Computer Assisted Mechanics and Engineering Sciences, 4: 521–532, 1997.

Polynomial representation of hybrid finite elements

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(Received July 22, 1996)

We introduce the hybrid-Trefftz FE formulations for linear statics of solids as well as for linear (slow velocity) steady fluid dynamics. Moving least square procedure is given to obtain continuous secondary fields (such as stresses for solids), which improves the results. For nonlinear problems the governing equations are satisfied in the discrete least square residual form. Also for such problems the hybrid FE formulation is shown.

1. Introduction

In this paper, we introduce different hybrid FE approaches for elastostatic problems and for stationary fluid dynamics. If the governing equations describing the continuum behaviour inside the domain are linear, we can use polynomial approximation to satisfy them in a strong sense (hybrid-Trefftz FE formulation). Moreover, special functions which will satisfy also the local boundary conditions, can be used to improve the element properties. Satisfaction of interelement continuity defines the connection between internal and boundary fields of primary variables and the discretized form for the FE formulation is obtained. The hybrid-Trefftz FE formulations for both linear elastostatics and slow incompressible stationary fluid flow are given.

In nonlinear problems the governing PDE describing the behaviour of the continuum are nonlinear and it is not possible to satisfy them in the strong sense. The hybrid formulation is possible for linearized incremental form of governing equations in strong sense, however, the computational procedure is much more complicated for single iteration step than it is in linear case. So we give also another approach using discrete least squares formulation in order to satisfy the governing equations.

Navier–Stokes equations describe the behaviour of fluid continuum. Although the mass continuity is a linear equation, the momentum equations containing convection and diffusion terms are nonlinear. In this case we can combine the approaches and satisfy the mass continuity in strong sense and momentum equations in the discrete least square form. The formulation of hybrid elements is then formulated in an iterative form.

In notation, we often use a right superscript with miscellaneous meaning. To avoid confusion, any quantity raised to any power is enclosed in parentheses. Thus, the square of x_i is denoted by $(x_i)^2$, which is different from x_i^2.

2. Governing equations for elastostatics of solids and for stationary fluid dynamics

We will consider an isotropic solid. For general large displacement, large rotation problems, the governing equations describing the elastostatic behaviour of the continuum are the Lame–Navier equations expressing the equilibrium in displacements u_i

$$\bar{b}_i + (\lambda+\mu)\, u_{j,ji} + \mu\, u_{i,jj} + (\lambda + 2\mu)\, u_{j,ik}\, u_{j,k} + \mu\, u_{j,kk}\, u_{j,i}$$
$$+(\lambda+\mu)\, u_{j,jk}\, u_{i,k} + \mu\, u_{k,jj}\, u_{i,k} + (\lambda+2\mu)\, u_{j,kl}\, u_{j,k}\, u_{i,l}$$
$$+\mu\, u_{j,kk}\, u_{j,l}\, u_{i,l} + \mu\,(u_{k,j} + u_{j,k})\, u_{i,kj} + \lambda\, u_{k,k}\, u_{i,jj}$$
$$+\mu\, u_{l,j}\, u_{l,k}\, u_{i,kj} + \lambda\, u_{k,l}\, u_{k,l}\, u_{i,jj} = 0\,, \tag{1}$$

where λ and μ are Lame constants, $\bar{b}_i$ are components of the body force vector. The index after a comma denotes the differentiation according to the initial (undeformed) coordinate direction and Einstein's summation is used for repeated indices in tensor variables or in their products. We use Cartesian coordinate system x_i. With a bar we denote functions, which will be supposed to be given. The indices are equal to 1, 2 for 2D and 1, 2, 3 for 3D problems.

The Green–Lagrange strain tensor is

$$\varepsilon_{ij}^{\mathrm{G-L}} = \frac{1}{2}\,(u_{i,j} + u_{j,i} + u_{k,i}\, u_{k,j}) \tag{2}$$

and corresponding second Piola–Kirchhoff stress tensor is

$$\sigma_{ij}^{\mathrm{P-K}} = \mu\,(u_{i,j} + u_{j,i}) + \lambda\,\delta_{ij}\, u_{k,k} + \lambda\,\delta_{ij}\, u_{k,l}\, u_{k,l} + \mu\, u_{k,i}\, u_{k,j}\,, \tag{3}$$

where δ_{ij} is the Cronecker delta.

For small displacement problems the equilibrium equations, strains and stresses are linear functions of displacements and we can neglect products of their derivatives. Thus, the equation (1) will contain the first three terms of left side, the strain tensor two terms and the stress tensor first three terms of their right hand sides. All equations (1)–(3) are then linear functions of displacements.

The boundary conditions are in the form

$$u_i = \bar{u}_i \quad \text{on} \quad \Gamma_u\,, \tag{4}$$

where displacements $\bar{u}_i$ are given, or

$$t_i \equiv n_j\, \sigma_{ij} = \bar{t}_i \quad \text{on} \quad \Gamma_t\,, \tag{5}$$

where tractions $\bar{t}_i$ are given. n_i is a unit normal vector component. Also mixed boundary conditions, where some components of displacements and another components of tractions are given, can be considered (e.g. symmetry conditions).

