Computer Assisted Mechanics and Engineering Sciences, 3: 205–213, 1996.

Optimization of shallow Schwedler domes

Grzegorz W. Duda
Institute of Civil Engineering, Technical University of Wrocław, Poland

(Received July 31, 1995)

The paper deals with the optimization of regular space trusses with fixed external dimensions under uniform snow and dead-weight loading. Attempts are being made to find such a number of truss joints which minimise the material volume. The set of constraints imposed on a structure includes the effect of the loss of stability of the compressed bar. The statical problem of shallow domes including geometrical nonlinearity was solved by using the Newton-Raphson iteration procedure.

1. Introduction

The Schwedler dome is a regular space truss well known in technical publications [6]. The present paper deals with Schwedler domes with joints which lie on a regular net of M meridian and N latitudinal circles of a spherical surface with the base diameter D and height F. The bars connected at any joint of a truss make the configuration shown in Fig. 1 which is repeated many times in the structure. The domes are supported by means of an elastic ring and subjected to a fixed axi-symmetric load.

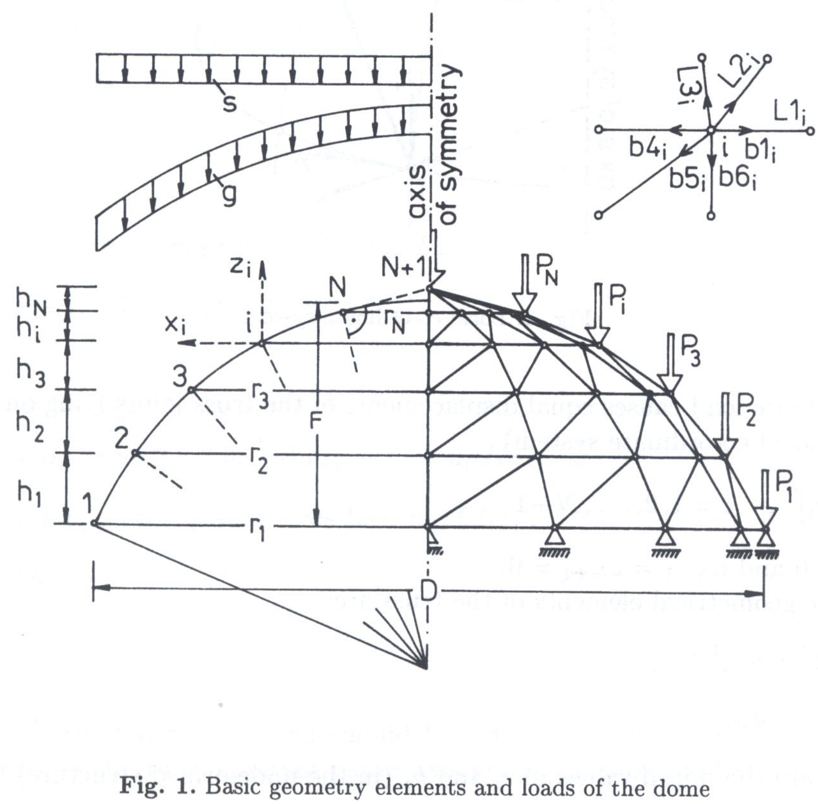

Fig. 1. Basic geometry elements and loads of the dome

The integers M and N will determine the length of the members of the dome, the resultant forces acting on the joints, the axial forces and cross-sectional areas of its members. The material volume of the structure, considered as the objective function of the optimization problem, also depends on the value of M and N. A search is being made for such values of the numbers $M = M^V$ and $N = N^V$ which would minimise the theoretical material volume V of the structure ($V = V_{\min}$). The optimization mentioned above presents a nonlinear programming problem. It was solved numerically by a systematic search for the points (M, N) of the feasible region of solutions, [2]. In the calculation of bar forces in a shallow Schwedler dome, it is required to account for the effects of the geometrical nonlinearities. This problem was solved by the stiffness method, using the Newton-Raphson iterative formula. The general problems of geometry, statical analysis and construction of bar domes are given in [6]. Problems of nonlinear analysis of bar structures are discussed in many papers, see e.g. [1, 4, 5, 7, 8]. The foundations of optimum design in civil engineering are given, among others, in [2] and [3].

