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A mathematical model for processes of structure analysis

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Generally, path-following algorithms are used for the history analysis of structures. Now, a new approach is presented for solving the problem by parametric optimization.

The optimization problem is solved in a direct product of function spaces. The necessary conditions of the stationarity of a curve are examined. A method is presented for determining a piece of a continuous component of the Karush-Kuhn-Tucker stationary curve depending on one parameter which transforms the problem into the space l^2.

1. Introduction

It is usual in mathematical physics to approximate nonequilibrium processes by means of local equilibrium states near to equilibrium [12]. In the paper, a mathematical model and computational algorithms are presented for an analysis of the state variables in the local equilibrium state.

In statics, the state variables caused by time-independent external loads are assumed as functions of space variables. The aim is to determine the time-independent equilibrium state. In dynamics, the external loads and the state variables are time-dependent but it is supposed that every function can be given as a product of a time- and a place-dependent functions. In determining the change of state in the structure, it is a very strong assumption.

Generally, the finite elements method is used for solving static and dynamic problems. The equation system of FEM is the Karush-Kuhn-Tucker system of a quadratic programming problem. This quadratic programming problem can be written on the basis of the stationary theorems of energy functions.

Different problems in plasticity can be solved by tools of mathematical programming but the time-dependence of the state variables is taken into consideration at some given instants of time (step by step methods). In this way, the state variables determined in discretized points of time are independent of each other.

Our aim is to elaborate a mathematical tool which takes into consideration the time-dependence of the state variables without the above assumptions; futhermore, is more suitable to model and solve the problems in local equilibrium state. The problems are formulated as a problem pair of mathematical programming.

The process of producing a mechanical model is presented by a simple example.

Let us determine the deflection function of a simply supported beam in the state of equilibrium. Its length is l, the stiffness is EI, the static load intensity is p at every point of the beam. The differential equation of the problem with the corresponding boundary conditions is

$$EI\frac{\mathrm{d}^4u(x)}{\mathrm{d}x^4} = p,$$

$$u(x)_{|x=0} = 0, \quad u(x)_{|x=l} = 0, \quad u''(x)_{|x=0} = 0, \quad u''(x)_{|x=l} = 0.$$

The solution of this differential equation and of the following variational problem is the same

$$\min\left\{\int_0^l\left[\tfrac{1}{2}EI\left(\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2}\right)^2 - u(x)p\right]\mathrm{d}x \;\middle|\; u''(x)_{|x=0}=0\,,\;\; u''(x)_{|x=l}=0\right\}.$$

Discretizing the beam into finite elements and choosing the shape functions which satisfy the boundary conditions, the first n terms of the approximation function system are

$$u(x)\approx\sum_{i=1}^{n}a_iN_i(x),$$

$$N_i(x)_{|x=0}=0\,,\quad N_i(x)_{|x=l}=0\,,\quad N_i''(x)_{|x=0}=0\,,\quad N_i''(x)_{|x=l}=0\,.$$

The approximation of the variational problem is

$$\min\left\{\int_0^l\left[\tfrac{1}{2}EI\left(\frac{\mathrm{d}^2\sum_{i=1}^{n}a_iN_i(x)}{\mathrm{d}x^2}\right)^2\sum_{i=1}^{n}pa_iN_i(x)\right]\mathrm{d}x\,,\quad N_2(x)\in C^2\right\}.$$

Differentiating the functional with respect to the unknown coefficients a_i (generalized Fourier coefficients), the solution of the problem is given by the following linear equation system

$$\sum_{i=1}^{n}a_iEI_i\int_0^l N_i''(x)N_j''(x)\,\mathrm{d}x - p\int_0^l N_j(x)\,\mathrm{d}x=0\,,\quad j=1,\ldots,n\,.$$

After integration over the finite elements, the form of equation system in matrix notation is

$$-\mathbf{EI\,Qa}+\hat{\mathbf{p}}=\mathbf{0}\,,$$

where $Q_{ij}=\int_0^l N_i''(x)N_j''(x)\,\mathrm{d}x$, $\hat{p}_j=p\int_0^l N_j(x)\,\mathrm{d}x$.

The matrix $\mathbf{Q}$ is symmetric, it can be decomposed into a matrix product $\mathbf{Q}=\mathbf{G}^*\mathbf{G}$. Stiffness EI of the elements can be written in the form of a diagonal matrix and so we get the well-known form of the equation system of a simple supported beam

$$\mathbf{G}^*-\mathbf{EI\,Ga}+\hat{\mathbf{p}}=\mathbf{0}\,.$$

Introducing the unknown vector $\mathbf{b}=-\mathbf{EI\,Ga}$, the equation system is

$$\mathbf{G}^*\mathbf{b}+\hat{\mathbf{p}}\qquad=\mathbf{0}\,,$$
$$\mathbf{Ga}+\mathbf{EI}^{-1}\mathbf{b}=\mathbf{0}\,.$$

This system is the Kuhn-Tucker optimality system of the following quadratic programming problem pair:

$$\left\{\min\tfrac{1}{2}\mathbf{bEI}^{-1}\mathbf{b}\quad|\quad\mathbf{G}^*\mathbf{b}+\hat{\mathbf{p}}=\mathbf{0}\right\}$$

and

$$\left\{\max-\tfrac{1}{2}\mathbf{bEI}^{-1}\mathbf{b}+\hat{\mathbf{p}}^*\mathbf{a}\quad|\quad\mathbf{Ga}+\mathbf{EI}^{-1}\mathbf{b}=\mathbf{0}\right\}.$$

If the external load is time-dependent, the deflection function is time-dependent as well. If the external load acts in the interval of time $[t_1,t_2]$, the form of the differential equation and boundary conditions are

$$EI\frac{\mathrm{d}^4u(x,t)}{\mathrm{d}x^4} = p(t)\,,$$

$$u(x,t)_{|x=0} = 0\,, \quad u(x,t)_{|x=l} = 0\,, \quad u''(x,t)_{|x=0} = 0\,, \quad u''(x,t)_{|x=l} = 0\,, \quad \forall\, t \in [t_1, t_2]\,.$$

The differential equation corresponding to a parametric variational form

$$\min \int_0^l \left[\tfrac{1}{2}EI\left(\frac{\mathrm{d}^2u(x,t)}{\mathrm{d}x^2}\right)^2 - u(x,t)p(x,t)\right] \mathrm{d}x \;\mid\; u''(x,t)_{|x=0} = 0\,, \quad u''(x,t)_{|x=l} = 0\,, \quad \forall\, t \in [t_1, t_2]\,,$$

and the boundary conditions have to be satisfied at every instant of time.

For numerical solution the functions $u(x,t)$ are approximated by the system of shape functions. If the shape functions depend on x only, the parameter t appears in the coefficients:

$$u(x,t) \approx \sum_{i=1}^{n} a_i(t)N_i(x),$$

$$N_i(x)_{|x=0} = 0\,, \quad N_i(x)_{|x=l} = 0\,, \quad N_i''(x)_{|x=0} = 0\,, \quad N_i''(x)_{|x=l} = 0\,.$$

The approximate solution of the problem is represented by the solution of the following linear equation system at every value of the parameter t in the interval $[t_1, t_2]$:

$$\sum_{i=1}^{n} a_i(t)EI_i \int_0^l N_i''(x)N_j''(x)\,\mathrm{d}x - p(t)\int_0^l N_j(x)\,\mathrm{d}x = 0\,, \quad j = 1,\dots,n \quad \forall\, t \in [t_1, t_2]\,.$$

After integration, the equation system in a matrix form is

$$\mathbf{G}^* - \mathbf{EI}\,\mathbf{G}\mathbf{a}(t) + \hat{\mathbf{p}}(t) = \mathbf{0}\,, \quad \forall\, t \in [t_1, t_2]\,.$$

Introducing the unknown components of the vector $\mathbf{b}(t) = -\mathbf{EI}\,\mathbf{G}\mathbf{a}(t)$, the following equation system has to be solved:

$$\mathbf{G}^*\mathbf{b}(t) + \hat{\mathbf{p}}(t) = \mathbf{0}\,,$$
$$\mathbf{G}\mathbf{a}(t) + \mathbf{EI}^{-1}\mathbf{b}(t) = \mathbf{0}\,, \quad \forall\, t \in [t_1, t_2]\,.$$

This system is the Kuhn–Tucker optimality system of the following mathematical programming problem pair:

$$\left\{\min \tfrac{1}{2}\mathbf{b}(t)\mathbf{EI}^{-1}\mathbf{b}(t) \;\mid\; \mathbf{G}^*\mathbf{b}(t) + \hat{\mathbf{p}}(t) = \mathbf{0}\,, \forall\, t \in [t_1, t_2]\right\}$$

and

$$\left\{\max -\tfrac{1}{2}\mathbf{b}(t)\mathbf{EI}^{-1}\mathbf{b}(t) + \hat{\mathbf{p}}^*(t)\mathbf{a}(t) \;\mid\; \mathbf{G}\mathbf{a}(t) + \mathbf{EI}^{-1}\mathbf{b}(t) = \mathbf{0}\,, \quad \forall\, t \in [t_1, t_2]\right\}.$$

The most common problem which is described in the local equilibrium state is the plastic analysis of structures [11]. In the simplest case, when only the strain and complementary strain energies are taken into consideration, the elastic-plastic analysis of the structure is described by the following set of problems:

(i) From a given set of stress-rates which satisfy the equilibrium equations, the yield and boundary conditions, only the stress-rates will be determined and the rate of the complementary strain energy will assume a minimal value. In this case, the compatibility equations are also satisfied.

(ii) From a given set of strain-rates satisfying the compatibility equations, boundary conditions, and the positivity of plastic rate multipliers, only the strain-rate will be determined, and the rate of the strain energy will assume a minimal value. The equilibrium equations are satisfied at this point.

This problem is solved supposing that the rate of state variables is independent of time [4]. In our approximation this restriction is not necessary and the method presented takes into consideration the time-dependence of the rate of state variables.

