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On the systematic construction of trial functions for hybrid Trefftz shell elements

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In this paper the possibility of applying the Trefftz-method to thick and thin shells is discussed. A mixed variational formulation is used in which the assumed strain and stress functions are derived from the three-dimensional solution representation for the displacement field. For the construction of the linearly independent Trefftz trial functions both the Neuber/Papkovich solution representation and a complex variable approach of the author are considered. The difficulty in constructing the solution functions for the displacement field consists of two problems: i) How can we choose the functions in order to have a symmetric structure in the displacement field and not to bias the solution in any direction? ii) How can we avoid to get linearly dependent terms for displacements, strains and stresses when seeking polynomial solution terms?

1. Introduction

For a series of different problems the solutions of the governing differential equations have been studied in the literature. For an application in the Trefftz-method [1] we are interested in a series of linearly independent functions satisfying the homogeneous differential equations and in "particular solutions" of the inhomogeneous differential equations. The linearly independent homogeneous solution functions have free parameters which have to be evaluated in the Trefftz algorithm. On the other hand the particular solutions usually do not have free parameters. Trefftz trial functions and their application for Finite Element Methods and Boundary Element procedures have been studied for example by Herrera [2–4], Jirousek and co-workers [5–14], Kita and Kamiya [15], Melenk [16, 17], Piltner [18–27], Piltner and Taylor [28, 29], Ruoff [30], Stein [31–33], Szybiński [34, 35, 13], Teixeira de Freitas [36], Tong [37], Wróblewski [38, 39], Zieliński [40–43], and Zienkiewicz [44–46]. Overview articles on the Trefftz method have been presented, for example, by Kita and Kamiya [15] and Zieliński [40]. More references on the Trefftz-method can be found in the special journal issue on the "Trefftz Method: 70 Years" [47].

For two dimensional elasticity problems the complex representation of Muskhelishvili [48] and Kolosov is very useful for the construction of Trefftz-trial functions since the solution is given in terms of arbitrary complex functions. Since the governing differential equations are satisfied for any functions of a complex variable we have a tool to construct sets of infinite linearly independent functions in different coordinates. The concept of Muskhelishvili to express solutions in terms of arbitrary complex valued functions can be extended to thick plate [21–23] and three-dimensional elasticity problems [24–26].

For the type of shell element considered in this paper we want to utilize stress functions which satisfy both the equilibrium and the compatibility equations. Such functions can be derived from a 3-dimensional solution representation for the displacement field.

The amount of literature on finite shell elements is quite huge. Many finite elements for the analysis of shells have been proposed. A list of books, monographs, conference proceedings and

survey papers on shells can be found in references [49, 50]. For this paper the transformation between the cartesian coordinates x, y, z and the natural coordinates ξ, η, ζ as well as the form of the displacement field u, v, w are chosen as described in references [51, 52]. For the mixed variational formulation considered in this paper the assumed stresses and strains are derived as Trefftz functions. In references [51, 52] a modified constitutive relationship is used whereas in the current approach the three-dimensional constitutive equations are used.

2. Variational formulation

For the derivation of a shell element the following modified Hu–Washizu variational formulation [28] is considered:

$$\Pi(\tilde{\mathbf{u}}, \boldsymbol{\varepsilon}^i, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}) = \int_V \tfrac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{E}\boldsymbol{\varepsilon}\,\mathrm{d}V - \int_V \tilde{\mathbf{u}}^{\mathrm{T}}\bar{\mathbf{f}}\,\mathrm{d}V - \int_S \tilde{\mathbf{u}}^{\mathrm{T}}\bar{\mathbf{T}}\,\mathrm{d}S - \int_V \boldsymbol{\sigma}^{\mathrm{T}}(\boldsymbol{\varepsilon} - \mathbf{D}\tilde{\mathbf{u}} - \boldsymbol{\varepsilon}^i)\,\mathrm{d}V\,, \tag{1}$$

where $\tilde{\mathbf{u}}$ is a compatible displacement field involving shape functions, nodal displacements and nodal rotations. $\boldsymbol{\varepsilon}^{\mathbf{i}}$ is the enhanced strain field satisfying the condition that each enhanced strain term is orthogonal to proper chosen reference stresses. This condition can be written as

$$\int_V (\delta\boldsymbol{\sigma}^*)^{\mathrm{T}}\boldsymbol{\varepsilon}^{\mathbf{i}}\,\mathrm{d}V = \int_V (\delta\boldsymbol{\varepsilon}^i)^{\mathrm{T}}\boldsymbol{\sigma}^*\,\mathrm{d}V = 0\,. \tag{2}$$

The original version of the "enhanced strain" method was proposed by Simo and Rifai [53]. One possibility of deriving enhanced strain terms is to use an appropriate incompatible displacement field. The use of incompatible displacements for the construction of admissible strains was first applied successfully for the four node plane strain/stress element by Taylor/Beresford and Wilson [54] (element QM6). Further discussion on the use of incompatible displacements can be found in reference [55].