The governing equations for stationary Newtonian fluid expressed in velocity components v_i and pressure p are the mass conservation equation

$$(\rho\, v_i)_{,i} = 0 \tag{6}$$

and the momentum (Navier–Stokes) equation

$$\rho\, v_j\, v_{i,j} = -p_{,j}\,\delta_{ij} + \bar{b}_i + \mu\,(v_{i,jj} + v_{j,ij}) + \left(\kappa - \frac{2}{3}\,\mu\right) v_{j,ij}\,, \tag{7}$$

where ρ is the fluid density, μ and κ are the shear and bulk viscosity, respectively, $\bar{b}_i$ is the body force vector similar as that for solids.

Boundary conditions can be given in velocity components

$$v_i = \bar{v}_i \tag{8}$$

and/or in traction components or pressure

$$t_i = \bar{t}_i\,, \qquad p = \bar{p}\,, \tag{9}$$

where

$$t_i = n_j \, \sigma_{ij} \tag{10}$$

and

$$\begin{aligned} \sigma_{ij} &= -p \, \delta_{ij} + \tau_{ij} \,, \\ \tau_{ij} &= \mu \, (v_{i,j} + v_{j,i}) + \delta_{ij} \left(\kappa - \frac{2}{3} \mu \right) v_{k,k} \,, \end{aligned} \tag{11}$$

σ_{ij} and τ_{ij} are the stress and deviatoric stress tensor components, respectively.

Navier (momentum) equations are linear only for slow velocities. Then the left hand side of (7) is equal to zero.

3. Hybrid FE formulation for elastostatics

We use a hybrid FE formulation in which the primary field satisfies all governing equation inside the element for linear problems and is approximated by polynomials [1]. Another field of primary variables (displacements) is defined on interelement boundaries only.

For linear elastostatic problems the displacements inside the element are approximated by

$$\{u\} = [A] \, \{c\} + \{u^0\} = \{u^{\mathrm{h}}\} + \{u^0\} \,, \tag{12}$$

where $[A]$ is a matrix of polynomial terms of given order, $\{c\}$ are unknown coefficients and $\{u^0\}$ are functions satisfying the nonhomogeneous part of equilibrium equations (1), which are now in the form

$$(\lambda + \mu) \, u_{j,ji} + \mu \, u_{i,jj} = -\bar{b}_i . \tag{13}$$

We can include into $\{u^0\}$ also special functions satisfying both homogeneous equation (13) inside the element and local boundary conditions (like local load conditions on the boundary [1], etc.).

Note, that the polynomials must be full in order to satisfy properly the governing equations. We use polynomials up to 6th order for 3D problems and up to 11th order for 2D problems.

Let us examine the first part of $\{u\}$ in (12)

$$\{u^{\mathrm{h}}\} = [A] \, \{c\} . \tag{14}$$

The corresponding strains can be obtained from partial derivatives of $[A]$ in the form (we omit here the upper index h for simplicity)

$$\{\varepsilon\} = [E] \, \{c\} \,, \tag{15}$$

where matrix $[E]$ contains the derivatives of $[A]$. The stress vector is obtained from

$$\{\sigma\} = [D] \, [E] \, \{c\} = [D\text{-}E] \, \{c\} \,, \tag{16}$$

$[D\text{-}E]$ is symbolic notation for the matrix product $[D] \, [E]$.

Substituting the stress components into the homogeneous solution of (13) we obtain

$$[Q] \, \{c\} = \{0\} \,, \tag{17}$$

where $[Q]$ contains the derivatives of $[D\text{-}E]$. We split the matrix $[Q]$ and the vector of coefficients $\{c\}$ into two parts — dependent $[Q^{\mathrm{d}}]$, $\{c^{\mathrm{d}}\}$ and independent $[Q^{\mathrm{i}}]$, $\{c^{\mathrm{i}}\}$ ones, so that we compare the terms by polynomials with equal polynomial terms $(x_1)^\alpha \, (x_2)^\beta \, (x_3)^\gamma$ for 3D or $(x_1)^\alpha \, (x_2)^\beta$ for 2D and obtain

$$[Q^{\mathrm{d}}] \, \{c^{\mathrm{d}}\} + [Q^{\mathrm{i}}] \, \{c^{\mathrm{i}}\} = \{0\} \,, \tag{18}$$

where $[Q^{\mathrm{d}}]$ is a square matrix. From (18) we can express

$$\{c^{\mathrm{d}}\} = -[Q^{\mathrm{d}}]^{-1}\,[Q^{\mathrm{i}}]\,\{c^{\mathrm{i}}\}. \tag{19}$$

All the calculations (14)–(19) are performed symbolically for all elements and for the same material model once only.