The state variables (e.g. stress, strain, etc.) are given in a vector space by vector–scalar functions in the case of equilibrium state. For a discretized structure, the vector space in question is supposed to be an n-dimensional space in a global coordinate system according to the number of nodes (n). Every node of the element is defined by a position vector (see Fig. 1). A state vector described in the local coordinate system is attached to each position vector. The number of independent components is the sum of the number of degrees of freedom (s) belonging to the nodes. If the number of degrees of freedom is the same in each node, the dimension of the vector space is ns.

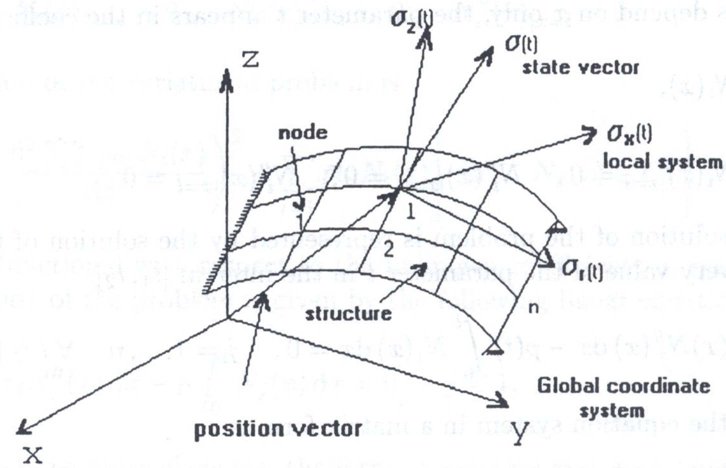

Fig. 1

In the local equilibrium state, the rates of the state variables are given in both local and global coordinate systems at the time-dependent problems, and they are vector–vector functions which depend on the time. Within the context of a small displacement theory, the position vectors are supposed to be time-independent.

The process of the change of state is described by a stationary curve in the local equilibrium state of the structure. Our aim is to determine this curve or its discrete points. It is proved by means of the differential geometry that stationary curves depending on a parameter are discontinuous, and only a part of a continuous component can be determined [13].

Nowadays, one of the most important topics in the optimization theory is the structural investigation of the Karush-Kuhn-Tucker stationary curve depending on one or more parameters providing the possibility of the sensitivity analysis of optimality problems. The differential topology is one of the possible ways to answer such questions in case of a parametric problem [7, 8, 13, 14]. To determine one of the parametric stationary curves, the path-following algorithms are used, a recent application of which are the interior point algorithms [17].

In the paper a new method is presented for determining a continuous component piece of the Karush-Kuhn-Tucker stationary curve depending on one parameter.

In section 2, the mathematical description of the space is presented; the general problem is posed in section 3. The optimality conditions are proved in l^2 space, therefore, we deal with the transformation between L^2 and l^2 spaces in section 4. The Fritz-John conditions are presented in function spaces in section 5, and in the case of nonlinear functions in section 6. In section 7 the Fritz–John conditions are transformed from L^2 space to l^2 space and a method is presented to solve the problem. The use of the functional derivative is shown in section 8. Finally, two basic examples illustrate the results.

2. Mathematical description of the space

Consider an arbitrary, s-dimensional vector–scalar function. Let every component of this function be an element of the space $L^2(0,1)$, (space of quadratically integrable functions [18]). As the $L^2(0,1)$, is a Hilbert space, we can choose a base P_i ($i = 1,\dots,\infty$) which is orthonormed on the interval [0,1] and in which every element of Hilbert space can be written as follows:

$$x(t) = \sum_{i=1}^{\infty} \alpha_i P_i(t)\,, \quad P_i(t) \in L^2(0,1), \quad \alpha \in R\,, \quad i = 1,\dots,\infty, \quad t \in [0,1] \tag{1}$$

(R is the one-dimensional Euclidean space, i.e., the set of real numbers).

The description of the state variables concerning a node is performed in the direct product of the corresponding Hilbert spaces given in the form of

$L_1^2 \times L_2^2 \times \ldots \times L_s^2$.

An element of this space can be written as

$$\mathbf{r}(t) = \sum_{j=1}^{s} x_j(t)\mathbf{e}_j = \sum_{j=1}^{s}\left(\sum_{l=1}^{\infty}\alpha_{ij}P_i^j(t)\right)\mathbf{e}_j\,,$$
$$\alpha_{ij} \in R\,, \quad P_i^j(t) \in L_j^2([0,1])\,, \quad t \in [0,1]\,, \quad i = 1,\dots,\infty\,, \quad j = 1,\dots,s\,, \tag{2}$$

where $\mathbf{e}_j$ ($j = 1,\dots,s$) are the unit vectors of the s-dimensional Euclidean space and $P_i^j(t)$ denotes the i-th base component ordered to the j-th axis of the local coordinate system.

The space assigned to the lth node is

$$\mathcal{L}^l : \left(R^3 \times L_1^2 \times L_2^2 \times \ldots \times L_s^2\right), \quad l = 1,\dots,n\,, \tag{3}$$

an element of which is:

$$\mathbf{y}^l(t) = \left(z_1^l,\, z_2^l,\, z_3^l,\, x_1^l(t),\, x_2^l(t),\, \dots,\, x_s^l(t),\right), \quad t\in[0,1]\,,$$

where z_i^l denotes the i-th coordinate ($i = 1,2,3$) of the position vector belonging to the l-th node ($l = 1,\dots,n$). This means that the problems concerning the local equilibrium state of the structure are described in the space

$$\mathcal{L} = \left(R^3 \times L_1^2 \times L_2^2 \times \ldots \times L_s^2\right)^n \tag{4}$$

If the position vectors of the structure are independent of time, the change of state can be described in the space:

$$\mathcal{F} = \left(L_1^2 \times L_2^2 \times \ldots \times L_s^2\right)^n\,. \tag{5}$$

Let us include the variables into a time-dependent vector

$$Y(t) = \left[x_1^1(t),\, x_2^1(t),\, \dots,\, x_s^1(t),\, x_1^2(t),\, x_2^2(t),\, \dots,\, x_s^2(t),\, \dots,\, x_1^n(t),\, x_2^n(t),\, \dots,\, x_s^n(t)\right]\,,\ t \in [0,1]\,.$$

The vector $\mathbf{Y}(t)$ can be written in the form of

$$\mathbf{Y}(t) = \boldsymbol{\alpha}^*\mathbf{B}(t)\,, \tag{6}$$

where the vector

$$\boldsymbol{\alpha} = \left[\alpha_{11}^1,\dots,\alpha_{\infty1}^1,\alpha_{12}^1,\dots,\alpha_{\infty2}^1,\alpha_{1s}^1,\dots,\alpha_{\infty s}^1,\alpha_{11}^n,\dots,\alpha_{\infty1}^n,\alpha_{12}^n,\dots,\alpha_{\infty2}^n,\alpha_{1s}^n,\dots,\alpha_{\infty s}^n\right]\,,$$
$$\alpha \in R\,, \quad j = 1,\dots,s\,, \quad i = 1,\dots,\infty\,,$$

contains the coefficients of the functions, and $\mathbf{B}(t)$ is a hyperdiagonal matrix a block of which is the following column vector:

$$\mathbf{B}_j(t) = \left[P_i^j(t),\ i = 1,\dots,\infty\right]\,. \tag{7}$$

By matrix notation, a vector is a column vector, and the transpose is denoted by the symbol *.

3. Description of the general problem

The time-dependent elastic, dynamic, elasto-plastic state-change analysis of the structures can be formulated as the following optimization problem depending on parameter t:

$$\begin{aligned} &\min f(\mathbf{y})\,, \\ &g_i(\mathbf{y}) \leq 0\,, \quad i = 1, \ldots, m\,, \\ &h_j(\mathbf{y}) = 0\,, \quad j = 1, \ldots, l\,, \\ &\mathbf{y} \in R^{ns}\,, \quad f, g_i, h_j : R^{ns} \to R\,, \\ &\mathbf{y} = \mathbf{x}(t)\,, \quad \text{at } \forall \text{ fix } t\,, \quad \mathbf{x}(t) \in \mathcal{F}\,, \quad t \in [0,1]. \end{aligned} \tag{8}$$

This problem series should be interpreted so that in case of $\forall\, t \in [0,1]$, the obtained nonlinear optimization problem (8) has a stationary point and thereby the curve $\mathbf{x}(t) \in \mathcal{F}$ is defined. The purpose is to ensure the theoretical manageability and numerical solvability of the above optimization problem. Such problems arise in parametric optimization [7–9, 13].

The classical nonlinear optimization problem interpreted in the n-dimensional Euclidean space is as follows:

$$\min\{f(\mathbf{x}) \mid g_i(\mathbf{x}) \leq 0, \quad h_j(\mathbf{x}) = 0, \quad i = 1, ..., m, \quad j = 1, ..., l; \quad \mathbf{x} \in R^n\}\,,$$

$$f(\mathbf{x}),\ g_i(\mathbf{x}),\ h_j(\mathbf{x}) \in C^1\,, \tag{9}$$

where the variables are the components of $\mathbf{x}$. The first-order characteristics of optimality is achieved by the Karush–Kuhn–Tucker theorem [2].

In a Banach space, the optimization problem can be written in a similar way, i.e.:

$$\min\{f(\mathbf{x}) \mid g_i(\mathbf{x}) \leq 0, \quad h_j(\mathbf{x}) = 0, \quad i = 1, ..., m, \quad j = 1, ..., l; \quad \mathbf{x} \in B\}, \tag{10}$$

where B is a linear, normalized space (Banach space). The Kuhn-Tucker theorem can be used in this case, too [3]. For the theoretical investigation of the problem (8), optimality conditions should be proved first. To this end we use the results related to Banach spaces.

4. Transformation between the $(R^3 \times L_1^2 \times L_2^2 \times ... \times L_s^2)^n$ and $(R^3 \times l_1^2 \times l_2^2 \times ... \times l_s^2)^n$ spaces

Consider the variables of the problem in the direct product of spaces l^2 instead of in the direct product of spaces $L^2(0,1)$, by the isomorphism theorem [15, 18]. This transformation is possible if the functions g_i, h_j and f are suitable for the linear and metric structures of the space.