The strain field $\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{u}$ is obtained from a three-dimensional displacement field $\mathbf{u}$ satisfying the Navier equations

$$\mathbf{D}^{\mathrm{T}}\mathbf{E}\mathbf{D}\mathbf{u} = \mathbf{0}\,, \tag{3}$$

where $\mathbf{D}$ is a differential operator matrix. The assumed stresses $\boldsymbol{\sigma}$ satisfy the equilibrium equations and can be obtained from the constitutive relationship $\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}$.

Carrying out the variation in (1) the following equations are obtained:

$$-\int_V \delta\boldsymbol{\sigma}^{\mathrm{T}}\left[\boldsymbol{\varepsilon} - \mathbf{D}\tilde{\mathbf{u}} - \boldsymbol{\varepsilon}^i\right]\mathrm{d}V = 0\,, \tag{4}$$

$$\int_V \delta\boldsymbol{\varepsilon}^{\mathrm{T}}\left[\mathbf{E}\boldsymbol{\varepsilon} - \boldsymbol{\sigma}\right]\mathrm{d}V = 0\,, \tag{5}$$

$$\int_V \delta(\mathbf{D}\tilde{\mathbf{u}})^{\mathrm{T}}\boldsymbol{\sigma}\,\mathrm{d}V = \int_V \delta\tilde{\mathbf{u}}^{\mathrm{T}}\bar{\mathbf{f}}\,\mathrm{d}V + \int_S \delta\tilde{\mathbf{u}}^{\mathrm{T}}\bar{\mathbf{T}}\,\mathrm{d}S\,, \tag{6}$$

$$\int_V (\delta\boldsymbol{\varepsilon}^i)^{\mathrm{T}}\boldsymbol{\sigma}\,\mathrm{d}V = 0\,. \tag{7}$$

The displacement, strain and stress fields are chosen in the following form:

$$\begin{aligned} \tilde{\mathbf{u}} &= \mathbf{N}\mathbf{q}\,, \\ \boldsymbol{\varepsilon} &= \mathbf{A}\boldsymbol{\beta}\,, \\ \boldsymbol{\sigma} &= \mathbf{E}\mathbf{A}\boldsymbol{\beta} = \mathbf{P}\boldsymbol{\beta}\,, \\ \boldsymbol{\varepsilon}^{\mathbf{i}} &= \mathbf{B}^{\mathbf{i}}\boldsymbol{\lambda}\,. \end{aligned} \tag{8}$$

where $\mathbf{N} = \mathbf{N}(\zeta, \eta)$ is the matrix of compatible shape functions and $\mathbf{q}$ contains the nodal displacements and rotations of the shell element. The vectors $\boldsymbol{\beta}, \boldsymbol{\lambda}$ are strain/stress and enhanced strain parameters, respectively. For the proposed approach appropriate Trefftz functions are chosen for the fields $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$. Note that for the current approach a variational formulation with domain integrals is used instead of a formulation with boundary integrals, usually chosen for Trefftz-type finite elements. Of course also here such a formulation with boundary integrals could be used. However, for the element under consideration we have six surface portions for the shell element on which boundary integrals have to be evaluated. It appears to be simpler in this case just to evaluated one volume integral instead of six surface integrals in order to get the finite element stiffness matrix.

In order to specify the entries of the shape function matrix $\mathbf{N}$ we first define the relationship between the cartesian coordinates x, y, z and the natural coordinates of the shell element denoted ξ, η, and ζ. The chosen coordinate transformation involves the unit normal vectors at the element nodes. For each node we need to specify the components of the normal vector

$$\mathbf{V}_n^T = [V_{nx}, V_{ny}, V_{nz}]\,. \tag{9}$$

At each node we construct two vectors

$$\mathbf{V}_1^T = [V_{1x}, V_{1y}, V_{1z}] \tag{10}$$

and

$$\mathbf{V}_2^T = [V_{2x}, V_{2y}, V_{2z}]\,, \tag{11}$$

which are orthogonal to the given normal vector $\mathbf{V}_n$. This is achieved by evaluating the following cross products:

$$\mathbf{V}_1 = \frac{\mathbf{e} \times \mathbf{V}_n}{\|\mathbf{e} \times \mathbf{V}_n\|} \tag{12}$$

and

$$\mathbf{V}_2 = \mathbf{V}_n \times \mathbf{V}_1\,. \tag{13}$$

The vector $\mathbf{e}$ has to be chosen such that $\mathbf{e}$ and $\mathbf{V}_n$ are not parallel. This can be achieved by using either

$$\mathbf{e}^{\mathrm{T}} = [0, 1, 0] \tag{14}$$

or

$$\mathbf{e}^{\mathrm{T}} = [1, 0, 0]\,. \tag{15}$$

With the above procedure we get for each node k the three orthonormal vectors $\mathbf{V}_1^k$, $\mathbf{V}_2^k$, $\mathbf{V}_n^k$.