2. Geometry and loading of a dome

The net of N latitudinal circles on a sphere is created on the basis of a uniform angular division of the circle in the meridian plane. The basic geometrical elements of the dome are the radii of latitudinal circles r_i and the vertical distance between two adjacent latitudinal planes h_i, $i = 1, 2, \ldots, N$. These elements are shown in Fig. 1. In a regular net of M meridian circles, the angular distance between two adjacent meridian planes is constant, and it is equal to 2φ, where $\varphi = \pi/M$. Using r_i, h_i and φ it is possible to determine the length of bars $L1_i$, $L2_i$, $L3_i$ and their direction cosines. In each joint of the truss, a local system of cylindrical co-ordinates $x_i\, y_i\, z_i$ was used, see Figs. 1 and 2.

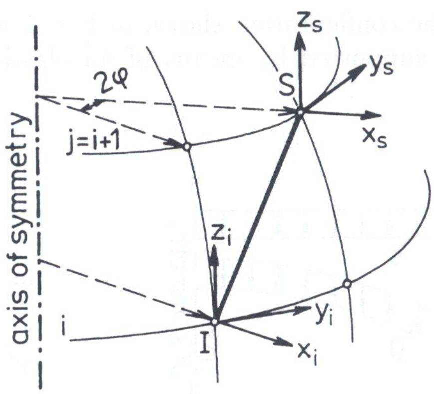

Fig. 2. Local co-ordinate systems

Any axi-symmetric load causes equal displacements of the truss joints lying on the same latitudinal circle i (in local co-ordinate system)

$$\mathbf{q}_i = [u_i,\, v_i,\, w_i]^{\mathrm{T}}, \quad i = 1, 2, \ldots, N{+}1, \tag{1}$$

where $v_1 = w_1 = 0$ and $u_{N+1} = v_{N+1} = 0$.

Thus the basic geometrical elements of the truss are:

$$\begin{aligned} r_i &= [(r_i^0 + u_i)^2 + {v_i}^2]^{1/2}, \\ h_i &= h_i^0 + w_{i+1} - w_i, \end{aligned} \tag{2}$$

where r_i^0 and h_i^0 are the initial values of r_i and h_i (in the undeformed structure).

The lengths of deformed members are computed according to the following formulae:

$$L1_i = 2r_i \sin\varphi\,, \quad i = 1,2,\ldots,N\,, \tag{3}$$

$$L2_i = \Big\{\big[(r^0_{i+1}+u_{i+1})\cos 2\varphi - v_{i+1}\sin 2\varphi - r^0_i - u_i\big]^2 + \big[(r^0_{i+1}+u_{i+1})\sin 2\varphi + v_{i+1}\cos 2\varphi - v_i\big]^2 + {h_i}^2\Big\}^{1/2}, \quad i = 1,2,\ldots,N-1\,, \tag{4}$$

$$L3_i = \Big\{(r^0_{i+1} - r^0_i + u_{i+1})^2 + (v_{i+1}-v_i)^2 + {h_i}^2\Big\}^{1/2}, \quad i = 1,2,\ldots,N\,. \tag{5}$$

Their direction cosines are computed as the components of the unit vectors shown in Fig. 1, relative to local co-ordinates system

$$\mathbf{b\nu}_i = [b\nu_{xi},\, b\nu_{yi},\, b\nu_{zi}]^{\mathrm{T}}, \quad \nu = 1,2,\ldots,6 \tag{6}$$

and they are as follows:

$$\begin{aligned} b1_{xi} &= [(r^0_i + u_i)\cos 2\varphi - v_i \sin 2\varphi - r^0_i - u_i]/L1_i\,,\\ b1_{yi} &= [(r^0_i + u_i)\sin 2\varphi + v_i\cos 2\varphi - v_i]/L1_i\,,\\ b1_{zi} &= 0\,, \end{aligned} \tag{7}$$

$$\begin{aligned} b2_{xi} &= [(r^0_{i+1} + u_{i+1})\cos 2\varphi - v_{i+1}\sin 2\varphi - r^0_i - u_i]/L2_i\,,\\ b2_{yi} &= [(r^0_{i+1} + u_{i+1})\sin 2\varphi + v_{i+1}\cos 2\varphi - v_i]/L2_i\,,\\ b2_{zi} &= h_i/L2_i\,, \end{aligned} \tag{8}$$

$$\begin{aligned} b3_{xi} &= (r^0_{i+1} - r^0_i + u_{i+1} - u_i)/L3_i\,,\\ b3_{yi} &= (v_{i+1} - v_i)/L3_i\,,\\ b3_{zi} &= h_i/L3_i\,, \end{aligned} \tag{9}$$