Let an arbitrary function $q(t) \in L^2$ be given with the orthonormed polynomial system P_i, $(i = 1, ..., \infty)$. The infinite-dimensional vector $\mathbf{q} \in l_2$ can be written in the following form, assuming that $\sum_{i=1}^{\infty} \alpha_i^2 < \infty$:

$$q(t) = \alpha_0 \sum_{i=1}^{\infty} \alpha_i P_i(t)\,, \quad t \in [0,1]\,, \quad \alpha_i \in R; \quad \mathbf{q} = \alpha_0 \sum_{i=1}^{\infty} \alpha_i \mathbf{e}_i\,, \quad \alpha_i \in R\,, \tag{11}$$

where $\mathbf{e}_i$ is the i-th unit vector of l^2 space.

Let us define a space $f = (R^3 \times l_1^2 \times l_2^2 \times \ldots \times l_s^2)^n$ and a hypermatrix $\mathbf{b}$. Their structures should correspond to those of the space $\mathcal{F}$ and matrix $\mathbf{B}(t)$, respectively. Instead of the element $P_i^j(t)$ in the matrix $[\mathbf{B}(t)]$, the unit vectors $\mathbf{e}_i^j$ of f correspond to them for every pair of indices i and j.

In what follows we will show the transformation between the spaces $\mathcal{F}$ and f in the case of linear functions, derivatives and integrals with respect to the time parameter, the power and nonlinear functions $\mathbf{x}_j^k(t)$, $k = 1, ..., n$, $j = 1, ..., s$ belonging to the k-th node related to the variables.

i. The transformations in the case of addition, multiplication by a scalar, and scalar product, are as follows:

$$\begin{aligned} &\mathbf{x}(t) + \mathbf{y}(t) = \boldsymbol{\alpha}^*\mathbf{B}(t) + \boldsymbol{\beta}^*\mathbf{B}(t) = (\boldsymbol{\alpha} + \boldsymbol{\beta})^*\mathbf{B}(t), \\ &c\mathbf{x}(t) = c\boldsymbol{\alpha}^*\mathbf{B}(t), \\ &\int_0^1 \mathbf{x}(t)^*\mathbf{y}(t)\,dt = \int_0^1 \boldsymbol{\alpha}^*[\mathbf{B}(t)][\boldsymbol{\beta}^*[\mathbf{B}(t)]]\,dt = \boldsymbol{\alpha}^*\int_0^1 \mathbf{B}(t)\,\mathbf{B}(t)^*\,dt\,\boldsymbol{\beta} = \boldsymbol{\alpha}^*\boldsymbol{\beta}, \end{aligned} \tag{12}$$

where $\mathbf{x}(t)$, $\mathbf{y}(t) \in \mathcal{F}$, $t \in [0, 1]$ are ns-dimensional vector functions and c is a scalar (by scalar multiplication, the orthonormality of the polynomial system $P_i^j(t)$ is taken into account in the interval $[0, 1]$) and

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= \boldsymbol{\alpha}^*\mathbf{b} + \boldsymbol{\beta}^*\mathbf{b} = (\boldsymbol{\alpha} + \boldsymbol{\beta})^*\mathbf{b}, \\ c\mathbf{x} &= c\boldsymbol{\alpha}^*\mathbf{b}, \\ \mathbf{x}^*\mathbf{y} &= \boldsymbol{\alpha}^*\mathbf{b}\,[\boldsymbol{\beta}^*\mathbf{b}] = \boldsymbol{\alpha}^*\mathbf{b}\mathbf{b}^*\boldsymbol{\beta} = \boldsymbol{\alpha}^*\boldsymbol{\beta}, \end{aligned} \tag{13}$$

where $\mathbf{x}$ and $\mathbf{y}$ are composed of infinitely many ns-dimensional vectors and c is a scalar.

ii. If the derivatives with respect to the parameter t (denoted by the dot over the symbol) were in linear relations in the space $\mathcal{F}$, the derived functions would be described in the base of the space by means of generalized Fourier series:

$$\dot{P}(t) = \sum_{k=1}^{\infty} \gamma_{ik} P_k(t), \qquad \text{where} \quad \gamma_{ik} = \int_0^1 P_k(t)\dot{P}_i(t)\,dt. \tag{14}$$

The derivative of the j-th element of vector $\mathbf{x}(t)$ is in the space $\mathcal{F}$:

$$\dot{x}_j(t) = \sum_{k=1}^{\infty} \alpha_{ij}\dot{P}_k^j(t) = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \gamma_{ki}\alpha_{ij}P_k^j(t) = \sum_{k=1}^{\infty} \beta_{kj}P_k^j(t), \tag{15}$$

where $\beta_{kj} = \sum_{i=1}^{\infty} \gamma_{ki}\alpha_{ij}$. The derivative vector with respect to t may be written, using the matrix notation, in the form:

$$\dot{\mathbf{x}}(t) = \boldsymbol{\beta}^*\mathbf{B}(t).$$

Transforming this result into the space f, we obtain

$$\dot{x}_j = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \gamma_{ki}\alpha_{ij}\mathbf{e}_k^j = \sum_{k=1}^{\infty} \beta_{kj}\mathbf{e}_k^j, \quad \text{or} \quad \dot{\mathbf{x}} = \boldsymbol{\beta}^*[\mathbf{b}]. \tag{16}$$

iii. In a similar way, if the one-parameter integral of the basis-function is in linear relations, then it can be written by the generalized Fourier series of the integral function as follows:

$$\int_0^t P_i(\tau)\mathrm{d}\tau = \sum_{k=1}^{\infty} \nu_{ik}P_k(t), \quad \text{where} \quad \nu_{ik} = \int_0^1 \left(\int_0^t P_i(\tau)\mathrm{d}\tau\, P_k(t)\right) \mathrm{d}t. \tag{17}$$

The integral of the j-th component of vector $\mathbf{x}(t)$ belonging to the space $\mathcal{F}$ can be written in the base of the space:

$$\tilde{x}_j(t) = \int_0^t x_j(\tau)\mathrm{d}\tau = \int_0^t \sum_{i=1}^{\infty} \alpha_{ij}P_i^j(\tau)\,\mathrm{d}\tau = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \nu_{ki}\alpha_{ij}P_k^j(t) = \sum_{k=1}^{\infty} \eta_{kj}P_k^j(t), \tag{18}$$

where $\eta_{kj} = \sum_{i=1}^{\infty} \nu_{ki}\alpha_{ij}$, or in matrix notation:

$$\tilde{\mathbf{x}}(t) = \int_0^t \mathbf{x}(\tau)\mathrm{d}\tau = \boldsymbol{\eta}^*[\mathbf{B}].$$

The result (18) can be transformed into the space f:

$$\tilde{x}_j = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \nu_{ki}\alpha_{ij}\mathbf{e}_k^j = \sum_{k=1}^{\infty} \eta_{k1}\mathbf{e}_k^j, \quad \text{or} \quad \tilde{\mathbf{x}} = \boldsymbol{\eta}^*[\mathbf{b}]. \tag{19}$$

iv. To transform the equality relation between the spaces $\mathcal{F}$ and f, the generalized Fourier series are formed for all terms of equality according to the base functions $P_i^j(t)$, $i = 1, ..., \infty$, $j = 1, ..., ns$. Let $\mathbf{x}(t) = \mathbf{0}$. The generalized Fourier series of $\mathbf{x}(t)$ in terms of the base functions $P_i^j(t)$, $i = 1, ..., \infty$, $j = 1, ..., ns$ is

$$x_j(t) = \sum_{i=1}^{\infty} \alpha_{ij} P_i^j(t), \quad \alpha_{ij} = \int_0^1 \mathbf{x}_j(t) P_i^j(t)\,\mathrm{d}t. \tag{20}$$

Using (12), in the case of the j-th component of the vector $\mathbf{x}(t)$, the equality is

$$\sum_{i=1}^{\infty} \alpha_{ij} P_i^j(t) = 0. \tag{21}$$

The equality (21) is satisfied if $\alpha_{ij} = 0$, $i = 1, ..., \infty$, $j = 1, ..., ns$.

In the space f in the case of the i-th element of j-th vector, the equality has the form

$$\sum_{i=1}^{\infty} \alpha_{ij}\mathbf{e}_i^j = 0, \tag{22}$$

that is, $\alpha_{ij} = 0$, $i = 1, ..., \infty$, $j = 1, ..., ns$.

v. In case of inequalities, a problem has arisen due to the fact that the isomorphism between the spaces $\mathcal{L}$ and l is not order-preserving, i.e., there is no function system in the $\mathcal{L}^2$ space ensuring the inequality $g(t) \geq 0$, $g(t) \in \mathcal{L}^2[0, 1]$ iff the Fourier coefficients are nonnegative [6].

Lemma 1. *The isomorphism between the spaces $\mathcal{L}^2$ and l^2 is order-preserving iff $\int_0^1 g(t)P_n(t)\,\mathrm{d}t \geq 0$, $i = 1, ..., \infty$, and $g(t) \geq 0$ hold together.*

Proof: The main idea of the proof was suggested by I. Dancs. If the statement is true, then $P_i(t) \geq 0$, $i = 1, ..., \infty$, $t \in [0, 1]$ almost everywhere. By contradiction, let us suppose that there exists an index i and a set A with positive measure such that $P_i(t) < 0$ in it. Let us denote the characteristic function of the set A by $\chi_{A_i} \in L^2([0, 1])$. By the condition, we obtain $\int_0^1 \chi_{A_i} P(t)\,\mathrm{d}t < 0$, which contradicts the order-preserving property.