For the shell element the following relationships between the coordinates are used:

$$\begin{aligned} x &= \sum_{k=1}^{n} N_k(\xi,\eta)x_k + \frac{\zeta}{2}t\sum_{k=1}^{n} N_k(\xi,\eta)V_{nx}^k\,, \\ y &= \sum_{k=1}^{n} N_k(\xi,\eta)y_k + \frac{\zeta}{2}t\sum_{k=1}^{n} N_k(\xi,\eta)V_{ny}^k\,, \\ z &= \sum_{k=1}^{n} N_k(\xi,\eta)z_k + \frac{\zeta}{2}t\sum_{k=1}^{n} N_k(\xi,\eta)V_{nz}^k\,. \end{aligned} \tag{16}$$

Using shape functions N_k the compatible displacement field $\tilde{\mathbf{u}}$ for a shell element is assumed in the following form:

$$\begin{aligned}
\tilde{u}(\xi,\eta,\zeta) &= \sum_{k=1}^{n} N_k(\xi,\eta)u_k + \frac{\zeta}{2}\sum_{k=1}^{n} N_k(\xi,\eta)[-V_{2x}^k\alpha_k + V_{1x}^k\beta_k]\,,\\
\tilde{v}(\xi,\eta,\zeta) &= \sum_{k=1}^{n} N_k(\xi,\eta)v_k + \frac{\zeta}{2}\sum_{k=1}^{n} N_k(\xi,\eta)[-V_{2y}^k\alpha_k + V_{1y}^k\beta_k]\,,\\
\tilde{w}(\xi,\eta,\zeta) &= \sum_{k=1}^{n} N_k(\xi,\eta)w_k + \frac{\zeta}{2}\sum_{k=1}^{n} N_k(\xi,\eta)[-V_{2z}^k\alpha_k + V_{1z}^k\beta_k]\,,
\end{aligned} \tag{17}$$

where

u_k, v_k, w_k = nodal displacements,

α_k, β_k = nodal rotations.

The variation of Π with respect to $\boldsymbol{\beta}$, $\mathbf{q}$ and $\boldsymbol{\lambda}$ gives us the following system of equations:

$$\begin{bmatrix} -\mathbf{H} & \mathbf{L} & \mathbf{L}^i \\ \mathbf{L}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{L}^{i\mathrm{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta} \\ \mathbf{q} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_{\mathrm{ext}} \\ \mathbf{0} \end{bmatrix}, \tag{18}$$

where

$$\begin{aligned}
\mathbf{H} &= \int_V \mathbf{A}^{\mathrm{T}}\mathbf{E}\mathbf{A}\,\mathrm{d}V = \int_V \mathbf{P}^{\mathrm{T}}\mathbf{E}^{-1}\mathbf{P}\,\mathrm{d}V\,,\\
\mathbf{L} &= \int_V \mathbf{A}^{\mathrm{T}}\mathbf{E}\mathbf{B}\,\mathrm{d}V = \int_V \mathbf{P}^{\mathrm{T}}\mathbf{B}\,\mathrm{d}V\,,\\
\mathbf{L}^i &= \int_V \mathbf{A}^{\mathrm{T}}\mathbf{E}\mathbf{B}^{\mathbf{i}}\,\mathrm{d}V = \int_V \mathbf{P}^{\mathrm{T}}\mathbf{B}^{\mathbf{i}}\,\mathrm{d}V\,,\\
\mathbf{f}_{\mathrm{ext}} &= \int_V \mathbf{N}^{\mathrm{T}}\bar{\mathbf{f}}\,\mathrm{d}V + \int_S \mathbf{N}^{\mathrm{T}}\bar{\mathbf{T}}\,\mathrm{d}S\,,
\end{aligned} \tag{19}$$

where $\mathbf{B}$ is the strain matrix obtained from the compatible displacement field $\tilde{\mathbf{u}}$ (i.e. $\mathbf{B} = \mathbf{DN}$). After eliminating the stress parameters $\boldsymbol{\beta}$ at the element level we obtain the following reduced system of equations:

$$\begin{bmatrix} \mathbf{K} & \boldsymbol{\Gamma}^{\mathrm{T}} \\ \boldsymbol{\Gamma} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\mathrm{ext}} \\ \mathbf{0} \end{bmatrix}, \tag{20}$$

where

$$\begin{aligned}
\mathbf{K} &= \mathbf{L}^{\mathrm{T}}\mathbf{H}^{-1}\mathbf{L}\,,\\
\boldsymbol{\Gamma} &= \mathbf{L}^{\mathbf{i}\mathrm{T}}\mathbf{H}^{-1}\mathbf{L}\,,\\
\mathbf{Q} &= \mathbf{L}^{\mathbf{i}\mathrm{T}}\mathbf{H}^{-1}\mathbf{L}^{\mathbf{i}}\,.
\end{aligned} \tag{21}$$

