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Hierarchic finite elements for thin plates and shells

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We consider the numerical approximation of thin plate and shell structures. The plate model is described following the Reissner–Mindlin assumptions while the shell is described using the Naghdi formulation. It is well known that the numerical approximation with standard finite elements suffers of the so-called *locking* phenomenon, i.e., the numerical solution degenerates as the thickness of the structure becomes smaller. Plates exhibit shear locking and shells show both shear and membrane locking. Several techniques to avoid the numerical locking have been proposed. Here we solve the problems using a family of high order hierarchic finite elements. We present several numerical results that show the robustness of the finite elements, able to avoid in many circumstances the locking behavior.

1. Introduction

In recent years high order finite elements, both as separate elements or in the so-called p-version of the finite element method, have been introduced and succesfully applied in several fields, e.g., for elasticity and Kirchhoff plate problems (see [1, 3, 7, 15]). In previous papers (see [8–10]) we have dealt with the Reissner–Mindlin plate problem combining its *plain* formulation with the use of *hierarchic high order* finite elements. In the papers [5,6] we have considered the shell model arising from the Naghdi formulation. A displacement finite element scheme has been developed using continuous finite element of hierarchic type of degree ranging from one to four. It is known that, despite its simple approach, the discretization of the Reissner–Mindlin model is not straightforward both in plate problems and in its extension to shell problems. The inclusion of transverse shear strain effect in the finite element models introduce an undesirable numerical effect, the so called *shear locking* phenomenon. Finite element schemes for shell problems also suffer of the so-called *membrane locking*, i.e., the finite element approximation of the membrane component of the energy is unstable with respect to the thickness of the shell.

High order finite elements strongly reduce the locking phenomenon in the plate problem and the numerical performances are quite effective for all practical values of thickness. In the case of the shell model, the use of the finite elements of hierarchic type, especially for the elements of higher degree, shows a good agreement with all the available benchmark results. To keep low the number of degrees of freedom we have used finite elements of Serendipity type, where the number of internal shape functions is highly reduced. A natural idea is to test *complete* finite elements, i.e., elements using complete polynomial approximation spaces, for comparing their performances against Serendipity elements. Such a problem has been recently addressed and carefully analyzed by Babuška and Elman [2] in the framework of the so-called $h-p$ version of the finite element method. Here we consider an intermediate family of finite elements obtained adding suitable bubbles functions. We focus our attention on the performances of these finite elements applied to the Reissner–Mindlin plate and Naghdi shell problems. Due to the hierarchic structure the various finite elements of the family are close one to each other. A remarkable improvement of the quality of the results has been achieved. When cost versus accuracy is considered, intermediate elements are more convenient: the gain in convergence outnumbers the increase of cost due to the larger number of degrees of freedom. The elements exhibit a weaker form of locking and produce satisfactory results.

The outline of the paper is the following. In Section 2 we recall the features of the finite elements of hierarchic type and introduce the new family of hierarchic finite elements. Sections 3 is devoted to a brief presentation of the plate bending problem in the Reissner–Mindlin formulation. In Section 4 we recall the Naghdi shell problem. Finally, in Sections 5 and 6 several numerical results are reported for both plate and shell problems.

2. THE HIERARCHIC FINITE ELEMENTS

In [5, 13] we have introduced a family of rectangular hierarchic finite elements to overcome the locking of the numerical approximation of the Reissner–Mindlin plate and the Naghdi shell, respectively. More precisely the finite elements have the following features:

- only polynomial functions are used to construct the approximation spaces;
- the same spaces are used to approximate normal deflection and rotations;
- the family is hierarchic, i.e., new shape functions are added to increase the degree of approximation, leaving unchanged the previous functions;
- the functions used are based on the family of Legendre polynomials; this class presents good properties from the point of view of roundoff error accumulation;
- the orthogonality properties of Legendre polynomials are transferred to the elementary stiffness matrix;
- finite elements of Serendipity type are used in order to minimize the number of internal degrees of freedom.

We denote the elements of the family with the names S4, S8, S12, S17, where S stands for *Serendipity* element and the number refers the degrees of freedom (of each field). A modification of the previous family of finite elements is obtained adding suitable internal functions. Only the number of internal functions is modified, while nodal and side functions are left unchanged. The internal functions are frequently named *bubble functions*, since they vanish along the whole boundary of the finite element and are different from zero only in the interior of the element. The use of bubble functions is known to be rather effective (see, e.g., [2]). This family of modified elements is intermediate between the Serendipity and complete family. We have named the new elements B9, B15, B22, where B stands for *bubble*. We observe that the element corresponding to $p=1$ does not include internal degrees of freedom, since the space of bilinear functions is fully described by four shape functions. Therefore the elements is the same in both old and new family. The family {B4, B9, B15, B22} is still hierarchic, i.e., the stiffness matrices of lower degree are submatrices of the one of higher order.