In this way we obtained the reduced vector of unknown coefficients $\{c^{\mathrm{i}}\}$. We denote it by $\{q\}$ i.e. $\{q\} \equiv \{c^{\mathrm{i}}\}$ and after splitting also the matrix $[A]$ in corresponding way, we have

$$\{u\} = ([A^{\mathrm{i}}] - [A^{\mathrm{d}}]\,[Q^{\mathrm{d}}]^{-1}\,[Q^{\mathrm{i}}])\,\{q\} + \{u^0\} = [B]\{q\} + \{u^0\} \tag{20}$$

and from (20) we can find strains, stresses and tractions on the element boundaries

$$\{\varepsilon\} = [E]\,\{q\} + \{\varepsilon^0\}\,, \tag{21a}$$

$$\{\sigma\} = [D\text{-}E]\,\{q\} + \{\sigma^0\}\,, \tag{21b}$$

$$\{t\} = [S]\,\{q\} + \{t^0\}\,, \tag{21c}$$

where

$$[S] = [T]\,[D\text{-}E] \tag{22}$$

and $[T]$ is a matrix of unit normal cosines transforming the stress vector into the surface traction vector

$$\{t\} = [T]\,\{\sigma\}. \tag{23}$$

All matrices $[B]$, $[E]$, $[D\text{-}E]$ are obtained symbolically in polynomial form once for each material.

The particular vectors $\{u^0\}$, $\{\varepsilon^0\}$, $\{\sigma^0\}$ and $\{t^0\}$ are obtained in similar way if the body forces $\bar{b}_i$ in (13) are expressed in polynomial form.

However, we include into $\{u^0\}$ also other important cases, such as concentrated loads, continuous tractions on the plane or curved boundaries, etc., when we choose the function $\{u^0\}$ so, that it will satisfy not only the equilibrium equations, but also local boundary conditions. We can find such functions for quite general tractions from the half plane solution for straight line boundaries and from circular domain or an infinite domain with a circular hole for curved boundaries [2]. Similarly we can find expressions for 3D problems from Boussinesq type solution [3]. Especially simple task is to find such solution for straight boundaries, where using the Boussinesq solution (B.S.) the boundary conditions are satisfied on the whole straight boundary. The solution can be used for curved boundaries also, but in that case the boundary conditions are satisfied by the B.S. in the point of applied load only. It is, however, a good approximation as we demonstrate in the example of a ring in plane stress conditions loaded by concentrated forces on opposite sides of inner radius (Fig. 1). As the problem is double symmetric, only one quarter of the problem was solved numerically.

The displacement field (20) and corresponding tractions (21c) will be incompatible between the elements. To impose the compatibility between elements we choose another independent displacement field $\{\tilde{u}\}$ defined on the element boundaries only and interpolated by means of shape functions $[N]$ and displacements $\{U^{\mathrm{e}}\}$ in nodal points of the element.

$$\{\tilde{u}\} = [N]\,\{U^{\mathrm{e}}\}\,. \tag{24}$$

The compatibility between the internal variables $\{u\}$ and $\{t\}$ and corresponding boundary variables will be enforced by the variation [4, 5]

$$-\int_{\Gamma_t} \delta\{u\}^{\mathrm{T}}\,(\{t\} - \{\bar{t}\})\,\mathrm{d}\Gamma + \int_{\Gamma_u} \delta\{t\}^{\mathrm{T}}\,(\{u\} - \{\bar{u}\})\,\mathrm{d}\Gamma + \int_{\Gamma_{\mathrm{i}}} \delta\{t\}^{\mathrm{T}}\,(\{u\} - \{\tilde{u}\})\,\mathrm{d}\Gamma = 0\,, \tag{25}$$

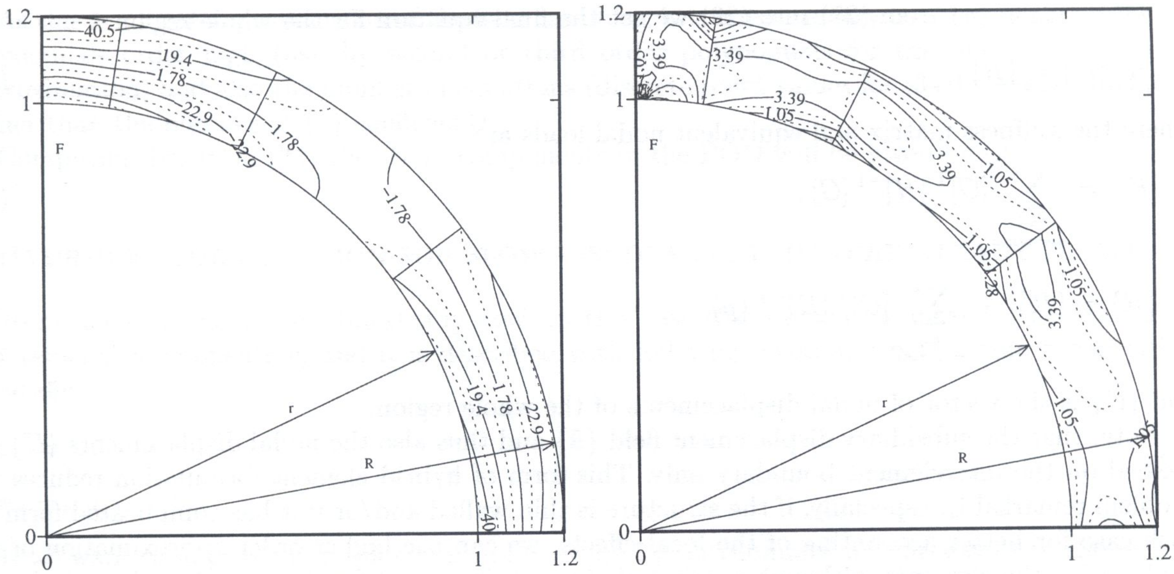


Fig. 1. Stress field obtained from displacement field approximated by polynomial of the 4th order inside the element; shape functions of the 2nd order on the interelement boundaries; $\mu = 0.3$

where Γ_t, Γ_u and Γ_i are parts of the element boundaries, where tractions $\bar{t}$ and displacements $\bar{u}$ are given and interelement boundaries, respectively (mixed boundary conditions of Γ_t and Γ_u are also possible, e.g. for symmetry conditions). δ denotes the variation and the right superscript T denotes the transposition.