The vector $\boldsymbol{\lambda}$ contains only internal element parameters which are not associated to any nodes. Therefore the enhanced strain parameters $\boldsymbol{\lambda}$ can be eliminated at the element level. A static condensation procedure leads to the the following reduced system of equations

$$\mathbf{kq} = \mathbf{f}_{\mathrm{ext}}\,, \tag{22}$$

where the element stiffness matrix is given by

$$\mathbf{k} = \mathbf{K} - \mathbf{\Gamma}^{\mathrm{T}}\mathbf{Q}^{-1}\mathbf{\Gamma}\,. \tag{23}$$

In the next sections the systematic construction of Trefftz functions for the strains $\boldsymbol{\varepsilon}$ and the associated stresses $\boldsymbol{\sigma}$ is discussed. Since the functions are derived from a displacement field, solution representations are considered first and then a method of obtaining linearly independent polynomial terms is discussed.

3. Solution representations for three-dimensional displacement fields

Two solution representation are considered: The Neuber–Papkovich solution representation and the author's solution representation which is given in terms of six arbitrary complex valued functions. The Neuber–Papkovich solution representation for the three-dimensional displacement field can be given in the form

$$\begin{aligned} 2\mu u &= -F_x + 4(1-\nu)H_1\,,\\ 2\mu v &= -F_y + 4(1-\nu)H_2\,,\\ 2\mu w &= -F_z + 4(1-\nu)H_3\,, \end{aligned} \tag{24}$$

where

$$2\mu = E/(1+\nu) \tag{25}$$

and

$$F = H_0 + xH_1 + yH_2 + zH_3\,. \tag{26}$$

The four functions $H_0(x,y,z)$, $H_1(x,y,z)$, $H_2(x,y,z)$, $H_3(x,y,z)$ have to satisfy

$$\Delta H_j = 0. \tag{27}$$

In terms of derivatives of the space harmonic functions H_j the displacement field can be written as

$$\begin{aligned} 2\mu u &= -H_{0x} + (3-4\nu)H_1 - xH_{1x} - yH_{2x} - zH_{3x},\\ 2\mu v &= -H_{0y} + (3-4\nu)H_2 - xH_{1y} - yH_{2y} - zH_{3y},\\ 2\mu w &= -H_{0z} + (3-4\nu)H_3 - xH_{1z} - yH_{2z} - zH_{3z}. \end{aligned} \tag{28}$$

The according stresses can be calculated as

$$\begin{aligned} \sigma_{xx} = H_{0yy} + H_{0zz} &+ 2(1-\nu)H_{1x} + x(H_{1yy} + H_{1zz})\\ &+ 2\nu H_{2y} + y(H_{2yy} + H_{2zz})\\ &+ 2\nu H_{3z} + z(H_{3yy} + H_{3zz}),\\ \sigma_{yy} = H_{0xx} + H_{0zz} &+ 2\nu H_{1x} + x(H_{1xx} + H_{1zz})\\ &+ 2(1-\nu)H_{2y} + y(H_{2xx} + H_{2zz})\\ &+ 2\nu H_{3z} + z(H_{3xx} + H_{3zz}),\\ \sigma_{zz} = H_{0xx} + H_{0yy} &+ 2\nu H_{1x} + x(H_{1xx} + H_{1yy})\\ &+ 2\nu H_{2y} + y(H_{2xx} + H_{2yy})\\ &+ 2(1-\nu)H_{3z} + z(H_{3xx} + H_{3xx}), \end{aligned} \tag{29}$$

$$
\begin{aligned}
\tau_{xy} &= -H_{0xy} + (1-2\nu)H_{1y} - xH_{1xy} + (1-2\nu)H_{2x} - yH_{2xy} - zH_{3xy}, \\
\tau_{xz} &= -H_{0xz} + (1-2\nu)H_{1z} - xH_{1xz} - yH_{2xz} + (1-2\nu)H_{3x} - zH_{3xz}, \\
\tau_{yz} &= -H_{0yz} - xH_{1yz} + (1-2\nu)H_{2z} - yH_{2yz} + (1-2\nu)H_{3y} - zH_{3yz}.
\end{aligned} \tag{30}
$$