Let $\mathcal{P}_p$ denote the space of polynomial of degree less or equal p in the two variables. Let $\mathcal{S}_p$ be the polynomial spaces of the Serendipity hierarchic shape functions of degree p. Let $\mathcal{Q}_p$ be the standard complete space of polynomials of degree p in each of the two variables. Let $\mathcal{B}_p$ be the intermediate space, obtained by $\mathcal{S}_p$ adding only a limited number of bubble functions. Due to the construction, for a given degree p, the following relation holds between the finite dimensional spaces:

$$\mathcal{P}_p \subset \mathcal{S}_p \subset \mathcal{B}_p \subset \mathcal{Q}_p$$

and therefore

$$\dim \mathcal{P}_p \leq \dim \mathcal{S}_p \leq \dim \mathcal{B}_p \leq \dim \mathcal{Q}_p \,.$$

In Table 1 the dimension are given for some values of the degree p. Moreover, the number of internal functions is reported.

Table 1. Total number of shape and bubble functions for the spaces $\mathcal{S}_p$, $\mathcal{B}_p$, $\mathcal{Q}_p$

p	shape functions			bubble functions		
	$\mathcal{S}_p$	$\mathcal{B}_p$	$\mathcal{Q}_p$	$\mathcal{S}_p$	$\mathcal{B}_p$	$\mathcal{Q}_p$
1	4	4	4	0	0	0
2	8	9	9	0	1	1
3	12	15	16	0	3	4
4	17	22	25	1	6	9

For a general value of p the dimensions of each space are:

$$\dim \mathcal{S}_p = 4p + \tfrac{1}{2}(p-2)(p-3)$$

$$\dim \mathcal{B}_p = \tfrac{1}{2}p\,(p+7) = 4p + \tfrac{1}{2}\,p(p-1)$$

$$\dim \mathcal{Q}_p = (p+1)^2 = 4p + (p-1)^2$$

To obtain the space $\mathcal{B}_p$ we take all monomials of the space $\mathcal{S}_p$ and we add the following monomial terms:

$$p = 2 : \{x^2y^2\};$$

$$p = 3 : \{x^2y^2, x^3y^2, x^2y^3\};$$

$$p = 4 : \{x^3y^2, x^2y^3, x^4y^2, x^3y^3, x^2y^4\}.$$

In Table 2 the single monomials terms of the spaces $\mathcal{S}_p$, $\mathcal{B}_p$, $\mathcal{Q}_p$ are listed. More precisely, for the standard hierarchic space $\mathcal{S}_p$ the monomials are added to increase the degree p. For the spaces $\mathcal{B}_p$ and $\mathcal{Q}_p$ only the monomials to be added to the correspondent space $\mathcal{S}_p$ and $\mathcal{B}_p$, respectively, are listed.

Table 2. Monomial terms of the spaces $\mathcal{S}_p$, $\mathcal{B}_p$, $\mathcal{Q}_p$

p	*Serendipity* space $\mathcal{S}_p$	*intermediate* space $\mathcal{B}_p$	*complete* space $\mathcal{Q}_p$
1	$\{1, x, y, xy\}$		
2	$\{x^2, x^2y, xy^2, y^2\}$	$\{x^2y^2\}$	
3	$\{x^3, x^3y, xy^3, y^3\}$	$\{x^2y^2, x^3y^2, x^2y^3\}$	$\{x^3y^3\}$
4	$\{x^4, x^4y, xy^4, y^4, x^2y^2\}$	$\{x^2y^3, x^3y^2, x^3y^3, x^4y^2, x^2y^4\}$	$\{x^3y^4, x^4y^3, x^4y^4\}$

Let us describe the main features of the new elements compared with the old ones. We consider the standard square reference element $[-1,+1] \times [-1,+1]$.

Element B9. The classical Serendipity element of degree two has eight degrees of freedom. The improved element is obtained by adding the sole bubble function of degree two, i.e., the function

$$b_1(x,y) = (1-x^2)\,(1-y^2)$$

In Fig. 1 both Serendipity and complete element are shown. We recall that for the plate problem at each node three degrees of freedom are present, one for the displacement and two for the rotations. For the shell problem two other degrees of freedom have to be added to approximate the in-plane displacements.