The variation

$$\sum_{\substack{\text{all}\\ \text{elements}}} \int_{\Gamma_\mathrm{i}} \delta\{\tilde{u}\}^\mathrm{T}\{t\}\,\mathrm{d}\Gamma = 0 \tag{26}$$

enforces the equilibrium between the elements.

Substituting (20) and (21) into (25) we obtain

$$[H]\{q\} = [G]\{U^\mathrm{e}\} + \{p\}\,, \tag{27}$$

where

$$[H] = \int_{\Gamma_u\cup\Gamma_\mathrm{i}} [S]^\mathrm{T}[B]\,\mathrm{d}\Gamma - \int_{\Gamma_t} [B]^\mathrm{T}[S]\,\mathrm{d}\Gamma\,,$$

$$[G] = \int_{\Gamma_\mathrm{i}} [S]^\mathrm{T}[N]\,\mathrm{d}\Gamma\,, \tag{28}$$

$$\{p\} = \int_{\Gamma_t} [B]^\mathrm{T}\left(\{t^0\} - \{\bar{t}\}\right)\mathrm{d}\Gamma - \int_{\Gamma_u} [S]^\mathrm{T}\left(\{u^0\} - \{\bar{u}\}\right)\mathrm{d}\Gamma - \int_{\Gamma_\mathrm{i}} [S]^\mathrm{T}\{u^0\}\,\mathrm{d}\Gamma$$

and from (26) we get

$$\sum_{\substack{\text{all}\\ \text{elements}}} [G]^\mathrm{T}\{q\} = \{R^0\} \tag{29}$$

with

$$\{R^0\} = -\sum_{\substack{\text{all}\\ \text{elements}}} \int_{\Gamma_\mathrm{i}} [N]^\mathrm{T}\{t^0\}\,\mathrm{d}\Gamma. \tag{30}$$

Substituting $\{q\}$ from (27) into (29) we get the final equation for the whole region

$$[K]\{U\} = \{R\}, \tag{31}$$

where the stiffness matrix and equivalent nodal loads are

$$[K] = \sum_{\substack{\text{all}\\ \text{elements}}} [G]^{\mathrm{T}}[H]^{-1}[G], \tag{32}$$

$$\{R\} = \{R^0\} - \sum_{\substack{\text{all}\\ \text{elements}}} [G]^{\mathrm{T}}[H]^{-1}\{p\} \tag{33}$$

and $\{U\}$ is the vector of nodal displacements of the whole region.

Note, that the subsidiary displacement field $\{\tilde{u}\}$ and thus also the nodal displacements $\{U\}$ are defined on the interelement boundary only. This form of hybrid element formulation reduces the problem remarkably, especially, if the structure is thin walled and/or if it has complicated form. In such case for better accounting of the local effects, we can use higher order approximation of the fields inside the elements, although number of element degrees of freedom can be relatively low.

Having obtained the displacements in the whole structure from the solution (31) we get the polynomial coefficients in elements from (27) and stresses from (21b). As we can see from examples given in Fig. 1, the stresses between the elements are incompatible (the interelement equilibrium and boundary conditions are satisfied in the weak sense only). The best continuity between the elements is obtained in tangential stress components, i.e. the components which are not contained in the boundary and interelement conditions. It is well known, that hybrid FEM formulations give better results inside the elements and worst in a thin layer around by the element boundaries.

The results can be much improved and smooth stress can be obtained, if we calculate the stresses from displacements obtained in nodal points and boundary conditions by moving least square interpolants [6], however, in our case we use the interpolation polynomials (20)–(21) not over one element, but over element patches and add corresponding nodal displacements and boundary conditions to obtain enough equations to resolve the unknown polynomial coefficients $\{q\}$ as follows

$$\sum_I w(x_I)\left([B(x_I)]\{q\} + \{u^0\} - \{u\}\right)^2 + \sum_J w(x_J)\left([S(x_J)]\{q\} + \{t^0\} - \{t\}\right)^2 \overset{!}{=} \min, \tag{34}$$

where $w(x)$ is weighting function, which is a positive continuous function and is maximum at the point where we calculate the stress and decreases monotonically with increasing distance away from this point. It can be an exponential function of the form

$$w(x_I) = \mathrm{e}^{-c\left(\frac{r}{r_{\mathrm{m}}}\right)^2}, \tag{35a}$$

where c is a constant, r is the distance between the point of the domain (POD) and the point I where the boundary condition or nodal point displacements are taken into account, and r_{m} is the radius defining the domain. The domain is defined by the sphere (3D) or the circle (2D) around the POD with this radius.