Because of orthogonality, the sets $A_1 = \{t \,|\, P_1(t) \neq 0 \,|\, P_1(t) > 0\}$, $i = 1, ..., \infty$ are disjoint almost everywhere. The Fourier series of χ_{A_i} is $\chi_{A_i} = P_i(t) \int_0^1 \chi_{A_i} P_i(t)\,\mathrm{d}t$ from which $P_i(t) = \dfrac{1}{\int_0^1 \chi_{A_i} P_i(t)\,\mathrm{d}t}\chi_{A_i} = \alpha_i \chi_{A_i}$ and $\int_0^1 P_i^2(t)\,\mathrm{d}t = \int_0^1 \alpha_i^2 \chi_A^2\,\mathrm{d}t = \alpha_i^2 \mu(A_i) = 1$, i.e., $P_i(t) = \dfrac{1}{\sqrt{\mu(A_i)}}\chi_{A_i}$.

The question is whether the system $\{P_i(t)\}$ is complete. For Lebesgue measure there exists an index i and the subsets A_i' and A_i'' of A_i with positive measure such that

$$A_i = A_i' \cup A_i'', \quad A_i' \cap A_i'' = 0 \quad \text{and} \quad \mu(A_i') = f(A_i'') = 1/2\mu(A_i).$$

$$\text{Let} \quad f(t) = \begin{cases} 1, & \text{if } t \in A_i', \\ -1, & \text{if } t \in A_i'', \\ 0, & \text{if } t \notin A_i, \end{cases}$$

$$\text{then} \quad \int_0^1 f(t) P_i(t)\, \mathrm{d}t = \frac{1}{\sqrt{\mu(A_i)}} \int_0^1 (\chi_{A_i'} - \chi_{A_i''}) \chi_{A_i}\, \mathrm{d}t = 0$$

which means the system $P_i(t)$, $i = 1, ..., \infty$ is not complete. ■

Because of Lemma 1, we have to approximate the inequalities only. In computations, finite numbers of base functions are taken into consideration from the basis of $\mathcal{F}$. If the basis of $\mathcal{F}$ is approximated by step functions, the inequalities can be computed step by step.

In practice, the function values of the finite number of the basis of $\mathcal{F}$ are computed at the points (t_r, $r = 1, ..., m$, where m is the number of points in time) chosen by the user. The inequalities at these points are independent of t, they contain the Fourier coefficients and constants only.

Let $\mathbf{x}(t) \leq 0$. The function $\mathbf{x}(t)$ is expanded into the generalized Fourier series of the function system $P_i(t)$, $i = 1, ..., \infty$.

$$\sum_{i=1}^{\infty} \alpha_{ij} P_i^j(t_r) \leq \mathbf{0}, \quad r = 1, \ldots, m. \tag{23}$$

The error in the approximation depends on the given instant of time.

5. Optimality conditions

Let the space $\mathcal{F} = (R^3 \times L_1^2 \times L_2^2 \times ... \times L_s^2)^n$ and $\mathbf{x}(t) \in \mathcal{F}$, $F : R^{ns} \times R \Rightarrow R$, $G^k : R^{ns} \times R \Rightarrow R$ for $k = 1, ..., q$, $H^j : R^{ns} \times R \Rightarrow R$ for $j = 1, ..., m$. Suppose F, H^j and G^k are continuously differentiable according to the first ns variables in $\mathcal{F}$. Consider the problem

$$\begin{aligned} &\min F(\mathbf{y}), \\ &G^k(\mathbf{y}) \leq 0, \quad k = 1, \ldots, q, \\ &H^j(\mathbf{y}) = 0, \quad j = 1, \ldots, m, \quad \mathbf{y} \in R^{ns}, \\ &\mathbf{y} = \mathbf{x}(t) \in \mathcal{F}, \quad \forall\, \text{fix}\, t, \quad t \in [0,1]. \end{aligned} \tag{24}$$

Let $\alpha_{il} \in R$, $i = 1, ..., \infty$, $l = 1, ..., ns$, $P_i^l(t) \in L_l^2[0,1]$, $l = 1, ..., ns$, $x_l(t) = \sum_{i=1}^{\infty} \alpha_{il} P_i^l(t)$ if $\sum_{i=1}^{\infty} \alpha_{il}^2 < \infty$, $l = 1, ..., ns$, $t \in [0,1]$. The matrix $[\mathbf{B}(t)]$ contains $P_i^l(t)$ according to the structure of $\mathcal{F}$. Let $F(\boldsymbol{\alpha}^* \mathbf{B}(t)) : R \Rightarrow R$, $G^k(\boldsymbol{\alpha}^* \mathbf{B}(t)) : R^{ns} \Rightarrow R$ for $k = 1, ..., q$, $H^j(\boldsymbol{\alpha}^* \mathbf{B}(t)) : R \Rightarrow R$ for $j = 1, ..., m$. Consider the problem

$$\begin{aligned} &\min F(\boldsymbol{\alpha}^* \mathbf{B}(t)), \\ &G^k(\boldsymbol{\alpha}^* \mathbf{B}(t)) \leq 0, \quad k = 1, \ldots, q, \\ &H^j(\boldsymbol{\alpha}^* \mathbf{B}(t)) = 0, \quad j = 1, \ldots, m, \\ &\alpha_{il} \in R, \quad i = 1, \ldots, \infty, \quad l = 1, \ldots, ns, \quad P_i^l(t) \in L_l^2[0,1], \end{aligned} \tag{25}$$

where α_{il}, $i = 1, ..., \infty$, $l = 1, ..., ns$ are variables. Let us introduce the notations of the Lagrangian functions

$$\mathcal{L}(\mathbf{y}) = \eta(t)F(\mathbf{y}) + \sum_{j=1}^{m} \lambda_j(t)H^j(\mathbf{y}) + \sum_{k=1}^{q} \mu_k(t)G^k(\mathbf{y}) .$$

$$\hat{\mathcal{L}}(\mathbf{y}) = \hat{\eta}(t)F(\boldsymbol{\alpha}^*\mathbf{B}(t)) + \sum_{j=1}^{m} \hat{\lambda}_j(t)H^j(\boldsymbol{\alpha}^*\mathbf{B}(t)) + \sum_{k=1}^{q} \hat{\mu}_k(t)G^k(\boldsymbol{\alpha}^*\mathbf{B}(t)) .$$

It is well-known in the optimality theory that Fritz-John and Karush-Kuhn-Tucker stationary curves depending on one parameter constitute a disjoint set consisting of continuous components [14]. The structural analysis of continuous components provides conditions for the fulfilment of the continuity of a part of stationary curves [7, 8, 14]. In the following part, we suppose that by "running" the parameter t in $[0, 1]$, the multipliers $\lambda_k(t)$ and $\mu_k(t)$ are continuous functions of t. Function $\mathcal{L}(\mathbf{y})$ can be taken as the generalization of the Lagrangian function belonging to problem (9).

Lemma 2. *The Fritz-John condition of (24) coincides with that of (25).*

Proof: By fixing the parameter t at t_0 and introducing the notation $\mathbf{y}^0 = \mathbf{x}(t_0)$, the Fritz-John condition of (24) is

$$\begin{aligned} &\nabla_y \mathcal{L}(\mathbf{y}^0) = 0 , \\ &\mu_k(t_0)G^k(\mathbf{y}^0) = 0 , \quad \mu_k(t_0) \geq 0 , \quad k = 1, ..., q . \end{aligned} \tag{26}$$

The system (26) is satisfied at every $t \in [0, 1]$, so that

$$\begin{aligned} &\nabla_y \mathcal{L}(\mathbf{y}) = 0 , \\ &\mu_k(t)G^k(\mathbf{y}) = 0 , \quad \mu_k(t) \geq 0 , \quad k = 1, ..., q , \end{aligned} \tag{27}$$

where ∇_y denotes the gradient vector according to $\mathbf{y}$ of the functions F, G and H at every fixed $t \in [0, 1]$.

At point t_0 the Fritz-John system of problem (25) is

$$\nabla_y \hat{\mathcal{L}}(\boldsymbol{\alpha}^*\mathbf{B}(t_0))\mathbf{B}(t_0) = 0 ,$$

$$\hat{\mu}_k(t_0)G^k(\boldsymbol{\alpha}^*\mathbf{B}(t_0)) = 0 , \quad \hat{\mu}_k(t) \geq 0 , \quad k = 1, ..., q .$$

Multiplying the first equation system by the inverse of the matrix $\mathbf{B}(t_0)$, at the point t_0 the form of the Fritz-John system of problem (25) becomes

$$\begin{aligned} &\nabla_y \hat{\mathcal{L}}(\boldsymbol{\alpha}^*\mathbf{B}(t_0)) = 0 , \\ &\hat{\mu}_k(t_0)G^k(\boldsymbol{\alpha}^*\mathbf{B}(t_0)) = 0 , \quad \hat{\mu}_k(t) \geq 0 , \quad k = 1, ..., q . \end{aligned} \tag{28}$$

Equations (27) are equivalent to (28), i.e.,

$$\eta(t_0) = \hat{\eta}(t_0) , \quad \lambda_j(t_0) = \hat{\lambda}_j(t_0) , \quad \mu_k(t_0) = \hat{\mu}_k(t_0) .$$

By adapting Eqs. (28) to every value of t, the Fritz-John condition of (25) is

$$\begin{aligned} &\nabla_y \hat{\mathcal{L}}(\boldsymbol{\alpha}^*\mathbf{B}(t)) = 0 , \\ &\hat{\mu}_k(t)G^k(\boldsymbol{\alpha}^*\mathbf{B}(t)) = 0 , \quad \hat{\mu}_k(t) \geq 0 , \quad k = 1, ..., q . \end{aligned} \tag{29}$$

From (29) and (26) we have

$$\eta(t) = \hat{\eta}(t) , \quad \lambda_j(t) = \hat{\lambda}_j(t) , \quad \mu_k(t) = \hat{\mu}_k(t) , \tag{30}$$

i.e., the stationary functions of (24) and (25) coincide. ■

6. Fritz-John condition in the case of nonlinear functions

Lemma 3. *The use of Fourier series of F, G, and H in problem (24) does not change the Fritz-John condition.*

Proof. Let us introduce the following notations

$$f_r(\boldsymbol{\alpha}) = \int_0^1 F(\boldsymbol{\alpha}^*\mathbf{B}(t_0))P_r(t)\,\mathrm{d}t\,,$$
$$g_r^k(\boldsymbol{\alpha}) = \int_0^1 G^k(\boldsymbol{\alpha}^*\mathbf{B}(t_0))P_r(t)\,\mathrm{d}t\,, \tag{31}$$
$$h_r^j(\boldsymbol{\alpha}) = \int_0^1 H^j(\boldsymbol{\alpha}^*\mathbf{B}(t_0))P_r(t)\,\mathrm{d}t\,.$$

The order of integration and differentiation can be changed at forming the gradient vectors according to $\boldsymbol{\alpha}$ of the Fourier coefficients $f_r(\boldsymbol{\alpha})$, $g_r^k(\boldsymbol{\alpha})$ and $h_r^j(\boldsymbol{\alpha})$.