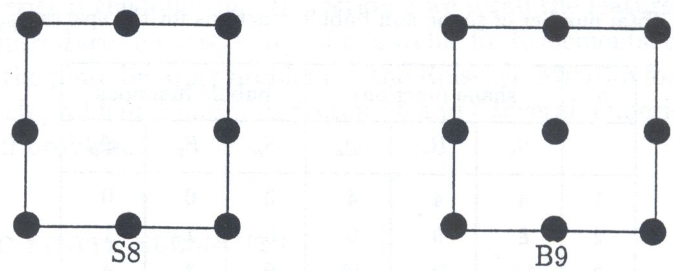

Fig. 1. Serendipity and complete element for $p = 2$

Element B15. The classical Serendipity element of degree three has twelwe degrees of freedom. The improved element is obtained by adding three bubble functions. Since we are dealing with hierarchic elements we keep the previous function $b_1(x, y)$ and to obtain B15 we add the following functions:

$$b_2(x, y) = x \times b_1(x, y)$$
$$b_3(x, y) = y \times b_1(x, y)$$

In a different way, the bubble functions we have added can be described as the product $\mathcal{P}_1 \otimes b_1(x, y)$. In Figure 2 the elements are shown.

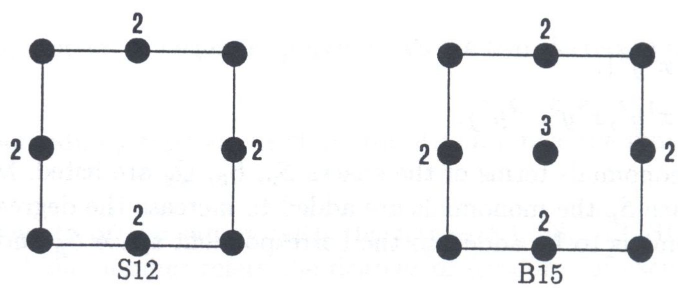

Fig. 2. Serendipity and intermediate element for $p = 3$

Element B22. The classical Serendipity element of degree four has seventeen degrees of freedom and includes the bubble function b_1. Adding five bubble functions we get the element B22. We keep all the previous bubble functions and add:

$$b_4(x, y) = xy \times b_1(x, y)$$
$$b_5(x, y) = x^2 \times b_1(x, y)$$
$$b_6(x, y) = y^2 \times b_1(x, y)$$

The set can be described as the product $\mathcal{P}_2 \otimes b_1(x, y)$. In Figure 3 the elements are shown.

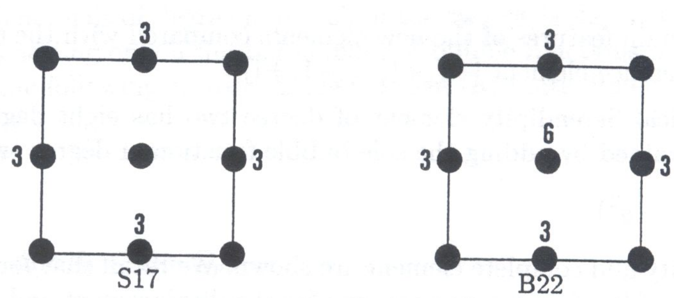

Fig. 3. Serendipity and intermediate element for $p = 4$

3. THE THIN PLATE PROBLEM

Hereafter we shortly recall the problem. Details can be found, e.g., in [4, 17]. We consider the Reissner–Mindlin assumptions for the plate bending problem. This plate theory takes into account the transverse shear deformations. The theory uses the hypothesis that particles of the plate originally on a line that is normal to the undeformed middle surface remain on a straight line during deformation, but this line is not necessarily normal to the deformed middle surface. Let Ω denotes the domain of the plate, E the Young's modulus, t the thickness of the plate, ν the Poisson's ratio, k the shear correction factor (usually taken as 5/6). Let $w(x,y)$ denote the vertical displacement, ϕ_x and ϕ_y the in-plane rotations. Taking into account the contribution of the external load $f(x,y)$, after scaling, normalizing some physical constant and setting $\vec{\phi} = (\phi_x, \phi_y)$ the problem can be stated as:

$$\begin{cases} \text{Find}(\vec{\phi}, w) \in \left(H_0^1(\Omega)\right)^2 \times H_0^1(\Omega) \text{ such that:} \\ \\ \displaystyle\min_{(\vec{\phi},w)\in\left(H_0^1(\Omega)\right)^2\times H_0^1(\Omega)} \left[\tfrac{1}{2}\, A(\vec{\phi},\vec{\phi}) + \tfrac{1}{2}\, t^{-2}\, \|\nabla w - \vec{\phi}\|_0^2 - (f, w)\right] \end{cases}$$

where $A(\vec{\phi}, \vec{\psi})$ is the form

$$A(\vec{\phi},\vec{\psi}) = \frac{E}{12(1-\nu^2)} \int_\Omega \Big\{ \phi_{x/x}\psi_{x/x} + \phi_{y/y}\psi_{y/y} + \nu\left(\phi_{x/x}\psi_{y/y} + \phi_{y/y}\psi_{x/x}\right)$$
$$+ \frac{1-\nu}{2}(\phi_{x/y} + \phi_{y/x})(\psi_{x/y} + \psi_{y/x}) \Big\}\, dxdy$$

