximation of the operator R, ($R(t)g(t) = C_n g' + A_n(t, g)$) defined by the formula:

$$\begin{aligned} R_\tau^i g_\tau &= R_\tau^i(g_\tau^i, g_\tau^{i-1}, g_\tau^{i-2}) \\ &= \frac{1}{2\tau} C_n(g_\tau^i - g_\tau^{i-2}) + A_n((i-1)\tau, g_\tau^{i-1}) + \frac{1}{2}(A_n')^{i-1}(g_\tau^{i-2} - 2g_\tau^{i-1} + g_\tau^i), \end{aligned}$$

with $(A_n')^i = A_n'(i\tau, u_\tau^i)$ (A_n' is the Jacobian matrix of the mapping $(t, g) \mapsto A_n(t, g)$).

Theorem below relating approximate solutions g_n of (26) to weak solutions u_n of (19)–(20) presents the extension of results of [19] (see also [17]) concerning the three dimensional sourceless filtration to the case of the one dimensional filtration flow with point sources.

Theorem 3. *Let conditions of Theorem 2 be satisfied. Then for any fixed $\tau > 0$, natural n and $\rho \in C^1([-t_0, 0]; \mathbb{R}^n)$, $\rho(0) = d_n$ the difference scheme (27) has the unique solution $g_{n\tau,\rho}$ approximating the solution g_n of (26) (i.e. of the Galerkin system (24), (25)) and in the consequence the solution u_n of (22), (23), in a sense that the function $u_{n\tau,\rho}(t) = \sum_{j=1}^n g_{j,n\tau,\rho}(t) w_j$, $t \in S$, satisfies*

$$\lim_{\tau \to 0} \|u_{n\tau,\rho} - u_n\|_{k\tau} = 0, \tag{28}$$

uniformly with respect to ρ in a bounded sets in $C^1([-t_0, 0]; \mathbb{R}^n)$. Moreover

$$\lim_{\substack{n \to \infty \\ \tau \to 0}} \|h - h_{n\tau,\rho}\|_\tau = 0, \tag{29}$$

where $\|u\|_\tau = \max\{\|u(i\tau)\|_{L^2(\Omega)} : i = 0, 1, \ldots, k\}$, $h_{n\tau,\rho} = u_{n\tau,\rho} + a$.

Outline of the proof. Since spaces V_n are finite dimensional, $B_n(t, g_n)$ f_{1n} inherit all properties of B and f stated in paragraph 5. Moreover by (iv), (v), $r_n(t, g_n)$ is continuous and bounded. Using (i)–(viii) one can prove (exactly in the same way as in the case of filtration without sources, see [19] for details) that R_τ approximates R, the system (27) is uniquely solvable and stable with respect to the right hand side, hence the modified version of the Lax–Filippov Theorem (compare e.g. [16, Ch. 2]) is applicable giving (28).

To prove (30), it will be shown firstly that the sequence of Galerkin approximations $\{u_n\}$ converges uniformly to the solution u of (19), (20), i.e.

$$u_n \to u \quad \text{in} \quad C^0([0,T]; L^2(\Omega)). \tag{30}$$

For the proof of (30), observe that since the problem (19), (20) has the unique solution, all converging subsequences of sequences $\{u_n\}$, $\{B(\cdot, u_n + a(\cdot))\}$ must have the same limits, which implies that

$$u_n \to u \quad \text{weakly in } X, \quad B(\cdot, u_n + a(\cdot)) \to B(\cdot, u + a(\cdot)) \quad \text{weakly in } X'.$$

Choose the sequence$\{v_n\} \subset C^1([0,T]; V_n)$ such that $\|v_n - u\|_X \to 0$, $\|v_n' - u'\|_{X'} \to 0$. Set (for simplicity arguments t are omitted)

$$I_1 = -\langle B(\cdot, a + u_n) - B(\cdot, a + u), u_n - u \rangle,$$

$$I_2 = -\langle B(\cdot, a + u_n) - B(\cdot, a + u), u - v_n \rangle, \qquad I_3 = \langle C(u - v_n)', u_n - v_n \rangle,$$

$$I_4 = \langle G(u_n + a) - G(u + a), u_n - u \rangle, \qquad I_5 = \langle G(u_n + a) - G(u + a), u - v_n \rangle$$