The Fritz-John condition of the problem (25) is

$$\hat{\eta}(t)\sum_{r=1}^{\infty}P_r(t)\nabla\mathrm{f}_r(\boldsymbol{\alpha}) + \sum_{j=1}^{m}\hat{\lambda}_j(t)\sum_{r=1}^{\infty}P_r(t)\nabla h_r^j(\boldsymbol{\alpha}) + \sum_{k=1}^{q}\hat{\mu}_k(t)\sum_{r=1}^{\infty}P_r(t)\nabla g_r^k(\boldsymbol{\alpha}) = 0\,. \tag{32}$$

Multiplying the Fritz-John optimality system (27) of the problem (24) by $\mathbf{B}(t) \neq \mathbf{0}$, we obtain

$$\eta(t)\nabla_y F(\mathbf{y})\mathbf{B}(t) + \sum_{j=1}^{m}\lambda_j(t)\nabla_y H^j(\mathbf{y})\mathbf{B}(t) + \sum_{k=1}^{q}\mu_k(t)\nabla_y G^k(\mathbf{y})\mathbf{B}(t) = 0\,. \tag{33}$$

The products of derivatives and polynomials are expanded into the generalized Fourier series with respect to $P_i(t)$

$$\hat{\eta}(t)\sum_{r=1}^{\infty}P_r(t)\nabla_\alpha\mathrm{f}_r(\boldsymbol{\alpha}) + \sum_{j=1}^{m}\hat{\lambda}_j(t)\sum_{r=1}^{\infty}P_r(t)\nabla_\alpha h_r^j(\boldsymbol{\alpha}) + \sum_{k=1}^{q}\hat{\mu}_k(t)\sum_{r=1}^{\infty}P_r(t)\nabla_\alpha g_r^k(\boldsymbol{\alpha}) = 0\,. \tag{34}$$

Equations (34) are identical with (32), i.e., the Fourier expansion does not change the stationary functions. ■

7. Transformation of Fritz-John condition between the spaces $\mathcal{F}$ and F

Consider the problem

$$\max F(\mathbf{y}) + \sum_{k=1}^{q}\mu_k(t)G^k(\mathbf{y}) + \sum_{j=1}^{m}\lambda_j(t)H^j(\mathbf{y})\,,$$

$$\eta(t)\nabla_y F(\mathbf{y}) + \sum_{j=1}^{m}\lambda_j(t)\nabla_y H^j(\mathbf{y}) + \sum_{k=1}^{q}\mu_k(t)\nabla_y G^k(\mathbf{y}) = 0\,. \tag{35}$$

$$\mu_k(t) \geq 0\,,\ k = 1,\dots,q; \quad \mathbf{y} = \mathbf{x}(t)\,, \quad \mathbf{x}(t) \in \mathcal{F}\,, \quad \forall\,\mathrm{fix}\,t \in [0,1]\,.$$

Lemma 4. *The optimality system of the problem (24) complies with the optimality system of (35), i.e., their stationary curves are the same.*

Proof: The Fritz-John optimality conditions of (24) are

$$G^k(\mathbf{x}(t)) \le 0\,, \quad k = 1, \dots, q\,,$$

$$H^j(\mathbf{x}(t)) = 0\,, \quad j = 1, \dots, m\,,$$

$$\eta(t)\nabla_y F(\mathbf{x}(t)) + \sum_{j=1}^{m} \lambda_j(t)\nabla H^j(\mathbf{x}(t)) + \sum_{k=1}^{q} \mu_k(t)\nabla G^k(\mathbf{x}(t)) = 0\,, \tag{36}$$

$$\mu_k \ge 0\,, \quad k = 1, \dots, q\,,$$

$$\mu_k G^k(\mathbf{x}(t))0\,, \quad k = 1, \dots, q\,, \quad \mathbf{x}(t) \in (L^2)^{ns}\,, \quad t \in [0,1]\,.$$

Let us use the relations (2) in (36), moreover the expansions of $\eta(t)$, $\lambda(t)$ and $\mu(t)$ with respect to $P_i(t)$:

$$G^k(\boldsymbol{\alpha}^*\mathbf{B}(t)) \le 0\,, \quad k = 1, \dots, q\,,$$

$$H^j(\boldsymbol{\alpha}^*\mathbf{B}(t)) = 0\,, \quad j = 1, \dots, m\,,$$

$$\sum_{s=1}^{\infty} \kappa_s P_s(t)\nabla F(\boldsymbol{\alpha}^*\mathbf{B}(t)) + \sum_{j=1}^{m}\sum_{s=1}^{\infty} \zeta_{sj} P_s^j(t)\nabla H^j(\boldsymbol{\alpha}^*\mathbf{B}(t)) + \sum_{k=1}^{q}\sum_{s=1}^{\infty} \theta_{sk} P_s^k(t)\nabla G^k(\boldsymbol{\alpha}^*\mathbf{B}(t)) = 0\,,$$

$$\sum_{s=1}^{\infty} \theta_{sk} P_s^k(t) \ge 0\,, \quad k = 1, \dots, q\,, \tag{37}$$

$$\sum_{s=1}^{\infty} \theta_{sk} P_s^k(t) G^k(\boldsymbol{\alpha}^*\mathbf{B}(t)) = 0\,,$$

$$P_i(t) \in L^2\,, t \in [0,1]\,,$$

where $\eta(t) = \sum_{s=1}^{\infty} \kappa_s P_s(t)$, $\lambda_j(t) = \sum_{s=1}^{\infty} \zeta_{sj} P_s^j(t)$, $\mu_k(t) = \sum_{s=1}^{\infty} \theta_{sk} P_s^k(t)$.

The variables in the necessary conditions are α_{il}, κ_i, ζ_{ij}, θ_{ik}, $l = 1, \dots, ns$, $j = 1, \dots, m$, $k = 1, \dots, q$, $i = 1, \dots, \infty$.

Consider the problem

$$\max F(\boldsymbol{\alpha}^*\mathbf{B}(t)) + \sum_{k=1}^{q}\sum_{s=1}^{\infty} \theta_{sk} P_s^k(t) G^k(\boldsymbol{\alpha}^*\mathbf{B}(t)) + \sum_{j=1}^{m}\sum_{s=1}^{\infty} \zeta_{sj} P_s^j(t) H^j(\boldsymbol{\alpha}^*\mathbf{B}(t))$$

$$\sum_{s=1}^{\infty} \kappa_s P_s(t)\nabla F(\boldsymbol{\alpha}^*\mathbf{B}(t)) + \sum_{j=1}^{m}\sum_{s=1}^{\infty} \zeta_{sj} P_s^j(t)\nabla H^j(\boldsymbol{\alpha}^*\mathbf{B}(t)) + \sum_{k=1}^{q}\sum_{s=1}^{\infty} \theta_{sk} P_s^k(t)\nabla G^k(\boldsymbol{\alpha}^*\mathbf{B}(t)) = 0\,,$$

$$\theta_{sk} P_s^k(t) \ge 0\,, \quad s = 1, \dots, \infty\,, \quad k = 1, \dots, q\,, \tag{38}$$

$$\alpha \in R\,, P_i(t) \in (L^2[0,1])^{ns}\,, \quad t \in [0,1]\,, \quad i = 1, \dots, \infty\,.$$

The Fritz-John system of (38) is equivalent to (36), so the stationary functions are the same. Returning to $x_l(t)$, $\mu_k(t)$, $\eta(t)$, $\lambda_j(t)$, we get the problem (35). ■

The problem (24) should be primary and (38) secondary.

Consequence: The system (36) can be transformed into the space f.

Proof: By using (32) and after the expansion of the nonlinear functions in the Fourier series, system (37) becomes

$$
\begin{aligned}
&\sum_{r=1}^{\infty} P_r(t) g_r^k(\alpha) \le 0\,, \quad k=1,\dots,q\,,\\
&\sum_{r=1}^{\infty} P_r(t) h_r^j(\alpha) = 0\,, \quad j=1,\dots,m\,,\\
&\sum_{s=1}^{\infty} \kappa_s P_s(t) \sum_{r=1}^{\infty} P_r(t) g_r^k(\alpha) \int_0^1 \nabla F(\mathbf{x}(t)) P_s^l(t) P_r(t)\,\mathrm{d}t\\
&\quad + \sum_{j=1}^{m}\sum_{s=1}^{\infty} \zeta_{sj} P_s^j(t) \sum_{r=1}^{\infty} P_r(t) \int_0^1 \nabla H^j(\mathbf{x}(t)) P_s^l(t) P_r(t)\,\mathrm{d}t\\
&\quad + \sum_{k=1}^{q}\sum_{s=1}^{\infty} \theta_{sk} P_s^k(t) P_s^j(t) \sum_{r=1}^{\infty} P_r(t) \int_0^1 \nabla G^k(\mathbf{x}(t)) P_s^l(t) P_r(t)\,\mathrm{d}t = 0\,,\\
&\sum_{s=1}^{\infty} \theta_{sk} P_s^k(t) \ge 0\,, \quad k=1,\dots,q\,,\\
&\sum_{s=1}^{\infty} \theta_{sk} P_s^k(t) G^k(\boldsymbol{\alpha}[\mathbf{B}(t)]^*) = 0\,,\\
&P_i(t) \in L^2(0,1)\,, \quad \forall\, \mathrm{fix}\, t\,, \quad t\in[0,1]\,,
\end{aligned}
\tag{39}
$$