Despite its simple approach the discretization of the Reissner–Mindlin model is not straightforward. The inclusion of transverse shear strain effect in standard finite element models introduces undesirable numerical effects. The approximate solution is very sensitive to the plate thickness and, for small thickness, it is very far from the true solution. The phenomenon is known as locking of the numerical solution.

4. THE SHELL PROBLEM

In recent years several types of numerical approximations have been proposed for dealing with shell problems. It is well known that a shell is a three-dimensional structure where one dimension, the thickness, is smaller compared with the remaining two dimensions. It can be derived from a thin plate by initially forming the middle plane to a curved surface. Under the action of external forces the shell, initially at rest, is subject to deformation according to the laws of the three–dimensional elasticity. Although the same assumptions regarding the transverse distribution of strains and stress are again valid, the way in which the shell supports external loads is quite different from that of a flat plate.

A class of models is based on the classical non linear shell theory and employs the notion of surface introduced by Cosserat. The inclusion of transverse shear strain effect in the finite element models introduce an undesirable numerical effect, the so called *shear locking* phenomenon. Among other Naghdi has developed this model (see [12]). We consider the shell model arising from the Naghdi formulation. The deformations field is represented by the displacements $\vec{u} = (u_1, u_2, u_3)$ of the midsurface of the shell and by the rotations $\widetilde{\theta} = (\theta_1, \theta_2)$. Let us use the convention that Greek indices range over 1 and 2. Denoting with V the space of the admissible deformations, the variational formulation of the Naghdi model is the following:

$$\begin{cases} \text{Find } (\vec{u},\widetilde{\theta}) \in V \text{ such that :} \\ \displaystyle\int_\Omega \frac{a^{\alpha\beta\gamma\delta}}{12} \Upsilon_{\alpha\beta}(\vec{u},\widetilde{\theta})\Upsilon_{\gamma\delta}(\vec{v},\widetilde{\psi})\,\sqrt{a}\,\mathrm{d}\xi_1\mathrm{d}\xi_2 + \frac{1}{t^2}\int_\Omega a^{\alpha\beta}\frac{E}{2(1+\nu)}\Sigma_\alpha(\vec{u},\widetilde{\theta})\Sigma_\beta(\vec{v},\widetilde{\psi})\,\sqrt{a}\,\mathrm{d}\xi_1\mathrm{d}\xi_2 \\ \displaystyle\quad + \frac{1}{t^2}\int_\Omega a^{\alpha\beta\gamma\delta}\Lambda_{\alpha\beta}(\vec{u})\Lambda_{\gamma\delta}(\vec{v})\,\sqrt{a}\,\mathrm{d}\xi_1\mathrm{d}\xi_2 = \int_\Omega \vec{f}\vec{v}\,\sqrt{a}\,\mathrm{d}\xi_1\mathrm{d}\xi_2 \qquad \forall(\vec{v},\widetilde{\psi}) \in V \end{cases}$$

where Υ is the change of curvature tensor, Σ is the transverse shear strain tensor, Λ is the membrane strain tensor and

$$a^{\alpha\beta\gamma\delta} = \frac{E}{2(1+\nu)}\left[a^{\alpha\gamma}a^{\beta\delta} + a^{\alpha\delta}a^{\beta\gamma} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\gamma\delta}\right]$$

We deal with the classical Scordelis–Lo problem, a test which is extremely useful for determining the ability of an element to accurately solve complex states of membrane strain.