Also weight functions of the polynomial type

$$w(x_I) = 1 - 3\left(\frac{r}{r_{\mathrm{m}}}\right)^2 + 2\left(\frac{r}{r_{\mathrm{m}}}\right)^3, \tag{35b}$$

$$w(x_I) = 1 - 2\left(\frac{r}{r_{\mathrm{m}}}\right)^2 + \left(\frac{r}{r_{\mathrm{m}}}\right)^4 \tag{35c}$$

are convenient for this purpose. The functions $\{u^0\}$ and $\{t^0\}$ are those as defined in (20) and (21c) and $\{u\}$ and $\{t\}$ are the nodal displacements calculated from (31), and prescribed displacements and tractions in the chosen discrete points of the boundaries.

As this is a local approximation of the field variables, it is not necessary to choose approximation polynomials order high (usually second or third order polynomials for the matrix $[B]$ is good approximation) and also the number of equations (discrete point values for (34)) need not be much higher than the number of $\{q\}$ coefficients.

Computing $\{q\}$ from (34) the stress components in the POD will be given by (21b).

4. Hybrid FE formulation for slow viscous flow (linear steady flow)

We consider slow incompressible steady flow. In this case the problem is linear in the pressure p and velocity components v_i and it is described with following mass and momentum conservation equations

$$\begin{aligned} & v_{i,i} = 0\,, \\ & p_{,i} - \mu\, v_{i,jj} = 0. \end{aligned} \tag{36}$$

If we want to approximate velocities and pressure by polynomials, it is necessary to choose one order lower polynomials for pressure then for velocity components, i.e.

$$\begin{aligned} \{v_i\} &= [A_i^v]\,\{c_i^v\}\,, \\ \{p\} &= [A^p]\,\{c^p\}. \end{aligned} \tag{37}$$

If we split again the polynomial terms on dependent and independent, we can express the dependent terms by independent so that (36) will be satisfied in strong sense. Denoting the vector of all independent coefficients from $\{c_i^v\}$ and $\{c^p\}$ by $\{q\}$ we can express

$$\begin{aligned} \{v\} &= [B^v]\,\{q\}\,, \\ \{p\} &= [B^p]\,\{q\}\,, \end{aligned} \tag{38}$$

where $\{v\}$ contains all velocity components. From velocities we can similarly as by solids express tractions (according to the relations given in the section 2) corresponding to the deviatoric stresses

$$\{t^v\} = [S^v]\,\{q\}. \tag{39}$$

The tractions from hydrostatic stress (pressure) are given by

$$t_i^p = -n_i\,p \tag{40}$$

thus

$$\{t^p\} = -\begin{bmatrix} n_1\,[B^p] & n_2\,[B^p] & n_3\,[B^p] \end{bmatrix}^{\mathrm{T}} \{q\} = -[B^{\mathrm{hd}}]\,\{q\}\,, \tag{41}$$

which give the resulting tractions in the form

$$\{t\} = \{t^v\} + \{t^p\} = \left([S^v] - [B^{\mathrm{hd}}]\right)\,\{q\}. \tag{42}$$

Further we define velocity and pressure field on the interelement boundaries

$$\begin{aligned} \{\tilde{v}\} &= [N^v]\{V^{\mathrm{e}}\}\,, \\ \{\tilde{p}\} &= [N^p]\{P^{\mathrm{e}}\}\,, \end{aligned} \tag{43}$$

where $[N^v]$ and $[N^p]$ are shape functions and $\{V^{\mathrm{e}}\}$ and $\{P^{\mathrm{e}}\}$ are vectors of velocity components and pressures in nodal points on interelement boundaries, respectively.

Variational formulation defines the compatibility of interelement and boundary field variables in the weak sense similarly as it was for solids. The formulation is similar than that for solid mechanics, but we have now kinetic energy of the fluid and power of external forces instead of deformation energy and work of external forces in solid mechanics [7, 8]. Compatibility of boundary fields of velocities and pressures on interelement boundaries is given by

$$\int_{\Gamma_i} \delta t_i\,(v_i - \tilde{v}_i)\,\mathrm{d}\Gamma + \int_{\Gamma_i} \delta v_{\{n\}}\,(p - \tilde{p})\,\mathrm{d}\Gamma + \int_{\Gamma_v} \delta t_i\,(v_i - \bar{v}_i)\,\mathrm{d}\Gamma - \int_{\Gamma_t} \delta v_i\,(t_i - \bar{t}_i)\,\mathrm{d}\Gamma = 0\,, \tag{44}$$

where Γ_v, Γ_t, Γ_i are parts of the element boundaries where velocity $\bar{v}_i$ and tractions $\bar{t}_i$ are given and interelement boundary, respectively. $v_{\{n\}}$ is the normal component of the velocity vector

$$v_{\{n\}} = v_i\, n_i \tag{45}$$

and n_i is the component of outer normal to the element surface.