The variables of (39) are α_{il}, κ_i, ζ_{ij}, θ_{ik}, $l = 1,...,ns$, $j = 1,...,m$, $k = 1,...,q$, $i = 1,...,\infty$. By introducing the notations

$$
\eta(t) = \sum_{s=1}^{\infty} \kappa_s P_s(t)\,, \quad \lambda_j(t) = \sum_{s=1}^{\infty} \zeta_{sj} P_s^j(t)\,, \quad \mu_k(t) = \sum_{s=1}^{\infty} \theta_{sk} P_s^k(t)\,.
$$

$$
\mathcal{C}(f_r(\alpha)) = \int_0^1 \nabla F(\boldsymbol{\alpha}^*[\mathbf{B}(t)]) P_r(t)\,\mathrm{d}t\,, \quad \mathcal{C}(g_r^k(\alpha)) = \int_0^1 \nabla G^k(\boldsymbol{\alpha}^*[\mathbf{B}(t)]) P_r(t)\,\mathrm{d}t\,,
$$

$$
\mathcal{C}(h_r^j(\alpha)) = \int_0^1 \nabla H^j(\boldsymbol{\alpha}^*[\mathbf{B}(t)]) P_r(t)\,\mathrm{d}t\,, \quad \mathcal{C}(\tilde{g}_r^k(\alpha)) = \int_0^1 P_i(t) G^k(\boldsymbol{\alpha}^*[\mathbf{B}(t)]) P_r(t)\,\mathrm{d}t\,,
$$

the system (39) can be written in a form where inequalities are taken into account at the given values of t only:

$$
\begin{aligned}
&\sum_{r=1}^{\infty} P_r(t_v) g_r^k(\alpha) \le 0\,, \quad k=1,\dots,q\,, \quad v=1,\dots,w\,,\\
&\sum_{r=1}^{\infty} P_r(t) h_r^j(\alpha) = 0\,, \quad j=1,\dots,m\,,\\
&\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} P_r(t) \left(P_s(t)\kappa_s \mathcal{C}(f_r(\alpha)) + \sum_{j=1}^{m} \zeta_{sj} P_s^j(t) \mathcal{C}(h_r^j(\alpha)) + \sum_{k=1}^{q} \theta_{sk} P_s^k(t) \mathcal{C}(g_r^k(\alpha)) \right) = 0\,,\\
&\qquad\qquad l=1,\dots,ns\,,\\
&\sum_{s=1}^{\infty} \theta_{sk} P_s^k(t_v) \ge 0\,, \quad k=1,\dots,q\,, \quad v=1,\dots,w\,,\\
&\sum_{s=1}^{\infty} \theta_{sk} \sum_{r=1}^{\infty} \mathcal{C}(\tilde{g}_r^k(\alpha)) = 0\,, \quad k=1,\dots,q\,, \quad l=1,\dots,ns\,,\\
&P_i(t) \in L^2([0,1])\,, \quad t\in[0,1]\,,
\end{aligned}
\tag{40}
$$

where w is the number of different points in which the inequalities are considered. At the third group of equations we again expand the polynomial product in the Fourier series:

$$P_r(t)P_s^j(t) = \sum_{z=1}^{\infty} P_z(t) \int_0^1 P_r(t)P_s^j(t)P_z(t)\,\mathrm{d}t = \sum_{s=1}^{\infty} P_z(t)P_{rsz}^j\,,$$

where $P_{rsz}^j = \int_0^1 P_r(t)P_z^j(t)P_s(t)\,\mathrm{d}t$.

Let us denote the value of the r-th polynomial belonging to the k-th basis in the v-th point in time by

$$p_{rv}^k = P_z^k(t_v)\,.$$

In the space f the system (40) is

$$\begin{aligned}
&p_{rv}^k g_r^k(\alpha) \le 0\,, \quad k = 1,\dots,q\,, \quad r = 1,\dots,\infty\,, \quad v = 1,\dots,w\,,\\
&h_r^j(\alpha) = 0\,, \quad j = 1,\dots,m\,, \quad r = 1,\dots,\infty\,,\\
&\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\left(p_{rzs}C(f_r(\alpha)) + \sum_{j=1}^{m} p_{rzs}^j \eta_z^j C(h_r^j(\alpha)) + \sum_{k=1}^{q} p_{rzs}^k \chi_z^k C(g_r^k(\alpha))\right) = 0\,,\\
&\qquad l = 1,\dots,ns\,, \quad i = 1,\dots,\infty\,, \quad z = 1,\dots,\infty\,,\\
&p_{rv}^k \theta_{sk} \ge 0\,, \quad k = 1,\dots,q\,, \quad z = 1,\dots,\infty\,, \quad v = 1,\dots,w\,,\\
&\sum_{z=1}^{\infty} \theta_{zk} C(\tilde{g}_r^k(\alpha)) = 0\,,
\end{aligned} \tag{41}$$

where $\eta_z^j = \zeta_{zj}/\kappa_z$, $\chi_z^k = \theta_{zk}/\kappa_z$. ■

If (8) contains equality constrains only, the solution can easily be computed by (41). The Fritz-John system is in this case as follows:

$$\begin{aligned}
&h_r^j(\alpha) = 0\,, \quad j = 1,\dots,m\,, \quad r = 1,\dots,\infty\,,\\
&\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\left(p_{rzs}C(f_r(\alpha)) + \sum_{j=1}^{m} p_{rzs}^j \eta_z^j C(h_r^j(\alpha))\right) = 0\,,\\
&\qquad l = 1,\dots,ns\,, \quad i = 1,\dots,\infty\,, \quad z = 1,\dots,\infty\,.
\end{aligned} \tag{42}$$

We multiply the first equality groups of (42) by $\sum_{s=1}^{\infty} p_{ris} \ne 0$ corresponding to the indices i and r:

$$\begin{aligned}
&\sum_{s=1}^{\infty} p_{ris} h_r^j(\alpha) = 0\,, \quad j = 1,\dots,m\,, \quad r = 1,\dots,\infty\,,\\
&\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\left(p_{rzs}C(f_r(\alpha)) + \sum_{j=1}^{m} p_{rzs}^j \eta_z^j C(h_r^j(\alpha))\right) = 0\,, \quad l = 1,\dots,ns\,, \quad z = 1,\dots,\infty\,.
\end{aligned} \tag{43}$$

The solution of (43) coincides with the solution of the following pair of problems:

$$\begin{aligned}
&\min \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\sum_{z=1}^{\infty} p_{rzs} f_r(\alpha)\,,\\
&\sum_{s=1}^{\infty} p_{ris}^j h_r^j(\alpha) = 0\,, \quad \alpha_{ij} \in R\,, \quad j = 1,\dots,m\,, \quad r = 1,\dots,\infty\,, \quad i = 1,\dots,\infty\,,
\end{aligned} \tag{44}$$

and

$$\max \sum_{s=1}^{\infty}\left(\sum_{r=1}^{\infty}\sum_{z=1}^{\infty} p_{rzs} f_r(\alpha) + \sum_{j=1}^{m} p^j_{rzs}\eta^j_z h^j_r(\alpha)\right),$$
$$\sum_{rzs=1}^{\infty}\left(p_{rzs}\nabla f_r(\alpha) + \sum_{j=1}^{m} p^j_{rzs}\eta^j_z \nabla h^j_r(\alpha)\right) = 0\,, \quad \alpha_{ij}\in R\,, \quad j=1,\dots,m\,, \quad s=1,\dots,\infty\,. \tag{45}$$

The problem (38) is dual to (25). If there are equalities only, (44) and (45) correspond to (25) and (38) in the space f, respectively. Note that (44) and (45) are nonlinear infinite-dimensional optimization problems.

If (8) contains inequality constraints, (41) should be solved approximately. The result is the linear combination of the Fourier coefficients $\boldsymbol{\alpha}$ determined by (41) with the elements of the basis.

If there are inequalities in (25) and (38), the same reasoning is not possible because the transformation of the inequalities into the space f does not satisfy the isomorphism theorem.

If the inequalities in (25) are discretized at a finite number of points, then it follows from the above results

$$\min \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\sum_{z=1}^{\infty} p_{rzs} f_r(\alpha)\,,$$
$$\sum_{s=1}^{\infty} p^j_{ris} h^j_r(\alpha) = 0\,, \quad j=1,\dots,m\,, \quad r=1,\dots,\infty\,, \quad i=1,\dots,\infty\,, \tag{46}$$
$$G^k(\boldsymbol{\alpha}^*\mathbf{B}(t_v)) \le 0\,, \quad k=1,\dots,q\,, \quad v=1,\dots,w\,, \quad \alpha_{ij}\in R\,, \quad t_v\in[0,1]\,,$$

and

$$\max\left(\sum_{s=1}^{\infty}\left(\sum_{r=1}^{\infty}\sum_{z=1}^{\infty} p_{rzs} f_r(\alpha) + \sum_{j=1}^{m} p^j_{rzs}\eta^j_z h^j_r(\alpha)\right) + \sum_{k=1}^{q}\sum_{v=1}^{w}\mu_k(t_v) G^k(\boldsymbol{\alpha}^*\mathbf{B}(t_v))\right),$$
$$\sum_{rzs=1}^{\infty}\left(p_{rzs}\nabla f_r(\alpha) + \sum_{j=1}^{m} p^j_{rzs}\eta^j_z\nabla h^j_r(\alpha)\right) + \sum_{k=1}^{q}\sum_{v=1}^{w}\mu_k(t_v)\nabla G^k(\boldsymbol{\alpha}^*\mathbf{B}(t_v)) = 0\,, \tag{47}$$
$$\mu_k(t_v) \ge 0\,,$$
$$\alpha_{ij}\in R\,, \quad t_v\in[0,1] \quad l=1,\dots,ns\,, \quad j=1,\dots,m\,, \quad s=1,\dots,\infty\,.$$

In practice, a finite number of functions are taken into consideration from the basis of $\mathcal{F}$ and the solution should be approximated only in this subspace. Consequently, computations should be made in the corresponding subspace of f, and a nonlinear inequality system or optimization problem may be solved in finite dimensions.