As an alternative to the real solution representation of Neuber–Papkovich the following complex solution representation of the author [25–27] can be used:

$$
\begin{aligned}
2\mu u &= \int \{\mathrm{Im}\,[\Psi_1 - 2\mathrm{i}x\Phi_1' + 2\,(3-4\nu)\,\Phi_1] \\
&\qquad + a_2\,\mathrm{Re}[-\Psi_2 + 2\mathrm{i}y\Phi_2'] + a_3\,\mathrm{Re}\,[-\Psi_3 + 2\mathrm{i}z\Phi_3']\}\;\mathrm{d}t\,, \\
2\mu v &= \int \{b_1\,\mathrm{Re}\,[-\Psi_1 + 2\mathrm{i}x\Phi_1'] \\
&\qquad + \mathrm{Im}[\Psi_2 - 2\mathrm{i}y\Phi_2' + 2(3-4\nu)\Phi_2] + b_3\,\mathrm{Re}\,[-\Psi_3 + 2\mathrm{i}z\Phi_3']\}\;\mathrm{d}t\,, \\
2\mu w &= \int \{c_1\,\mathrm{Re}\,[-\Psi_1 + 2\mathrm{i}x\Phi_1'] \\
&\qquad + c_2\,\mathrm{Re}\,[-\Psi_2 + 2\mathrm{i}y\Phi_2'] + \mathrm{Im}\,[\Psi_3 - 2\mathrm{i}z\Phi_3' + 2(3-4\nu)\Phi_3]\}\;\mathrm{d}t\,.
\end{aligned} \tag{31}
$$

The complex functions in this representation depend on the complex variables ζ_1, ζ_2, ζ_3 in the form $\Phi_k = \Phi_k(\zeta_k)$, $\Psi_k = \Psi_k(\zeta_k)$ and $k = 1, 2, 3$.

The complex variables are given as

$$
\begin{aligned}
\zeta_1 &= \mathrm{i}x + b_1(t)y + c_1(t)z\,, \\
\zeta_2 &= a_2(t)x + \mathrm{i}y + c_2(t)z\,, \\
\zeta_3 &= a_3(t)x + b_3(t)y + \mathrm{i}z\,,
\end{aligned} \tag{32}
$$

where the parameter functions $a_2(t)$, $a_3(t)$, $b_1(t)$, $b_3(t)$, $c_1(t)$, $c_2(t)$ have to satisfy the equations

$$
\begin{aligned}
b_1^2(t) + c_1^2(t) &= 1\,, \\
a_2^2(t) + c_2^2(t) &= 1\,, \\
a_3^2(t) + b_3^2(t) &= 1\,.
\end{aligned} \tag{33}
$$

Important examples for the choice of parameter functions are $a_3(t) = \cos t$, $b_3(t) = \sin t$.

For any choice for the six complex functions the resulting real displacements u, v, w automatically satisfy the governing differential equations. An example for the choice of complex functions for the 3-dimensional case is

$$
\Phi_3(\zeta_3) = \sum_{n=0}^{N} A_n \zeta_3^n + \sum_{n=1}^{N} \sum_{m=1}^{n} \left(B_n^m \zeta_3^n \cos mt + C_n^m \zeta_3^n \sin mt \right)\,, \tag{34}
$$

where

$$
\zeta_3 = iz - x\cos t - y\sin t\,. \tag{35}
$$

In this case, the upper and lower limits of integration in representation (31) are π and $-\pi$, respectively. Performing the integration with respect to the parameter variable t we systematically obtain all possible space harmonic polynomials which can be written in the form

$$
\int_{-\pi}^{\pi} (z + ix\cos t + iy\sin t)^n \begin{matrix}\cos\\ \sin\end{matrix}\, mt\,\mathrm{d}t = \frac{2\pi i^m n!}{(n+m)!} r^n P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\,, \tag{36}
$$

where P_n^m are Legendre functions. Alternatively we could construct the $(2n+1)$ linearly independent space harmonic polynomials of the order n by using either

$$\int_{-\pi}^{\pi} (x + iy\cos t + iz\sin t)^n \begin{matrix}\cos\\ \sin\end{matrix}\, mt\,\mathrm{d}t \tag{37}$$

or

$$\int_{-\pi}^{\pi} (y + iz\cos t + ix\sin t)^n \begin{matrix}\cos\\ \sin\end{matrix}\, mt\,\mathrm{d}t\,. \tag{38}$$