$$\begin{aligned} b4_{xi} &= [(r^0_i + u_i)\cos 2\varphi + v_i\sin 2\varphi - r^0_i - u_i]/L1_i\,,\\ b4_{yi} &= [-(r^0_i + u_i)\sin 2\varphi + v_i\cos 2\varphi - v_i]/L1_i\,,\\ b4_{zi} &= 0\,, \end{aligned} \tag{10}$$

$$\begin{aligned} b5_{xi} &= [(r^0_{i-1} + u_{i-1})\cos 2\varphi + v_{i-1}\sin 2\varphi - r^0_i - u_i]/L2_{i-1}\,,\\ b5_{yi} &= [-(r^0_{i-1} + u_{i-1})\sin 2\varphi + v_{i-1}\cos 2\varphi - v_i]/L2_{i-1}\,,\\ b5_{zi} &= -h_{i-1}/L2_{i-1}\,, \end{aligned} \tag{11}$$

$$\mathbf{b6}_i = -\mathbf{b3}_{i-1}\,. \tag{12}$$

A uniformly distributed gravity load should be transformed into a system of resultant forces acting on the truss joints. In the present paper axi-symmetric snow load s, and dead-weight g are considered. Due to this type of load, the forces $\mathbf{P}_i$ in the joints which lie on the same latitudinal ring i are equal, and these are the following:

$$\mathbf{P}_i = [P_{ix},\, P_{iy},\, P_{iz}]^{\mathrm{T}} = [0,\, 0,\, P_{iz}]^{\mathrm{T}}, \tag{13}$$

where

$$P_{iz}(g) = gF_{gi}\,, \quad P_{iz}(s) = sF_{si}\,, \tag{14}$$

and

$$F_{gi} = \sin\varphi \left\{ [\cos^2\varphi\,(r_i^0 - r_{i-1}^0)^2 + h_{i-1}{}^2]^{1/2}(2r_i^0 + r_{i-1}^0) + [\cos^2\varphi\,(r_i^0 - r_{i+1}^0)^2 + h_i{}^2]^{1/2}(2r_i^0 + r_{i+1}^0) \right\} /3\,, \tag{15}$$

$$F_{si} = \sin 2\varphi\,(r_{i-1}^0 - r_{i+1}^0)^2(r_{i-1}^0 + r_i^0 + r_{i+1}^0)/6\,. \tag{16}$$

3. Statical analysis

As the theoretical model of the dome, the space truss with pinned joints was used. The number w of degrees of freedom of the truss joints, in the case of axi-symmetric load, may be reduced to:

$$w = 3N - 1\,. \tag{17}$$

The geometrically nonlinear analysis of the structure was carried out by the stiffness method based on equation, [1]:

$$(\mathbf{K}_{\mathrm{E}} + \mathbf{K}_{\mathrm{G}})\mathbf{q} = \mathbf{P} \tag{18}$$

in which

$\mathbf{K}_{\mathrm{E}}(w,w)$ is the standard elastic stiffness matrix,

$\mathbf{K}_{\mathrm{G}}(w,w)$ is the geometrical stiffness matrix which depends on the current value of axial forces,

$\mathbf{q}(w)$ is the vector of joint displacements,

$\mathbf{P}(w)$ is the vector of joint loads.

The stiffnes matrix $\mathbf{K} = \mathbf{K}_{\mathrm{E}} + \mathbf{K}_{\mathrm{G}}$ is most frequently built in one Cartesian co-ordinate system. In the case presented above, it is convenient to use a multi-global co-ordinate system, similarly to Ref. 3. This leads to a considerable reduction of the number of degrees of freedom of the truss joints, according to formula (17).