8. FRITZ-JOHN CONDITION WITH FUNCTIONAL DERIVATIVE

We obtain the same result by writing the Fritz-John condition with the functional derivatives of the functions in L^2. In the case of mechanical processes, the rates of state variables are different from zero until the process reaches the state of equilibrium. In theoretical physics, the equilibrium state of structures is in limit positions only. Now, assume that the rates are different from zero. Multiplying and dividing by $\sum_{i=1}^{\infty}\alpha_{il}\dot{P}(t)\neq 0$, the first equation group of (27) is

$$\eta(t)\frac{\mathrm{d}\left(F\sum_{i=1}^{\infty}\alpha_i lP(t)\right)}{\mathrm{d}\left(\sum_{i=1}^{\infty}\alpha_i lP(t)\right)}\left(\sum_{i=1}^{\infty}\alpha_i l\dot{P}(t)\right)\frac{1}{\left(\sum_{i=1}^{\infty}\alpha_i l\dot{P}(t)\right)}$$

$$+\sum_{j=1}^{m}\lambda_j(t)\frac{\mathrm{d}\left(H^j\sum_{i=1}^{\infty}\alpha_i lP(t)\right)}{\mathrm{d}\left(\sum_{i=1}^{\infty}\alpha_i lP(t)\right)}\left(\sum_{i=1}^{\infty}\alpha_i l\dot{P}(t)\right)\frac{1}{\left(\sum_{i=1}^{\infty}\alpha_i l\dot{P}(t)\right)}$$

$$-\sum_{k=1}^{q}\mu_k(t)\frac{\mathrm{d}\left(G^k\sum_{i=1}^{\infty}\alpha_i lP(t)\right)}{\mathrm{d}\left(\sum_{i=1}^{\infty}\alpha_i lP(t)\right)}\left(\sum_{i=1}^{\infty}\alpha_i l\dot{P}(t)\right)\frac{1}{\left(\sum_{i=1}^{\infty}\alpha_i l\dot{P}(t)\right)},\quad l=1,\dots,ns. \tag{48}$$

The derivatives are the derivatives of F, G and H with respect to t. Thus (48) can be written in the following form:

$$\eta(t)\frac{\dot{F}(\mathbf{x}(t))}{\dot{x}_l(t)}+\sum_{j=1}^{m}\lambda_j(t)\frac{\dot{H}^j(\mathbf{x}(t))}{\dot{x}_l(t)}-\sum_{k=1}^{q}\mu_k(t)\frac{\dot{G}^k(\mathbf{x}(t))}{\dot{x}_l(t)}=0\,, \tag{49}$$

$$\dot{\mathbf{x}}(t)\neq 0\,, x(t)\in\left(L^2\right)^{sn},\quad t\in[0,1]\,,\quad l=1,\dots,ns\,.$$

The quotients of the derivatives with respect to the parameter appearing in (49), are the functional derivatives of F, G and H, [1].

The functional derivative is the generalization of Stieltjes derivative which was interpreted as the operation inverse to the Stieltjes integral in [4] and [19].

If $\dot{x}(t)\neq 0$, the Stieltjes derivative is

$$\frac{\mathrm{d}f(x(t))}{\mathrm{d}x(t)}=\frac{\dot{f}(x(t))}{\dot{x}(t)}=\nabla f(\mathbf{x}(t))\,.$$

The rate of the state variables in mechanical processes can be investigated by functional derivatives.

9. Examples

Two simple examples are presented to illustrate the above method. They have no mechanical meaning because in this case the dimensions would be too large for manual computations.

a. Consider the problem:

$$\begin{aligned}
&1 \qquad x_1(t)+4x_2(t)-5\sin t=0,\\
&6 \qquad -x_1(t)\leq 0,\\
&7 \qquad -x_2(t)\leq 0,\\
&\qquad\ \ \min\ x_1(t)-3x_2(t).
\end{aligned} \tag{50}$$

The Fritz-John optimality conditions of (50) are

$$
\begin{aligned}
&1 \quad x_1(t) + 4x_2(t) = 5\sin t,\\
&2 \quad u_1(t) - u_2(t) + 1 = 0,\\
&3 \quad 4u_1(t) - u_2(t) - 3 = 0,\\
&4 \quad u_2(t)x_1(t) = 0,\\
&5 \quad u_3(t)x_2(t) = 0,\\
&6 \quad x_1(t) \geq 0,\\
&7 \quad x_2(t) \geq 0,\\
&8 \quad u_2(t) \geq 0,\\
&9 \quad u_3(t) \geq 0.
\end{aligned}
\tag{51}
$$

The Fritz-John optimality conditions (51) can be written by using expression (2), and by considering the Fourier series up to the first four terms:

$$
\begin{aligned}
&1. \quad \Big(K_1 + \sum_{i=1,3} \alpha_{1i}\sin(it) + \sum_{i=2,4} \alpha_{1i}\cos(it) \Big)\\
&\qquad +4\Big(K_2 + \sum_{i=1,3} \alpha_{2i}\sin(it) + \sum_{i=2,4} \alpha_{2i}\cos(it) \Big) = 0 + 5\sin t\,,\\
&2. \quad \Big(k_1 + \sum_{i=1,3} \beta_{1i}\sin(it) + \sum_{i=2,4} \beta_{1i}\cos(it) \Big)\\
&\qquad -\Big(k_2 + \sum_{i=1,3} \beta_{2i}\sin(it) + \sum_{i=2,4} \beta_{2i}\cos(it) \Big) + 1 = 0\,,\\
&3. \quad 4\Big(k_1 + \sum_{i=1,3} \beta_{1i}\sin(it) + \sum_{i=2,4} \beta_{1i}\cos(it) \Big)\\
&\qquad -\Big(k_3 + \sum_{i=1,3} \beta_{2i}\sin(it) + \sum_{i=2,4} \beta_{2i}\cos(it) \Big) - 3 = 0\,,\\
&4. \quad \Big(k_2 + \sum_{i=1,3} \beta_{2i}\sin(it) + \sum_{i=2,4} \beta_{2i}\cos(it) \Big)\\
&\qquad +\Big(K + \sum_{i=1,3} \alpha_{1i}\sin(it) + \sum_{i=2,4} \alpha_{1i}\cos(it) \Big) = 0\,,\\
&5. \quad \Big(k_3 + \sum_{i=1,3} \beta_{3i}\sin(it) + \sum_{i=2,4} \beta_{3i}\cos(it) \Big)\\
&\qquad +\Big(K_2 + \sum_{i=1,3} \alpha_{2i}\sin(it) + \sum_{i=2,4} \alpha_{2i}\cos(it) \Big) = 0\,,\\
&6. \quad \Big(K_1 + \sum_{i=1,3} \alpha_{1i}\sin(it) + \sum_{i=2,4} \alpha_{1i}\cos(it) \Big) \geq 0\,,\\
&7. \quad \Big(K_1 + \sum_{i=1,3} \alpha_{2i}\sin(it) + \sum_{i=2,4} \alpha_{2i}\cos(it) \Big) \geq 0\,,\\
&8. \quad \Big(k_2 + \sum_{i=1,3} \beta_{2i}\sin(it) + \sum_{i=2,4} \beta_{2i}\cos(it) \Big) \geq 0\,,\\
&9. \quad \Big(k_2 + \sum_{i=1,3} \beta_{3i}\sin(it) + \sum_{i=2,4} \beta_{3i}\cos(it) \Big) \geq 0\,.
\end{aligned}
\tag{52}
$$

Formulating the Fritz-John optimality conditions in the space f, considering the values of the function at points $k\pi/4$, $k = 0, 1, \ldots, 7$ and computing the values of the integrals of the polynomial products (p_{ijk}), we obtain

$$\begin{array}{llllll}
1. & K_1+4K_2=0, & 2. & k_1-k_2+k_3+1=0, & 3. & 4k_1-k_3-3=0,\\
 & \alpha_{11}+4\alpha_{21}=5, & & \beta_{11}-\beta_{21}+\beta_{31}=0, & & 4\beta_{11}-\beta_{31}=0,\\
 & \alpha_{12}+4\alpha_{22}=0, & & \beta_{12}-\beta_{22}+\beta_{32}=0, & & 4\beta_{12}-\beta_{32}=0,\\
 & \alpha_{13}+4\alpha_{23}=0, & & \beta_{13}-\beta_{23}+\beta_{33}=0, & & 4\beta_{13}-\beta_{33}=0,\\
 & \alpha_{14}+4\alpha_{24}=0, & & \beta_{14}-\beta_{24}+\beta_{34}=0, & & 4\beta_{14}-\beta_{34}=0,
\end{array}$$

$$\begin{array}{ll}
4. & k_2K_2+\pi(k_2\alpha_{11}+K_1\beta_{21})+\pi/2(-\beta_{21}\alpha_{14}-\beta_{22}\alpha_{13}+\beta_{23}\alpha_{12}+\beta_{24}\alpha_{11})=0,\\
 & k_2K_2+\pi(k_2\alpha_{12}+K_1\beta_{22})+\pi/2(-\beta_{21}\alpha_{13}+\beta_{22}\alpha_{14}-\beta_{23}\alpha_{11}+\beta_{24}\alpha_{12})=0,\\
 & k_2K_2+\pi(k_2\alpha_{13}+K_1\beta_{23})+\pi/2(\beta_{21}\alpha_{12}-\beta_{22}\alpha_{11}+\beta_{23}\alpha_{14}-\beta_{24}\alpha_{13})=0,\\
 & k_2K_2+\pi(k_2\alpha_{14}+K_1\beta_{24})+\pi/2(\beta_{21}\alpha_{11}+\beta_{22}\alpha_{12}-\beta_{23}\alpha_{13}-\beta_{24}\alpha_{14})=0,
\end{array}$$