The result of the integral in equation (36) can also be written in terms of x, y, z for given values of n and m. For example, using a symbolic manipulation program we can easily compute the following integrals:

$$\begin{aligned}
&\int_{-\pi}^{\pi} (z + \mathrm{i}x\cos t + \mathrm{i}y\sin t)^3 \cos t\,\mathrm{d}t = \mathrm{i}\frac{3}{4}\pi(4z^2 - y^2 - x^2)x\,,\\
&\int_{-\pi}^{\pi} (z + \mathrm{i}x\cos t + \mathrm{i}y\sin t)^3 \cos 2t\,\mathrm{d}t = \frac{3}{2}\pi(y^2 - x^2)z\,,\\
&\int_{-\pi}^{\pi} (z + \mathrm{i}x\cos t + \mathrm{i}y\sin t)^3 \sin 2t\,\mathrm{d}t = -3\pi xyz\,.
\end{aligned} \tag{39}$$

Since there are only $(2n+1)$ linearly independent space harmonic functions of the order n the displacement field can have only $3(2n+1) = 6n+3$ linearly independent terms of the order n. However, it was found that it is not possible just to set one function to zero in the Neuber–Papkovich representation or to set three complex functions to zero in representation (31) in order to have three functions left. Having just three functions left in the real or in the complex solution representation will lead to linearly dependent polynomial solution terms. We have to use all functions in the solution representations (28) or (31). In reference [27] a relationship between the real and the complex solution representation was given. In order to get linearly independent polynomial solution terms for the displacement field we can choose complex power series for the functions Φ_j and Ψ_j and establish a relationship between the pairs of functions (Φ_1 and Ψ_1), (Φ_2 and Ψ_2), and (Φ_3 and Ψ_3). Using the real solution representation of Neuber/Papkovich we can decompose the function H_0 into three functions according to

$$\begin{aligned}
H_0(x,y,z) &= A_0(x,y,z) + B_0(x,y,z) + C_0(x,y,z)\\
&= \sum_j A_0^j(x,y,z)a_j + \sum_j B_0^j(x,y,z)b_j + \sum_j C_0^j(x,y,z)c_j
\end{aligned} \tag{40}$$

and impose a relationship between the pairs (H_1, A_0), (H_2, B_0), and (H_3, C_0). Practically this means we will use the same coefficients for the polynomials in H_1 and A_0, for example. The construction of the polynomial solution terms for the displacements will be shown explicitly in the next section.

4. Construction of polynomial solution terms for the displacement field.

The assumed displacement field can be written in the form

$$u = u_j\beta_j\,, \qquad v = v_j\beta_j\,, \qquad w = w_j\beta_j\,,$$

where u_j, v_j, w_j are Trefftz functions and the β_j are coefficients which have to be determined. For the hybrid element formulation we have to exclude the six rigid body terms from the list of trial functions. The remaining linear displacement terms and the according strain and stress terms can be written as:

$$
\begin{aligned}
u_1 &= (1-2\nu)x\,, & v_2 &= (1-2\nu)y\,, & w_3 &= (1-2\nu)z\,,\\
u_4 &= (1-2\nu)y\,, & v_4 &= (1-2\nu)x\,, & w_5 &= (1-2\nu)x\,,\\
u_5 &= (1-2\nu)z\,, & v_6 &= (1-2\nu)z\,, & w_6 &= (1-2\nu)y\,.
\end{aligned}
\tag{41}
$$

The constant strain terms are:

$$
\begin{aligned}
\varepsilon^1_{xx} &= 1-2\nu\,, & \gamma^4_{xy} &= 2(1-2\nu)\,,\\
\varepsilon^2_{yy} &= 1-2\nu\,, & \gamma^5_{xy} &= 2(1-2\nu)\,,\\
\varepsilon^3_{zz} &= 1-2\nu\,, & \gamma^6_{xy} &= 2(1-2\nu)\,.
\end{aligned}
\tag{42}
$$

The constant stress terms are:

$$
\begin{aligned}
\sigma^1_{xx} &= 1-\nu\,, & \sigma^2_{xx} &= \nu\,, & \sigma^3_{xx} &= \nu\,,\\
\sigma^1_{yy} &= \nu\,, & \sigma^2_{yy} &= 1-\nu\,, & \sigma^3_{yy} &= \nu\,,\\
\sigma^1_{zz} &= \nu\,, & \sigma^2_{zz} &= \nu\,, & \sigma^3_{zz} &= 1-\nu\,,\\
\tau^4_{xy} &= 1-2\nu\,,\\
\tau^5_{xz} &= 1-2\nu\,,\\
\tau^6_{yz} &= 1-2\nu\,.
\end{aligned}
\tag{43}
$$

All terms u_j, v_j, w_j ε^j_{xx}, etc. (for j=1,..,6) not listed above are zero.