Inequalities (18), $m c_1 \geq c_0 a_1$, (ii) and (v) imply that $I_1 + I_4 \leq 0$. From the continuity of the operator C, boundedness of the sequences $\{v_n\}$, $\{u_n\}$, $\{B(\cdot, a + u_n)\}$ and conditions (i), (v) it follows that

$$|I_2| \leq M_1 \|u_n - u\|_V \, \|u - v_n\|_V \qquad \leq K_1 \|u - v_n\|_V,$$

$$|I_3| \leq \|C\| \, \|(u - v_n)'\|_{V'} \|u_n - v_n\|_V \leq K_2 \|u' - v_n'\|_{V'},$$

$$|I_5| \leq 2a_2 \sqrt{h_{max} - h_{rec}} \|u - v_n\|_V \quad = K_3 \|u - v_n\|_V,$$

where K_i denote suitable positive constants. We have

$$\langle C(u_n - v_n)', u_n - v_n \rangle = \langle -B(\cdot, a + u_n) + B(\cdot, a + u), u_n - v_n \rangle = \sum_{j=1}^{5} I_j$$

and a simple calculation gives

$$\tfrac{1}{2}(\langle C(u_n(t) - v_n(t)), u_n(t) - v_n(t) \rangle - \langle C(u_n(0) - v_n(0)), u_n(0) - v_n(0) \rangle)$$

$$= \int_0^t \langle C(u_n(s) - v_n(s))', u_n(s) - v_n(s) \rangle \, \mathrm{d}s = \int_0^t \sum_{j=1}^{5} \mathrm{d}s \leq \int_0^t (I_2 + I_3 + I_4) \, \mathrm{d}s$$

$$\leq (K_1 + K_3) \|u - v_n\|_X + K_2 \|u' - v_n'\|_{X'}.$$

The last formula and (vii) yield for $t \in [0,T]$ the inequality

$$e_0 \|u_n(t) - v_n(t)\|_{L^2(\Omega)} \leq K_4 \left(e^0 \|u_n(0) - v_n(0)\|_{L^2(\Omega)} + \|u - v_n\|_X + \|u' - v_n'\|_{X'} \right),$$

from which (30) follows immediately.

The formula (29) is a consequence of (28) and (30).

Remark 2. Usually, in practical computations, the spaces V_n are chosen as the finite element spaces, but other choices are not excluded. In the case of the prelinear filtration problems the Lagrange simplex elements seem to be the most suitable (see [22] or [26] and the references therein).

Remark 3. Function ρ appearing as starting data in the discussed scheme can be obtained either from the experimental data or can be computed with the aid of a two–level difference scheme (e.g. Crank–Nicholson scheme). The latter approach leads to the method using two grid operators: Crank–Nicholson for computing the first value of u_τ and the 3–level one for determining u_τ at remaining points of the time grid (*cf.* [22] where such kind of complex schemes are considered and their convergence is discussed).

Remark 4. The assumption $\Omega \subset \mathbb{R}$ plays the crucial role in the model with nonlinear sources, since only in the one dimensional case $H^1(\Omega)$ can be embedded in $C^0(\Omega;\mathbb{R})$, permitting to neglect the size of the source and to describe the process by the equation with distributional coefficient.

In the case when in numerical computations Lagrange simplex elements are applied, it is strongly recommended to choose the space-grid over Ω in such a way that one, say x_s, of its nodes is located at x_0. For such a choice and the Lagrangean basis $\{w_i\}$ we have

$$w_i(x_s) = \begin{cases} 0 & \text{for} \quad i \neq s, \\ 1 & \text{for} \quad i = s, \end{cases}$$

which implies that

$$\langle G(u_n + a(t)), w_i \rangle = \begin{cases} 0 & \text{for} \quad i \neq s, \\ q(g_{sn}(t) + a(t, x_s)) & \text{for} \quad i = s, \end{cases}$$

i.e. the impact of the source upon the system is concentrated in a single (s-th) equation of the system (24).

Remark 5. Equations with distribution coefficients are not suitable for modelling point sources in two or three dimensional filtration domains. However, it is still possible to obtain the similar result as stated in Theorem 2 (using the same arguments) when the influence of the source on the sorrounding filtration field is described by the operator (we specify it for the two dimensional field, the three dimesional case is treated similarly)

$$G : H^1(\Omega) \to L^2(\mathbb{R}^2, \mathbb{R}_+) \quad G(h) = \psi(x) g(L(h))$$

where g is defined as previously, $\psi \in L^2(\mathbb{R}^2; \mathbb{R}_+)$,

$$\psi(x) = \begin{cases} 0 & \text{for} \quad |x - x_0| > r, \\ 1 & \text{for} \quad |x - x_0| \leq r \end{cases}$$

(r is a diameter of the source) and $L(h) = c_1 \int_{\mathbb{R}^2} \psi(x) h(x)\, dx$, $c_1 = (\pi r^2)^{-1}$.

In both two- or three-dimensional cases and when Lagrangean elements are used, the influence of the source on a system (24) is no longer concentrated at a single equation. It acts on the system through the group of equations. Their numbers correspond to the numbers of the shape functions not vanishing at x_0.