$$\begin{array}{ll}
5. & k_3K_2+\pi(k_3\alpha_{21}+K_2\beta_{31})+\pi/2(-\beta_{31}\alpha_{24}-\beta_{32}\alpha_{23}+\beta_{33}\alpha_{22}+\beta_{34}\alpha_{21})=0,\\
 & k_3K_2+\pi(k_3\alpha_{22}+K_2\beta_{32})+\pi/2(-\beta_{31}\alpha_{23}+\beta_{32}\alpha_{24}-\beta_{33}\alpha_{21}+\beta_{34}\alpha_{22})=0,\\
 & k_3K_2+\pi(k_3\alpha_{23}+K_2\beta_{33})+\pi/2(\beta_{31}\alpha_{22}-\beta_{32}\alpha_{21}+\beta_{33}\alpha_{24}-\beta_{34}\alpha_{23})=0,\\
 & k_3K_2+\pi(k_3\alpha_{24}+K_2\beta_{34})+\pi/2(\beta_{31}\alpha_{21}+\beta_{32}\alpha_{22}-\beta_{33}\alpha_{23}-\beta_{34}\alpha_{24})=0,
\end{array}$$

$$\begin{array}{llll}
6. & K_1+\alpha_{12}+\alpha_{14}\ge 0, & 7. & K_2+\alpha_{22}+\alpha_{24}\ge 0,\\
 & K_1+\sqrt{2}/2\alpha_{11}+\sqrt{2}/2\alpha_{12}+\alpha_{13}\ge 0, & & K_2+\sqrt{2}/2\alpha_{21}+\sqrt{2}/2\alpha_{22}+\alpha_{23}\ge 0,\\
 & K_1+\alpha_{11}-\alpha_{14}\ge 0, & & K_2+\alpha_{21}-\alpha_{24}\ge 0,\\
 & K_1+\sqrt{2}/2\alpha_{11}-\sqrt{2}/2\alpha_{12}-\alpha_{13}\ge 0, & & K_2+\sqrt{2}/2\alpha_{21}-\sqrt{2}/2\alpha_{22}-\alpha_{23}\ge 0,\\
 & K_1-\alpha_{12}+\alpha_{14}\ge 0, & & K_2-\alpha_{22}+\alpha_{24}\ge 0,\\
 & K_1-\sqrt{2}/2\alpha_{11}-\sqrt{2}/2\alpha_{12}+\alpha_{13}\ge 0, & & K_2-\sqrt{2}/2\alpha_{21}-\sqrt{2}/2\alpha_{22}+\alpha_{23}\ge 0,\\
 & K_1-\alpha_{11}-\alpha_{14}\ge 0, & & K_2-\alpha_{21}-\alpha_{24}\ge 0,\\
 & K_1-\sqrt{2}/2\alpha_{11}+\sqrt{2}/2\alpha_{12}-\alpha_{13}\ge 0, & & K_2-\sqrt{2}/2\alpha_{21}+\sqrt{2}/2\alpha_{22}-\alpha_{23}\ge 0,
\end{array}$$

$$\begin{array}{llll}
8. & k_2+\beta_{22}+\beta_{24}\ge 0, & 9. & k_3+\beta_{32}+\beta_{34}\ge 0,\\
 & k_2+\sqrt{2}/2\beta_{21}+\sqrt{2}/2\beta_{22}+\beta_{23}\ge 0, & & k_3+\sqrt{2}/2\beta_{31}+\sqrt{2}/2\beta_{32}+\beta_{33}\ge 0,\\
 & k_2+\beta_{21}-\beta_{24}\ge 0, & & k_3+\beta_{31}-\beta_{34}\ge 0,\\
 & k_2+\sqrt{2}/2\beta_{21}-\sqrt{2}/2\beta_{22}-\beta_{23}\ge 0, & & k_3+\sqrt{2}/2\beta_{31}-\sqrt{2}/2\beta_{32}-\beta_{33}\ge 0,\\
 & k_2-\beta_{22}+\beta_{24}\ge 0, & & k_3-\beta_{32}+\beta_{34}\ge 0,\\
 & k_2-\sqrt{2}/2\beta_{21}-\sqrt{2}/2\beta_{22}+\beta_{23}\ge 0, & & k_3-\sqrt{2}/2\beta_{31}-\sqrt{2}/2\beta_{32}+\beta_{33}\ge 0,\\
 & k_2-\beta_{21}-\beta_{24}\ge 0, & & k_3-\beta_{31}-\beta_{34}\ge 0,\\
 & k_2-\sqrt{2}/2\beta_{21}+\sqrt{2}/2\beta_{22}-\beta_{23}\ge 0, & & k_3-\sqrt{2}/2\beta_{31}+\sqrt{2}/2\beta_{32}-\beta_{33}\ge 0,
\end{array}$$

The approximation of the solution (50) is obtained by solving the system (53) for coefficients α and K:

$$\begin{aligned}
x_1 &= K_1+\sum_{i=1,3}\alpha_{1i}\sin(it)+\sum_{i=2,4}\alpha_{1i}\cos(it),\\
x_2 &= K_2+\sum_{i=1,3}\alpha_{2i}\sin(it)+\sum_{i=2,4}\alpha_{2i}\cos(it)\,.
\end{aligned} \tag{53}$$

b. Consider the problem:

$$\begin{array}{ll}
1 & x_1(t)+4x_2(t)-5\sin t=0\,,\\
 & \min x_1(t)x_2(t)\,.
\end{array} \tag{54}$$

The Fritz-John optimality conditions of (54) are

$$\begin{aligned} &1 \quad x_1(t)+4x_2(t) &&= 5\sin t,\\ &2 \quad u_1(t)+x_2(t) &&= 0,\\ &3 \quad 4u_1(t)+x_1(t) &&= 0. \end{aligned} \tag{55}$$

By using the expression (2) and considering the Fourier series up to the first four terms, (55) assumes the form

$$\begin{aligned} 1.\quad & \Big(K_1+\sum_{i=1,3}\alpha_{1i}\sin(it)+\sum_{i=2,4}\alpha_{1i}\cos(it)\Big)\\ & +4\Big(K_2+\sum_{i=1,3}\alpha_{2i}\sin(it)+\sum_{i=2,4}\alpha_{2i}\cos(it)\Big)=0+5\sin t,\\ 2.\quad & \Big(k_1+\sum_{i=1,3}\beta_{1i}\sin(it)+\sum_{i=2,4}\beta_{1i}\cos(it)\Big)\\ & -\Big(K_2+\sum_{i=1,3}\alpha_{2i}\sin(it)+\sum_{i=2,4}\alpha_{2i}\cos(it)\Big)+1=0,\\ 3.\quad & 4\Big(k_1+\sum_{i=1,3}\beta_{1i}\sin(it)+\sum_{i=2,4}\beta_{1i}\cos(it)\Big)\\ & -\Big(K_1+\sum_{i=1,3}\alpha_{1i}\sin(it)+\sum_{i=1,4}\alpha_{1i}\cos(it)\Big)-3=0. \end{aligned} \tag{56}$$

In the space f the optimality system (56) is

$$\begin{aligned} &1.\ K_1+4K_2=0, && 2.\ k_1+K_2=0, && 3.\ 4k_1+K_1=0,\\ &\alpha_{11}+4\alpha_{21}=5, && \beta_{11}+\alpha_{21}=0, && 4\beta_{11}+\alpha_{11}=0,\\ &\alpha_{12}+4\alpha_{22}=0, && \beta_{12}+\alpha_{22}=0, && 4\beta_{12}+\alpha_{12}=0,\\ &\alpha_{13}+4\alpha_{23}=0, && \beta_{13}+\alpha_{23}=0, && 4\beta_{13}+\alpha_{13}=0,\\ &\alpha_{14}+4\alpha_{24}=0, && \beta_{14}+\alpha_{24}=0, && 4\beta_{14}+\alpha_{14}=0, \end{aligned} \tag{57}$$

The following mathematical programming problem has the optimality system (57):

$$\begin{aligned} 1.\quad & K_1+4K_2=0,\\ & \alpha_{11}+4\alpha_{21}=5,\\ & \alpha_{12}+4\alpha_{22}=0,\\ & \alpha_{13}+4\alpha_{23}=0,\\ & \alpha_{14}+4\alpha_{24}=0,\\ & \min\,(K_1K_2+\alpha_{11}\alpha_{21}+\alpha_{12}\alpha_{22}+\alpha_{13}\alpha_{23}+\alpha_{14}\alpha_{24}) \end{aligned} \tag{58}$$

and also the dual problem of (58) has the optimality system (57):

$$\begin{aligned} &2.\ k_1+K_2=0, && 3.\ 4k_1+K_1=0,\\ &\beta_{11}+\alpha_{21}=0, && 4\beta_{11}+\alpha_{11}=0,\\ &\beta_{12}+\alpha_{22}=0, && 4\beta_{12}+\alpha_{12}=0,\\ &\beta_{13}+\alpha_{23}=0, && 4\beta_{13}+\alpha_{13}=0,\\ &\beta_{14}+\alpha_{24}=0, && 4\beta_{14}+\alpha_{14}=0,\\ &\min\,(-5\beta_{11}-K_1K_2-\alpha_{11}\alpha_{21}-\alpha_{12}\alpha_{22}-\alpha_{13}\alpha_{23}-\alpha_{14}\alpha_{24}). \end{aligned} \tag{59}$$

10. SUMMARY

A new method is presented for the analysis of the state change in the local equilibrium state. This method can be used if the rate of state variables depends on the time. The problem is solved as a

parametrical optimization problem instead of the "path following" method. It is proved that the Fritz-John type conditions are valid in L^2. The problem is transformed into space l^2 based on the isomorphism theorem. Inequality conditions are not order-preserving, therefore, only approximate solutions can be derived.

Finally, the original problem with inequalities formulated in the space L^2 can be transformed into one-parameter inequality system in the space l^2, or pairs of semi-infinite mathematical programming problem. If only equalities appear in mathematical programming problems, pairs of semi-infinite mathematical programming problem can be transformed in l^2 to the problem in the space L^2.

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