Using the complex representation of space harmonic functions discussed in the previous section we can obtain the following second and third order functions

$$
\begin{aligned}
H^1_3 &= -2z^2+y^2+x^2\,, & H^7_3 &= 3x(4z^2-y^2-x^2)\,,\\
H^2_3 &= xz\,, & H^8_3 &= 3y(4z^2-y^2-x^2)\,,\\
H^3_3 &= yz\,, & H^9_3 &= -(3y^2-3x^2)z\,,\\
H^4_3 &= x^2-y^2\,, & H^{10}_3 &= 3xyz\,,\\
H^5_3 &= xy\,, & H^{11}_3 &= x(3y^2-x^2)\,,\\
H^6_3 &= (3y^2+3x^2)z-2z^3\,, & H^{12}_3 &= y(y^2-3x^2)\,.
\end{aligned}
\tag{44}
$$

In order to get a symmetric structure in the representation of the displacements, strain and stresses the variables x, y, and z are cycled in the functions for H^j_3. First we take the functions (44) and change x to z, change y to x, and change z to y. This procedure gives the following set of harmonic functions:

$$
\begin{aligned}
H^1_2 &= -2y^2+x^2+z^2\,, & H^7_2 &= 3z(4y^2-x^2-z^2)\,,\\
H^2_2 &= zy\,, & H^8_2 &= 3x(4y^2-x^2-z^2)\,,\\
H^3_2 &= xy\,, & H^9_2 &= -(3x^2-3z^2)y\,,\\
H^4_2 &= z^2-x^2\,, & H^{10}_2 &= 3zxy\,,\\
H^5_2 &= zx\,, & H^{11}_2 &= z(3x^2-z^2)\,,\\
H^6_2 &= (3x^2+3z^2)y-2y^3\,, & H^{12}_2 &= x(x^2-3z^2)\,.
\end{aligned}
\tag{45}
$$

In order to get the function terms for H^j_1 we take the functions (44) and do the following: change x to y, change y to z, and change z to x. This gives the following set of functions:

$$
\begin{aligned}
H_1^1 &= -2x^2 + z^2 + y^2\,, & H_1^7 &= 3y(4x^2 - z^2 - y^2)\,,\\
H_1^2 &= yx\,, & H_1^8 &= 3z(4x^2 - z^2 - y^2)\,,\\
H_1^3 &= zx\,, & H_1^9 &= -(3z^2 - 3y^2)x\,,\\
H_1^4 &= y^2 - z^2\,, & H_1^{10} &= 3yzx\,,\\
H_1^5 &= yz\,, & H_1^{11} &= y(3z^2 - y^2)\,,\\
H_1^6 &= (3z^2 + 3y^2)x - 2x^3\,, & H_1^{12} &= z(z^2 - 3y^2)\,.
\end{aligned}
\tag{46}
$$

The solution representation for the displacements involve the derivatives H_{0x}, H_{0y}, and H_{0z} and the functions H_1, H_2, and H_3. In order to get the correct polynomial order for the displacement components u, v, w we have to choose the terms for the function H_0 one order higher than the polynomial terms in H_1, H_2, and H_3. In order to get a symmetric structure in our solution representation we decompose the function H_0 into three parts according to equation (40).

Because the six function entering the solution representation (i.e. the functions H_1, H_2, H_3, A_0, B_0, C_0) can not be completely independent a relationship between pairs of functions is established: For the polynomials in H_1 and A_0 we use the same coefficients. Accordingly we also use the same polynomial coefficients for the pairs H_2, B_0 and H_3, C_0. For the function C_0 the following polynomials are obtained from the complex representation of space harmonic functions:

$$
\begin{aligned}
C_0^1 &= -z(2z^2 - 3y^2 - 3x^2)\,,\\
C_0^2 &= 3x(4z^2 - y^2 - x^2)\,,\\
C_0^3 &= 3y(4z^2 - y^2 - x^2)\,,\\
C_0^4 &= -3(y - x)(y + x)z\,,\\
C_0^5 &= 3xyz\,,\\
C_0^6 &= (8z^4 - 24y^2z^2 - 24x^2z^2 + 3y^4 + 6x^2y^2 + 3x^4)/4\,,\\
C_0^7 &= xz(4z^2 - 3y^2 - 3x^2)\,,\\
C_0^8 &= yz(4z^2 - 3y^2 - 3x^2)\,,\\
C_0^9 &= (y - x)(y + x)(6z^2 - y^2 - x^2)/2\,,\\
C_0^{10} &= -xy(6z^2 - y^2 - x^2)\,,\\
C_0^{11} &= x(3y^2 - x^2)z\,,\\
C_0^{12} &= y(y^2 - 3x^2)z\,.
\end{aligned}
\tag{47}
$$