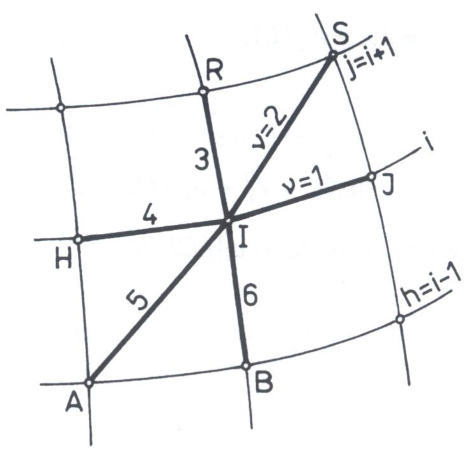

Fig. 3. Local notation of truss joints

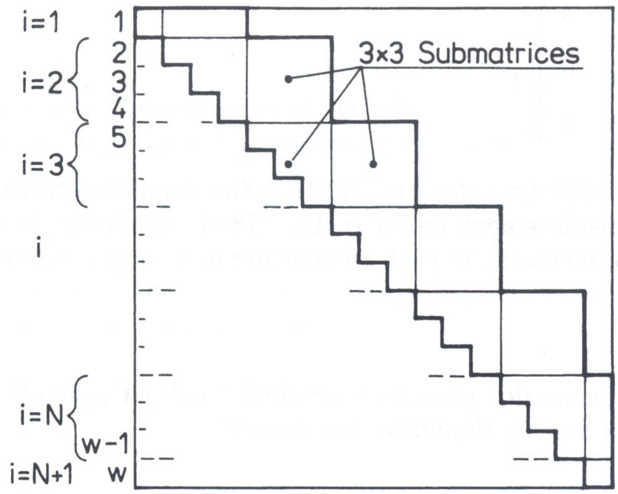

Fig. 4. Storage of stiffness matrix

As the matrix **K** is symmetric, only half of it needs to be stored. It is shown in Fig. 4. The matrix is composed of the submatrices $\mathbf{K}_{ij}(3,3)$. The submatrices $\mathbf{K}_{ii}$ and $\mathbf{K}_{ij}$, according to the signs of the truss joints shown in Fig. 3, are computed as follows:

$$\mathbf{K}_{ii} = \mathbf{K}_{\mathrm{IH}} + \mathbf{K}_{\mathrm{II}} + \mathbf{K}_{\mathrm{IJ}}\,, \quad i = 1, 2, \ldots, N-1\,, \tag{19}$$

$$\mathbf{K}_{ij} = \mathbf{K}_{\mathrm{IS}} + \mathbf{K}_{\mathrm{IR}}\,, \quad j = 2, 3, \ldots, N\,, \tag{20}$$

where respective submatrices are calculated from the formulae, [1]

$$\mathbf{K}_{\mathrm{II}} = \sum_{\nu=1}^{6} k\nu_i\, \mathbf{b}\boldsymbol{\nu}_i \cdot \mathbf{b}\boldsymbol{\nu}_i{}^{\mathrm{T}} + \sum_{\nu=1}^{6} n\nu_i(\mathbf{I}_3 - \mathbf{b}\boldsymbol{\nu}_i \cdot \mathbf{b}\boldsymbol{\nu}_i{}^{\mathrm{T}})\,, \tag{21}$$

$$\mathbf{K}_{\mathrm{IH}} = -k1_i \mathbf{b4}_i \cdot \mathbf{b4}_i{}^{\mathrm{T}} \cdot \mathbf{D} - n1_i(\mathbf{I}_3 - \mathbf{b4}_i \cdot \mathbf{b4}_i{}^{\mathrm{T}}) \cdot \mathbf{D}\,, \tag{22}$$

$$\mathbf{K}_{\mathrm{IJ}} = -k1_i \mathbf{b1}_i \cdot \mathbf{b1}_i{}^{\mathrm{T}} \cdot \mathbf{D}^{\mathrm{T}} - n1_i(\mathbf{I}_3 - \mathbf{b1}_i \cdot \mathbf{b1}_i{}^{\mathrm{T}}) \cdot \mathbf{D}^{\mathrm{T}}\,, \tag{23}$$

$$\mathbf{K}_{\mathrm{IS}} = -k2_i \mathbf{b2}_i \cdot \mathbf{b2}_i{}^{\mathrm{T}} \cdot \mathbf{D}^{\mathrm{T}} - n2_i(\mathbf{I}_3 - \mathbf{b2}_i \cdot \mathbf{b2}_i{}^{\mathrm{T}}) \cdot \mathbf{D}^{\mathrm{T}}\,, \tag{24}$$

$$\mathbf{K}_{\mathrm{IR}} = -k3_i \mathbf{b3}_i \cdot \mathbf{b3}_i{}^{\mathrm{T}} - n3_i(\mathbf{I}_3 - \mathbf{b3}_i \cdot \mathbf{b3}_i{}^{\mathrm{T}})\,, \tag{25}$$

$\mathbf{I}_3$ = unit matrix of order 3,

and

$$k\nu_i = EA\nu_i / L\nu_i\,, \qquad n\nu_i = N\nu_i / L\nu_i\,, \tag{26}$$

in which

E = modulus of elasticity,

$A\nu_i$ = cross-sectional areas of the members,

$L\nu_i$ = lengths of the members,

$N\nu_i$ = axial forces in the members.