7. Software for prelinear filtration problems

Computer codes for solving various engineering problems concerning the nonlinear filtration have been developed and refined giving the package MUBS (Multipurpose Underground Basin Simulator). Routines collected in the MUBS package can be divided into three groups according to their purposes.

The first group contains one- and two-dimensional codes for stationary and nonstationary Dupuit filtration (the free surface filtration). Both Dirichlet and Neumann boundary conditions are available as well as the special option for the well modelling. The classical Crank-Nicholson scheme is used in nonstationary cases.

Routines from the second group may be applied to the prelinear as well as to the linear Darcy filtration in whole completely saturated soils. Similar cases of boundary conditions and time-step scheme as in the first group are taken into consideration. Moreover fast linearized difference scheme [19] may be utilized.

The third group offers the method for the parameter identification (inverse problem solving) for prelinear flow fields (*cf.* [3]).

All three groups of codes have been thouroughly tested in application to real engineering problems (see e.g. [19]) and in application to simplified ones for which the distance between obtained numerical and accurate analytical solutions have been evaluated (see e.g. [18]).

Both PC and UNIX platform for MUBS numerical codes are acceptable. Codes for one-dimensional problems have been tested mainly, on PC 386 and SPARQ computers. Two-dimensional codes have been also run on VAXes and on CONVEX C3820 on which fast, completely vectorized VECLIB routines have been linked.

The MUBS computations on UNIX platform is supported by a powerful graphic environment called OCTOPUS [1] (*cf.* [4, 5]) which includes mesh generation modulus, various viewers and animation procedures for different type of solutions (*cf.* Fig. 3).

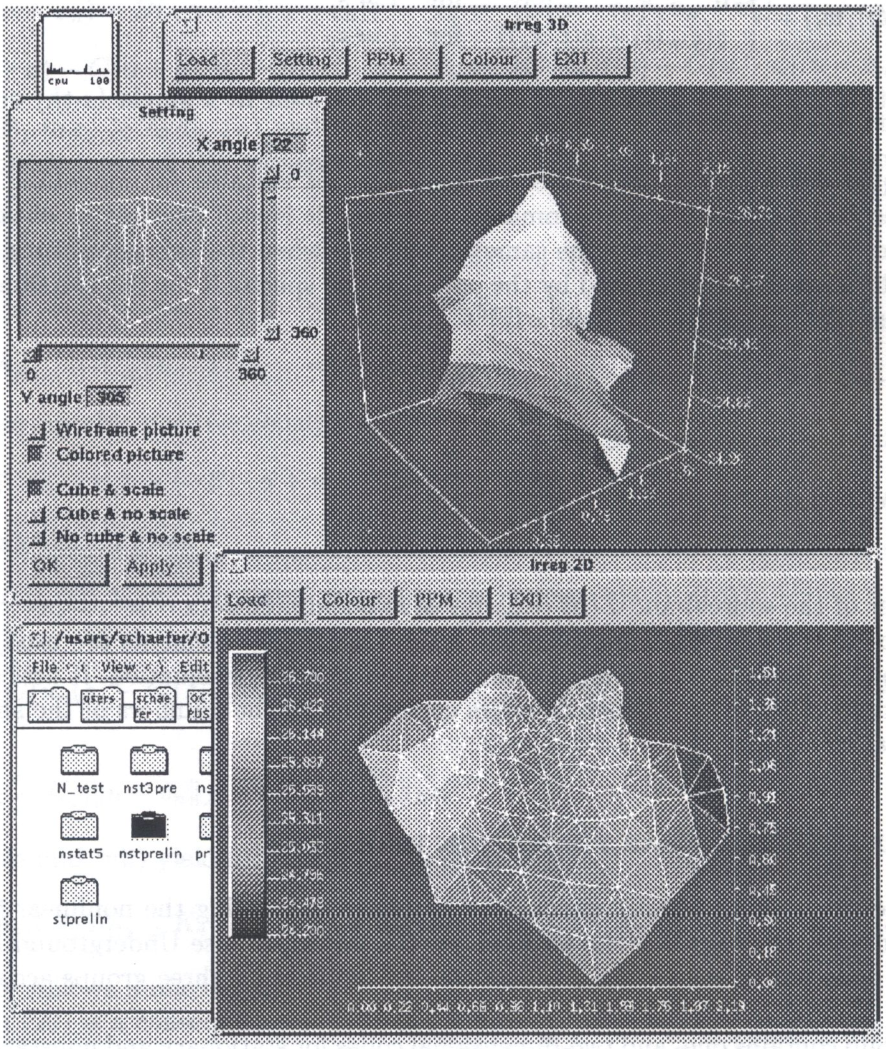

Fig. 3. Working on nonlinear filtration in OCTOPUS environment

[1] All numerical MUBS software as well as the graphical OCTOPUS elements mentioned above are obtainable for research purposes. The requests should be send by E-mail: `schaefer@ii.uj.edu.pl`

8. Computational example

The model presented above is illustrated by simulation of a simple vertical prelinear flow similar to the one described in paragraph 3. Input data are based on the measurements which have been performed at the testing site "Białośliwie" located in the valley of the Noteć river (*cf.* Wolski [25]).