Substituting corresponding terms from (38)–(43) into (44) we have

$$\left(\int_{\Gamma_i\cup\Gamma_v} [S^v]^{\mathrm{T}}\,[B^v]\,\mathrm{d}\Gamma + \int_{\Gamma_i} [B^{nv}]^{\mathrm{T}}\,[B^p]\,\mathrm{d}\Gamma - \int_{\Gamma_t} [B^v]^{\mathrm{T}}\,[S^v]\,\mathrm{d}\Gamma\right)\{q\} - \int_{\Gamma_i} [S^v]^{\mathrm{T}}\,[N^v]\,\mathrm{d}\Gamma\,\{V^{\mathrm{e}}\} - \int_{\Gamma_i} [B^{nv}]^{\mathrm{T}}\,[N^p]\,\mathrm{d}\Gamma\,\{P^{\mathrm{e}}\} - \int_{\Gamma_v} [S^v]^{\mathrm{T}}\,\{\bar{v}\}\,\mathrm{d}\Gamma + \int_{\Gamma_t} [B^v]^{\mathrm{T}}\,\{\bar{t}\}\,\mathrm{d}\Gamma = 0 \tag{46}$$

or

$$[H]\{q\} = [G^v]\,\{V^{\mathrm{e}}\} + [G^p]\,\{P^{\mathrm{e}}\} + \{r\}\,, \tag{47}$$

where

$$[B^{nv}] = \{n\}^{\mathrm{T}}\,[B^v]. \tag{48}$$

As the velocities and pressure are independent fields, the interelement equilibrium in the weak form will be satisfied by

$$\sum_{\substack{\text{all}\\ \text{elements}}} \int_{\Gamma_i} \delta\tilde{v}_i\, t_i\,\mathrm{d}\Gamma = 0 \tag{49}$$

and

$$\sum_{\substack{\text{all}\\ \text{elements}}} \int_{\Gamma_i} \delta\,(\tilde{p}\, n_i)\; v_i\,\mathrm{d}\Gamma \equiv \sum_{\substack{\text{all}\\ \text{elements}}} \int_{\Gamma_i} \delta\tilde{p}\, v_{\{n\}}\,\mathrm{d}\Gamma = 0 \tag{50}$$

and after substitution for corresponding field variables we have from (49)

$$\sum_{\substack{\text{all}\\ \text{elements}}} \int_{\Gamma_i} [N^v]^{\mathrm{T}}\,[S^v]\,\mathrm{d}\Gamma\,\{q\} \equiv \sum_{\substack{\text{all}\\ \text{elements}}} [G^v]^{\mathrm{T}}\,\{q\} = \{0\} \tag{51}$$

and from (50)

$$\sum_{\substack{\text{all}\\ \text{elements}}} \int_{\Gamma_i} [N^p]^{\mathrm{T}}\,[B^{nv}]\,\mathrm{d}\Gamma\,\{q\} \equiv \sum_{\substack{\text{all}\\ \text{elements}}} [G^p]^{\mathrm{T}}\,\{q\} = \{0\}. \tag{52}$$

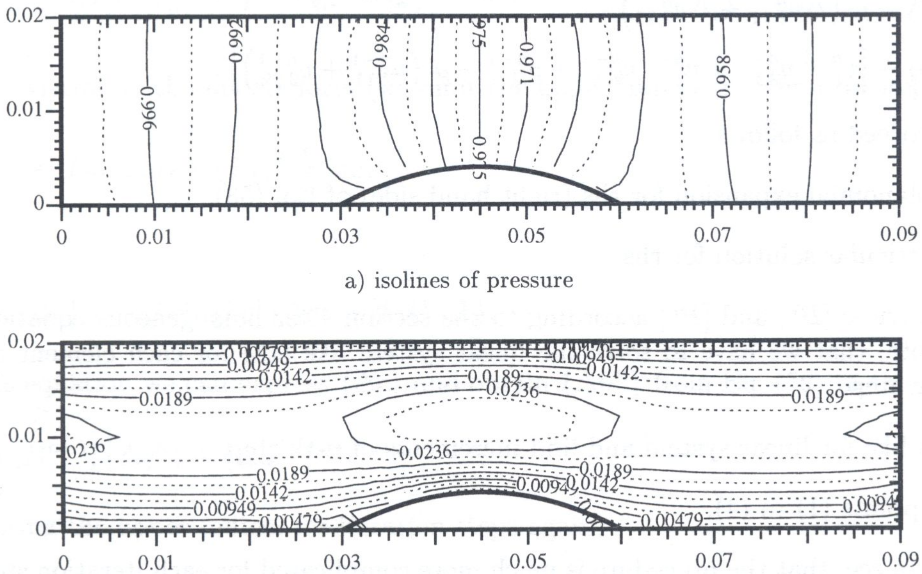
Substituting for $\{q\}$ from (47) into (51) and (52) yields

$$\sum_{\substack{\text{all}\\ \text{elements}}} \begin{bmatrix} [G^v]^{\mathrm{T}}[H]^{-1}[G^v] & [G^v]^{\mathrm{T}}[H]^{-1}[G^p] \\ [G^p]^{\mathrm{T}}[H]^{-1}[G^v] & [G^p]^{\mathrm{T}}[H]^{-1}[G^p] \end{bmatrix} \begin{Bmatrix} \{V^{\mathrm{e}}\} \\ \{P^{\mathrm{e}}\} \end{Bmatrix} = \sum_{\substack{\text{all}\\ \text{elements}}} \begin{Bmatrix} \{R^v\} \\ \{R^p\} \end{Bmatrix}, \tag{53}$$

where

$$\begin{aligned} \{R^v\} &= -[G^v]^{\mathrm{T}}[H]^{-1}\{r\}, \\ \{R^p\} &= -[G^p]^{\mathrm{T}}[H]^{-1}\{r\}. \end{aligned} \tag{54}$$