The transformation matrix $\mathbf{D}$ of order 3 is

$$\mathbf{D} = \begin{bmatrix} \cos 2\varphi & \sin 2\varphi & 0 \\ -\sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{27}$$

in which the angle $2\varphi = 2\pi/M$ (see also Fig. 2), M is the number of meridians. Since the stiffness matrices depend on the displacement vector $\mathbf{q}$, Eq. (18) is nonlinear. It was solved by using the Newton-Raphson iterative methods. In each current iterative step j, based on equation

$$(\mathbf{K}_{\mathrm{E}}^{j} + \mathbf{K}_{\mathrm{G}}^{j}) \cdot \mathrm{d}\mathbf{q}_j = \mathrm{d}\mathbf{P}_j\,, \tag{28}$$

the increment of displacements $\mathrm{d}\mathbf{q}_j$ caused by residual forces $\mathrm{d}\mathbf{P}_j$ should be computed, and next the current value of displacements should be determined,

$$\mathbf{q}_{j+1} = \mathbf{q}_j + \mathrm{d}\mathbf{q}_j\,. \tag{29}$$

This procedure may be repeated until the errors become as small as desired, according to formula

$$\mathrm{d}\mathbf{q}_r < \varepsilon \mathbf{q}_r\,, \tag{30}$$

where ε is the small factor. In the numerical example (Section 6) $\varepsilon = 1 \cdot 10^{-6}$ was used, and number r of iteration steps for domes with different M, N and F was changing between 3 and 12.

4. Determination of the cross-sections of the bars

Determination of the cross-sectional areas of the bars was made according to the limit state method design. For compressed bars the design procedure is based on the Euler and parabolic formula. Both these curves are shown in Fig. 5. The shape of the cross-section of a bar is described by the dimensionless factor $\psi = A^2/I$, where A is the cross-sectional area, and I is the second moment of area. For a circular tube with dimensions r, t (r — radius of the centre line, t — wall thickness), $\psi = 16\pi\eta/(4+\eta^2)$, where $\eta = t/r$.

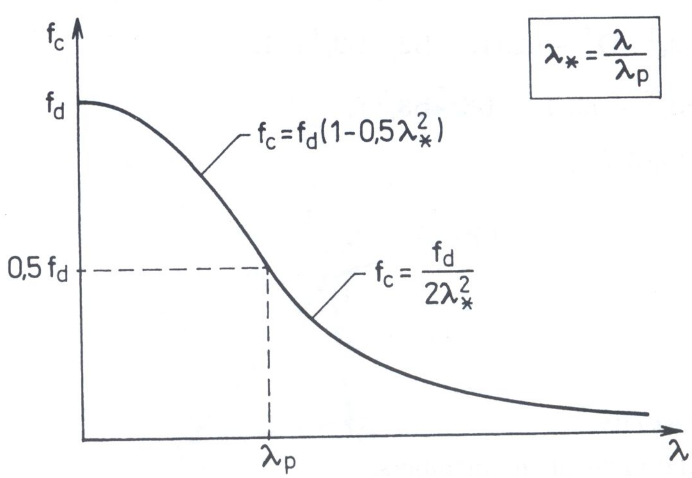

Fig. 5. Relationship between f_c and slenderness ratio λ

If the bar i is loaded by the tensile force N_i, its cross-sectional area is calculated from the formula:

$$A_i \geq N_i/f_\mathrm{d}\,, \tag{31}$$

where f_d is the design strength of the structural material.