The lower sediment part of depth 2.7m is gytia with hydraulic parameters $p = 0.75$, $M = 0.22 \cdot 10^{-5}$ m/s, $\theta = 0.86$, $s_0 = 2.8$. It is covered by the 3.1 m peat stratum with $p = 0.86$, $M = 0.3 \cdot 10^{-6}$ m/s, $\theta = 0.6$, $s_0 = 2.5$.

Both organic soil patterns are separated by a thin 0.2 m sand layer, which is dramatically more permeable, and drains its neighbourhood water content into the river bed. Transmission parameters (described in (4)) of the part of the sand layer between the central part of considered area and the river bed are as follows: $h_{rec} = 0.0$ m, $h_{lam} = 2.8$ m, $h_{max} = 4.0$ m, $a_1 = 0.5 \cdot 10^{-7}$ m/s, $a_2 = 0.7 \cdot 10^{-7}$ $\mathrm{m}^{\frac{3}{2}}$/s.

The upper loading of testing area has grown rapidly during first ten days of simulation. Resulting time rate of volume skeleton strains grew from zero at the beginning to about $0.2 \cdot 10^{-6}$ s^{-1} after the loading period and then decreased to $0.1 \cdot 10^{-8}$ s^{-1} at the end of the simulation (two months later). Filtration process in sediments has been also simulated by the artesian piezometric pressure $h_p = 2.04$ m appearing in the well permeable, stable, preconsolidated sand layer which constituted the basis of organic sediments.

The simulated evolution of piezometric pressure $h[\mathrm{m}H_2O]$ vs. time $t[\mathrm{s} \cdot 10^6]$ and depth $x[\mathrm{m}]$ ($x = 0$ on the terrain surface) is shown on Fig. 4. The numerical results are satisfactory close to the experimental data (*cf.* [17, 25]).

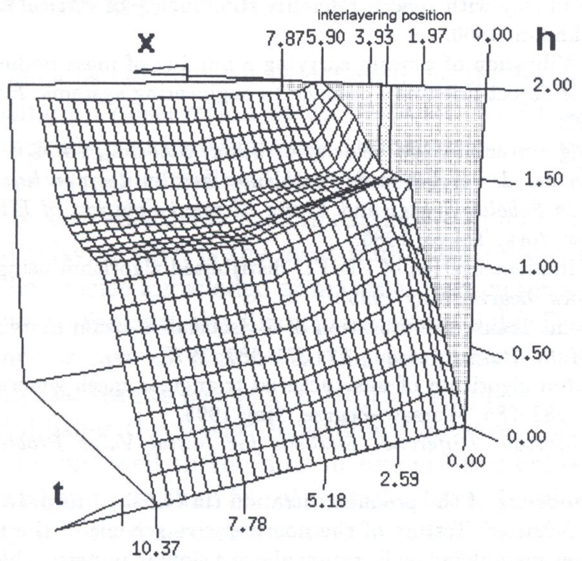

Fig. 4. The piezometric pressure h versus t (time) and x (space variable)

9. General conclusions

• The numerical model (6)–(8) of prelinear filtration based on local mass balance, new constitutive formula and FE/FD scheme is well posed from the mathematical standpoint (*cf.* Theorems 1 and 3), i.e. (6)–(8) has the unique solution being the limit of the sequence of approximate FE/FD solutions. The replacement of the classical filtration models by the nonlinear model using the formulae (4) and (5) leads to the visible improvement of the accuracy of obtained results.

• The well conditioning of proposed constitutive formula (4), (5) extends also to the non-classical cases of filtration with point sources. The discrete representation of small size outlets or supplies consumes considerably less computational time and RAM space than the standard element representation. The advantage of proposed approach becomes more apparent in two or three dimensional cases in which the lumped models of small sources (wells, interlayerings) provide the significant complication in FE network.

• Experiments show considerable advantage of the proposed FE/FD 3-level scheme over the classical ones, based on the 2–level schemes (see [17]).

• Current MUBS numerical algorithms are actually transformed to the parallel form by the use of various domain decomposition methods. Their distributed implementation is prepared and tested using low level RPC tools (PVM, Power C) as well as high level O-O techniques which will be delivered in OCTOPUS environment (*cf.* [9, 13, 15]). The second direction of development is to attach the mesh adoption algorithms in space as well as in time domains.

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