Now we take the functions (47) and change x to z, change y to x, and change z to y. This procedure gives the following set of harmonic functions:

$$
\begin{aligned}
B_0^1 &= -y(2y^2 - 3x^2 - 3z^2)\,,\\
B_0^2 &= 3z(4y^2 - x^2 - z^2)\,,\\
B_0^3 &= 3x(4y^2 - x^2 - z^2)\,,\\
B_0^4 &= -3(x - z)(x + z)y\,,\\
B_0^5 &= 3zxy\,,\\
B_0^6 &= (8y^4 - 24x^2y^2 - 24z^2y^2 + 3x^4 + 6z^2x^2 + 3z^4)/4\,,\\
B_0^7 &= zy(4y^2 - 3x^2 - 3z^2)\,,\\
B_0^8 &= xy(4y^2 - 3x^2 - 3z^2)\,,\\
B_0^9 &= (x - z)(x + z)(6y^2 - x^2 - z^2)/2\,,\\
B_0^{10} &= -zx(6y^2 - x^2 - z^2)\,,\\
B_0^{11} &= z(3x^2 - z^2)y\,,\\
B_0^{12} &= x(x^2 - 3z^2)y\,.
\end{aligned}
\tag{48}
$$

In order to get the function terms for A_0^j we take the functions (47) and do the following: change x to y, change y to z, and change z to x. This gives the following set of functions:

$$
\begin{aligned}
A_0^1 &= -x(2x^2 - 3z^2 - 3y^2)\,,\\
A_0^2 &= 3y(4x^2 - z^2 - y^2)\,,\\
A_0^3 &= 3z(4x^2 - z^2 - y^2)\,,\\
A_0^4 &= -3(z-y)(z+y)x\,,\\
A_0^5 &= 3yzx\,,\\
A_0^6 &= (8x^4 - 24z^2x^2 - 24y^2x^2 + 3z^4 + 6y^2z^2 + 3y^4)/4\,,\\
A_0^7 &= yx(4x^2 - 3z^2 - 3y^2)\,,\\
A_0^8 &= zx(4x^2 - 3z^2 - 3y^2)\,,\\
A_0^9 &= (z-y)(z+y)(6x^2 - z^2 - y^2)/2\,,\\
A_0^{10} &= -yz(6x^2 - z^2 - y^2)\,,\\
A_0^{11} &= y(3z^2 - y^2)x\,,\\
A_0^{12} &= z(z^2 - 3y^2)x\,.
\end{aligned}
\tag{49}
$$

Using the above defined functions we get the quadratic and cubic displacement terms as:

$$
\begin{aligned}
u_{j+6} &= (3-4\nu)H_1^j - xH_{1x}^j - A_{0x}^j\,,\\
u_{j+18} &= -yH_{2x}^j - B_{0x}^j\,,\\
u_{j+30} &= -zH_{3x}^j - C_{0x}^j\,,\\
v_{j+6} &= -xH_{1y}^j - A_{0y}^j\,,\\
v_{j+18} &= (3-4\nu)H_2^j - yH_{2y}^j - B_{0y}^j\,,\\
v_{j+30} &= -zH_{3y}^j - C_{0y}^j\,,\\
w_{j+6} &= -xH_{1z}^j - A_{0z}^j\,,\\
w_{j+18} &= -yH_{2z}^j - B_{0z}^j\,,\\
w_{j+30} &= (3-4\nu)H_3^j - zH_{3z}^j - C_{0z}^j\,,
\end{aligned}
\tag{50}
$$

where $j = 1, 2, \ldots, 12$. Using the above displacement terms we can get the associated strain and stress terms and use these terms in the variational formulation for the assumed fields $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$.

5. Concluding remarks

A modified Hu–Washizu variational formulation for the application of the Trefftz method in the analysis of shells has been proposed. The problem of constructing linearly independent polynomial solution terms in the sense of Trefftz which do not bias a particular direction has been addressed. The information obtained from the complex solution representation in terms of six complex functions was useful. After constructing space harmonic functions we have the option to use the Neuber–Papkovich representation and work with six instead of four real functions. The proper decomposition of one of the four harmonic functions in the Neuber–Papkovich representation follows from the complex representation and its relation to the real representation of Neuber–Papkovich. Linearly dependent functions are omitted by using relationships between three pairs of functions. The construction of higher order polynomials is straight forward. (Higher order functions have been derived but are not listed in this paper.) Not discussed in this paper are choices for the enhanced strain field which has the general task to make the element more flexible and to produce better results for displacements and stresses with coarse meshes. However, many possibilities exist to choose these enhanced strain terms and it not clear yet how to find an optimal set of enhanced

strains. Currently numerical tests are performed with the aim to get a well suited set of enhanced strains for a well performing low order shell element. Numerical results will be reported in a future publication.

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