For the bar j, compressed by a force N_j', one should compute first its characteristic length:

$$l_\mathrm{p}^j = \lambda_\mathrm{p}\sqrt{2N_j/(\psi f_\mathrm{d})}\,, \tag{32}$$

where $\lambda_\mathrm{p} = \pi\sqrt{E/(0.75 f_\mathrm{d})}$, $N_j = \mathrm{abs}(N_j')$, ψ is the shape factor, E is Young's modulus, and then its cross-sectional area, using one of the formulas:

$$A_j \geq \frac{l_j}{\lambda_\mathrm{p}}\sqrt{2\psi N_j/f_\mathrm{d}}\,, \qquad \text{if} \quad l_j \geq l_\mathrm{p}^j\,, \tag{33}$$

$$A_j \geq \frac{N_j}{f_\mathrm{d}} + \frac{1}{2}\psi(l_j/\lambda_\mathrm{p})^2\,, \quad \text{if} \quad l_j < l_\mathrm{p}^j\,, \tag{34}$$

where l_j is the real length of bar j. Eq. (33) is based on Euler's curvature, Eq. (34) — on the parabolic formula.

5. Formulation of the optimum design problem

In the above described optimization problem of the Schwedler dome, the constant parameters are:

- external dimensions of the spherical surface D and F,
- the value of the loads g and s,
- the design strength of the structure material f_d,
- the shape of the cross-section of the bar given by the dimensionless factor ψ.

The integer numbers M and N are decision variables. The objective function $V(M,N)$ expresses the theoretical volume V of the structural material. The criterion of the optimizatoin is formulated as:

$$V = V(M,N) = \min = V^V\,. \tag{35}$$

The set of restrictive conditions imposed on a structure consists of:

a) cross-sectional areas A_i of the tension members are determined from Eq. (31),

b) cross-sectional areas A_j of the compression members are determined from Eqs. (33) and (34),

c) $A_k \geq A_\mathrm{min}$ for all members,

d) the bar length is bounded by $l_\mathrm{min} < l_k < l_\mathrm{max}$.

Since the analytical formula of the objective function is unknown, its values are computed numerically from the formula $V = \sum_k l_k A_k$. The restriction d) (above) creates a discrete region of permissible solutions, in which the optimal structure is to be found (for example, see Fig. 6 in Section 6).