Equation (53) is resulting system of linear equations used for the resolution of nodal velocities and pressures for the whole region. Some numerical implementations and results were published in [10, 11]. An example of results is presented in Fig. 2.



a) isolines of pressure

b) isolines of $\sqrt{\{u\}\cdot\{u\}}$ field, $\{u\}$ is the velocity vector

Fig. 2. Pressure and velocity fields in 2D channel small velocity (linear flow), $\mu = 0.01$ Pa s

5. Hybrid FE formulation for nonlinear problems

The hybrid-Trefftz FE formulation is based on the satisfaction of all governing equations inside the region (element) in the strong sense. This is not possible in nonlinear (NL) problems, i.e. when the governing equations are NL.

We have several possibilities how to proceed in such case. Let's consider for simplicity 2D incompressible steady flow of the fluid.

$$v_{1,1} + v_{2,2} = 0. \tag{55}$$

The governing equations inside the region are the Navier–Stokes (N–S) equations

$$\begin{aligned} v_1\, v_{1,1} + v_2\, v_{1,2} + p_{,1} - \bar{b}_1 - \mu\,(v_{1,11} + v_{1,22}) &= 0\,, \\ v_2\, v_{2,1} + v_2\, v_{2,2} + p_{,2} - \bar{b}_2 - \mu\,(v_{2,11} + v_{2,22}) &= 0. \end{aligned} \tag{56}$$

The mass conservation equation (55) is linear, the momentum equations (56) are nonlinear. Suppose, that we obtained approximate solution in iterative procedure after $n-1$ steps and want to find a better solution in incremental form by linearizing the N–S equations.

Using the polynomial expansion for velocities and pressure, the equation (55) will be satisfied in strong sense. Then the incremental form of this problem will be

$$\Delta v^n_{1,1} + \Delta v^n_{2,2} = 0 \tag{57}$$

and

$$\begin{aligned} & v^{n-1}_{1,1}\,\Delta v^n_1 + v^{n-1}_1\,\Delta v^n_{1,1} + v^{n-1}_2\,\Delta v^n_{1,2} + v^{n-1}_{1,2}\,\Delta v^n_2 \\ & \qquad + p^n_{,1} - \mu\left(\Delta v^n_{1,11} + \Delta v^n_{1,22}\right) \\ & \qquad = \bar{b}_1 - \left[v^{n-1}_1\, v^{n-1}_{1,1} + v^{n-1}_2\, v^{n-1}_{1,2} + p^{n-1}_{,1} - \mu\left(v^{n-1}_{1,11} + v^{n-1}_{1,22}\right)\right], \\ & v^{n-1}_1\,\Delta v^n_{2,1} + v^{n-1}_{2,1}\,\Delta v^n_1 + v^{n-1}_{2,1}\,\Delta v^n_1 + v^{n-1}_{2,2}\,\Delta v^n_2 \\ & \qquad + p^n_{,2} - \mu\left(\Delta v^n_{2,11} + \Delta v^n_{2,22}\right) \\ & \qquad = \bar{b}_2 - \left[v^{n-1}_1\, v^{n-1}_{2,1} + v^{n-1}_2\, v^{n-1}_{2,2} + p^{n-1}_{,2} - \mu\left(v^{n-1}_{2,11} + v^{n-1}_{2,22}\right)\right]. \end{aligned} \tag{58}$$

We can proceed as follows:

(1) Find polynomial expansion for rhs (right hand side) of Eq. (58).

(2) Find particular solution for rhs.

(3) Find matrices $[B^v]$ and $[B^p]$ according to the section 4 for homogeneous equations (57) and (58). Note, that we have to solve the linear system like (18) for each element and for each iteration step.

(4) Proceed like for linear system and find increments of n-th step.

(5) Repeat points (1) to (4).

Thus we can see, that the procedure is much more complicated for each iteration step than that for linear analysis. Moreover, as we can see, if we use this procedure for some 3D problems like geometric nonlinearities, then Eq. (1) gives very complicated incremental expression with many terms (there are 36 polynomial expressions in it, i.e. 3 displacement components, 9 first and 24 second derivatives).

A more effective procedure for NL problems can be obtained, if the governing equations inside the element will be satisfied in discrete least square sense.