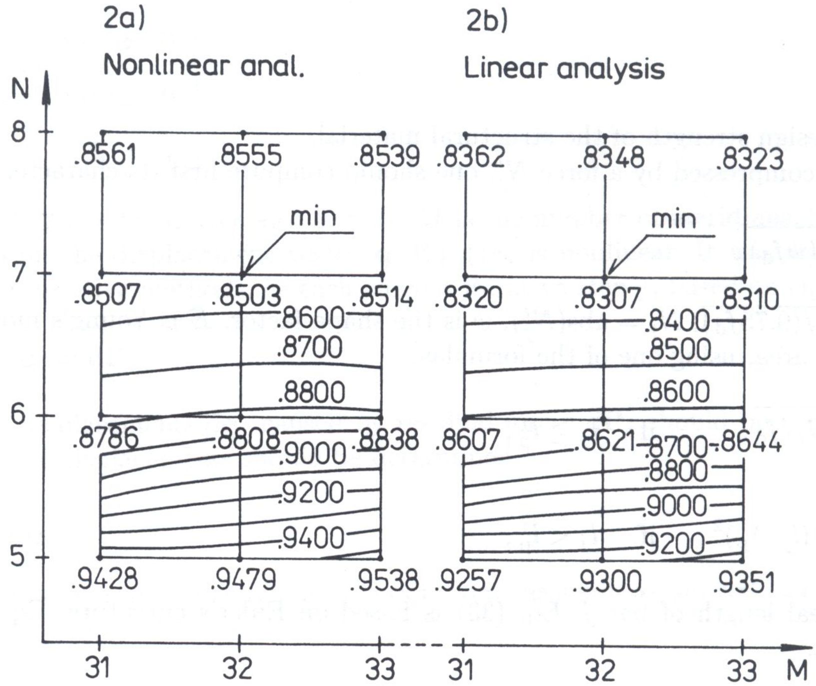

Fig. 6. Map of the objective function for the domes with $F = 4.0$ m

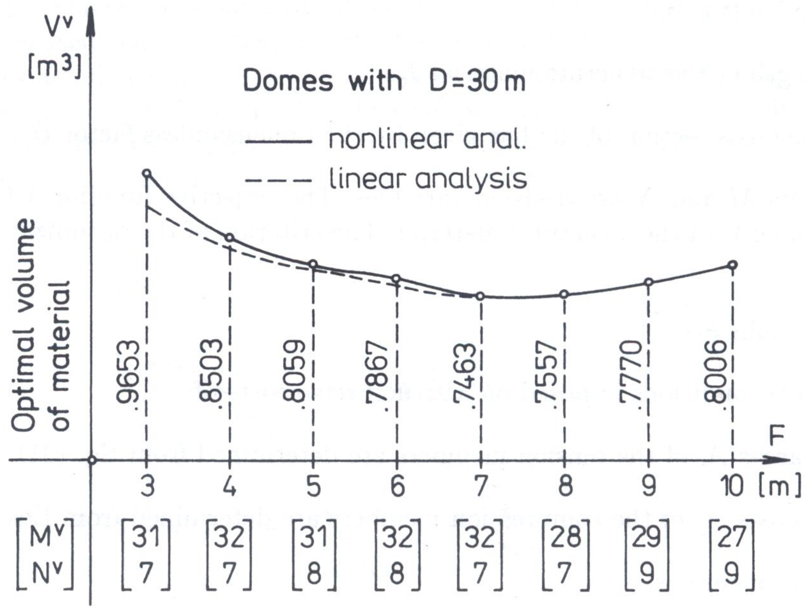

Fig. 7. Material volume V^V as a function of the dome height F

6. Numerical example

In order to investigate the described optimization problem, a computer program was written (in TB). The following values of constant parameters have been accepted: $D = 30$ m, $g = 0.95$ kN/m^2, $s = 0.80$ kN/m^2, $f_d = 210$ MPa (steel). It is assumed that the members of the structure are made of circular steel tubes described by the shape factor $\psi = 1.876$, of which the smallest cross-section is $A_{\min} = \max(A_0, \psi l^2/200^2)$, where $A_0 = 3.676$ cm^2, l is given in centimetres. The bar length was bounded by $l_{\min} = 0.40$ m and $l_{\max} = 4.0$ m.

For the fixed value of F the local optimization task was solved by means of a computer. Fig. 6 presents a map of the objective function (treated as continuous) for the domes having $F = 4.0$ m. The shape of the line V = constant is only probable. In Fig. 6a are presented the results of computations with allowance made for geometrical nonlinearity, and in Fig. 6b the results obtained according to linear statics are given. The optimal solution has been found for point $(M^V, N^V) = (32, 7)$, in which $V^V = 0.8503$ m^3 of steel. In the region of permissible solutions, the maximal difference of material volume between domes with different M and N is equal to 12.2%. The difference between the values V^V according to nonlinear and linear statical analysis is small and equal to 2.41%. Cutting the function $V(M, N)$ at the point $(M^V, N^V) = (32, 7)$ in the direction parallel to M axis and parallel to N axis, one can state that the function $V(M, N)$ is more sensitive to a change of the number of latitudinal circle N than to a change of the number of meridians M.

In further analysis the optimization of the dome with changing height F has been provided.The height F of the spherical surface was changing in the range 3.0 m to 10.0 m. Some results of the computation are shown in Fig. 7, where the local optimal structure is represented by one point. It has been found that the global minimum of the structure material is in the dome with ratio $F/D \approx 7/30$.

References

[1] J.H. Argyris. *Recent advances in matrix methods of structural analysis.* Pergamon Press, Oxford-London-New York-Paris, 1964.

[2] A.M. Brandt. *Foundations of optimum design in civil engineering.* PWN-Polish Scientific Publishers, Warszawa, Poland, Nijhoff Publishers, Dordrecht-Boston-London, 1989.

[3] E.J. Haug, J.S. Arora. *Applied optimal design mechanical and structural system.* A Wiley-Interscience Publication, John Wiley and Sons, New York-Chichester-Brisbane-Toronto, 1979.

[4] D.S. Jagannathan, H.J. Epstein, P. Christiano. Non-linear analysis of reticulated space truss. *J. of the Structural Division*, ASCE, **ST12**: 2641–2658, 1975.

[5] M. Kleiber. *Incremental finite element modelling in non-linear solid mechanics.* Polish Scientific Publishers, Warszawa, Poland, Ellis Horwood Limited, Chichester, England, 1989.

[6] Z.S. Makowski. *Analysis, design and construction of braced domes.* Nicholas Publishing Company, New York, 1984.

[7] J.S. Przemieniecki. *Theory of matrix structural analysis.* McGraw-Hill, New York, 1968.

[8] H. Rothert, T. Dickel, D. Renner. Snap-through buckling of reticulated space trusses. *J. of the Structural Division*, ASCE, **ST1**: 129–143, 1981.