We will choose again polynomial approximation of primary variables $\{u\}$ and split polynomial terms so, that the part of coefficients denoted by $\{c\}$ will be reduced in order to satisfy the governing equations and the other part $\{q\}$ will serve to fulfill the boundary and interelement continuity

$$\{u\} = [A^{\mathrm{d}}]\,\{c\} + [A^{\mathrm{i}}]\{q\}. \tag{59}$$

Let us denote

$$L(u) = 0 \tag{60}$$

the governing equations inside the domains (L is the differential operator). Let us have an approximate solution of the problem from $(n-1)$-th iteration step. We shall look for the next, improved approximation in n-th iteration step. For this we will use the Taylor expansion of L in the vicinity of $(n-1)$-th solution and use the first term of the expansion only

$$L(u) = L^{n-1}(u) + \frac{\partial L^{n-1}}{\partial\{c\}}\,\{\Delta c^n\} + \frac{\partial L^{n-1}}{\partial\{q\}}\,\{\Delta q^n\}. \tag{61}$$

We express the partial derivatives in finite differences in components as

$$\begin{aligned}\frac{\partial L^{n-1}}{\partial c_j} &\approx \frac{L(u(c_j^{n-1}+\delta c_j)) - L(u(c_j^{n-1}))}{\delta c_j} = L_{c.j}^{n-1}\,,\\ \frac{\partial L^{n-1}}{\partial q_j} &\approx \frac{L(u(q_j^{n-1}+\delta q_j)) - L(u(q_j^{n-1}))}{\delta q_j} = L_{q.j}^{n-1}\,,\end{aligned} \tag{62}$$

where δc_j and δq_j are very small increments of c_j and q_j, respectively. $L_{c_{ij}}$ and $L_{q_{ij}}$ will define the values $L_{c.j}$ and $L_{q.j}$ in the discrete point i.

We will require, that the square error of (60) will be minimum for the increments $\{\Delta c^n\}$, i.e.

$$\sum_i \left(L^{n-1}(u(x_i)) + L_{c_{ij}}^{n-1}\,\Delta c_j^n + L_{q_{ik}}^{n-1}\,\Delta q_k^n\right)^2 \overset{!}{=} \min \tag{63}$$

what can be written in the matrix form (we omit the upper indices for iteration step)

$$([Lc]\,\{\Delta c\} + [Lq]\,\{\Delta q\} + \{L\})^2 \overset{!}{=} \min \tag{64}$$

which gives

$$[Lc]^{\mathrm{T}}\,[Lc]\,\{\Delta c\} = -[Lc]^{\mathrm{T}}\,[Lq]\,\{\Delta q\} - [Lc]^{\mathrm{T}}\,\{L\}. \tag{65}$$

From this equation we can express $\{\Delta c\}$ and obtain $\{\Delta u^n\}$ from Eq. (59) in the form

$$\{\Delta u^n\} = [B^{n-1}]\,\{\Delta q^n\} + \{\Delta u^{0n}\}\,, \tag{66}$$

where (omitting the upper indices for iteration steps again)

$$[B] \quad = [A^{\mathrm{i}}] - [A^{\mathrm{d}}]\,\left([Lc]^{\mathrm{T}}\,[Lc]\right)^{-1}\,[Lc]^{\mathrm{T}}\,[Lq]\,, \tag{67}$$

$$\{\Delta u^0\} = -[A^{\mathrm{d}}]\,\left([Lc]^{\mathrm{T}}\,[Lc]\right)^{-1}\,[Lc]^{\mathrm{T}}\,\{L\}. \tag{68}$$

Further we proceed as in the linear case and compute $\{\Delta u^n\}$ in nodal points for the whole region and from it we have

$$\{u^n\} = \{u^{n-1}\} + \{\Delta u^n\}. \tag{69}$$

Note, that we can use this algorithm also for linear problems, and obtain the solution in one step. The solutions will be identical if we use the same polynomial expansion, but also splitting on d and i parts.

Although the problem of steady flow of the fluid is nonlinear, the continuity equation (conservation of mass) is linear. So we can satisfy this equation in a strong sense as we did by linear problems and thus reduce the problem and only the momentum equations will be solved iteratively.

As we mentioned, we receive the better efficiency, the better all governing equations and corresponding boundary conditions will be satisfied. In fluid mechanics it is necessary to use very fine meshes in the vicinity of solid boundaries, if we consider a viscous fluids, because of large gradients in primary variable fields at boundary layers. In hybrid formulations, we can use special functions of the approximate solution for boundary layers [9] in order to better satisfy the governing equations and boundary conditions at the same time.

6. Conclusions

The following conclusions can be made from this research:

- Polynomial representation can be used to formulate hybrid FE for both solids and fluids. If the problems are linear, the governing equations can be satisfied in strong sense and we obtain the hybrid-Trefftz formulation.
- For nonlinear problems the governing equations inside the element can be approximated by polynomials which satisfy the governing equations by minimizing their least squares of errors.
- Local effects, such as concentrated loads or shape irregularities, resulting in large gradients in element fields can be solved very effectively by large elements, if the local effects are conveniently modelled by additional functions satisfying locally all governing equations and boundary conditions. High accuracy of the solution can be achieved without fine meshes also in regions with very high gradients or with singular fields.
- The resulting system of equations contains the nodal points on interelement boundaries only, so that the problem can be considerably reduced especially if thin layers of the continuum are to be modelled, like it is by thin walled structures.

Further development of the method should be made especially in the following areas:

- special approximation functions for particular purposes
- application for nonlinear problems
- solution of transient problems

Acknowledgement

This research has been partly supported by the Grant No.1/629/94 of the Slovak Grant Agency for Science.

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