5. Numerical solution of a plate problem

As a plate problem we consider a unit square with uniform decomposition in quadrilateral elements. Due to the symmetry of the domain the computations have been performed on a quarter of plate only. Hereafter we present some results related to the clamped plate problem. The clamped plate is the most effective test to check the robustness of finite elements with respect to the locking effect. For a clamped plate we impose the following condition on the boundary $\partial\Omega$ of the plate

$$w(x,y) = \phi_x(x,y) = \phi_y(x,y) = 0 \quad \text{on } \partial\Omega$$

Several tests with different values of thickness have been performed to analyse reliability and robustness of finite elements. For each test, among others, displacement at the center C of the plate and the discrete strain energy have been computed. Let $w_{\mathrm{ex}}(C)$ denote the exact displacement at the center of the plate (see [16]) and $w_h(C)$ the finite element solution. The relative displacement error is defined as

$$\mathcal{E}_d = \frac{w_{\mathrm{ex}}(C) - w_h(C)}{w_{\mathrm{ex}}(C)} \times 100$$

The exact strain energy was not available. Using the computed values of the strain energy an extrapolation has been made in order to get an estimated value of the exact energy. Let us denote with E_{ex} the extrapolated value of the energy and let E_h be the computed discrete energy. The relative energy norm $||e||$ of the error $e = w_{\mathrm{ex}} - w_h$ can be expressed in the following way:

$$\mathcal{E}_e = ||e|| = \left(\frac{E_{\mathrm{ex}} - E_h}{E_{\mathrm{ex}}}\right)^{1/2} \times 100$$

Pointwise evaluation of the displacement error is, of course, a local error indicator whereas the energy norm error is a global indicator. Figures 4 and 5 give the displacement error $\mathcal{E}_d$ versus the number of degrees of freedom and the energy norm error $\mathcal{E}_e$ versus the reciprocal of the mesh size parameter, respectively. In each picture the dashed line refers to the Serendipity elements, the continuous line to the new "bubble plus" elements, except for the S4 element where only one line is present. We consider a thickness of value $t = 0.0001$ (very thin plate), for which some Serendipity elements show a locking behavior. The numerical results show that the performance of the "bubble plus" elements is substantially improved with respect to the Serendipity elements in the cases $p = 2$ and $p = 3$, the element S17 already exhibiting a good behavior. Always the locking phenomenon is kept under control. The cost of the increase of the number of degrees of freedom is negligible compared with the improvement of the results.

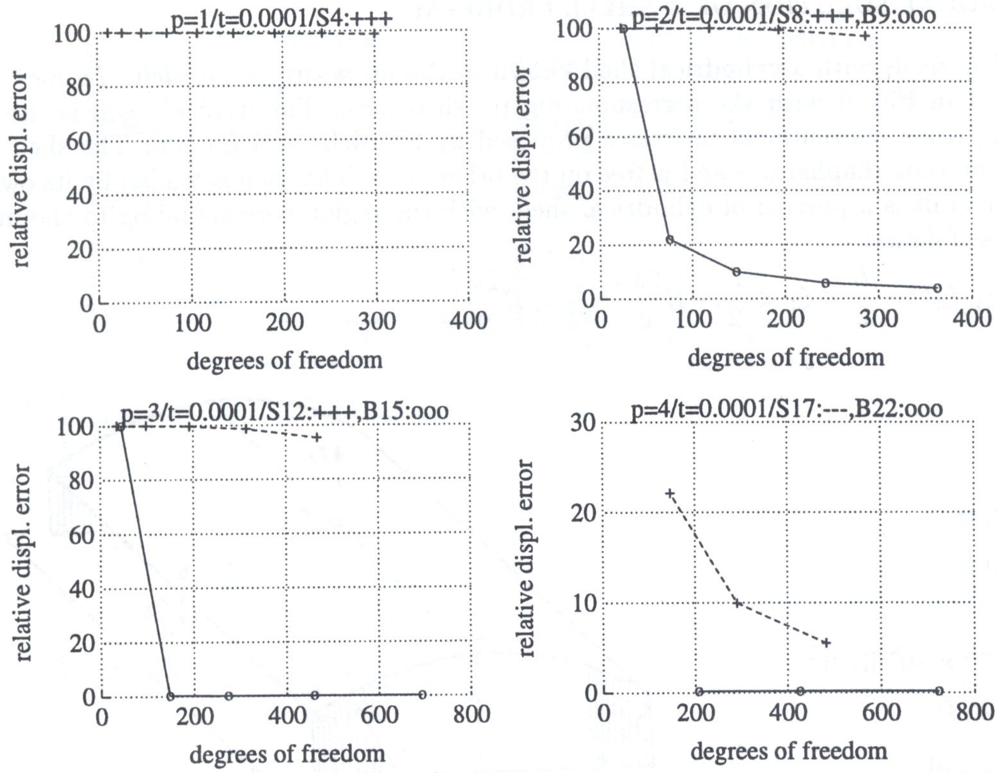

Fig. 4. Relative displacement error *vs.* d.o.f.; $t = 0.0001$; p=1–4.

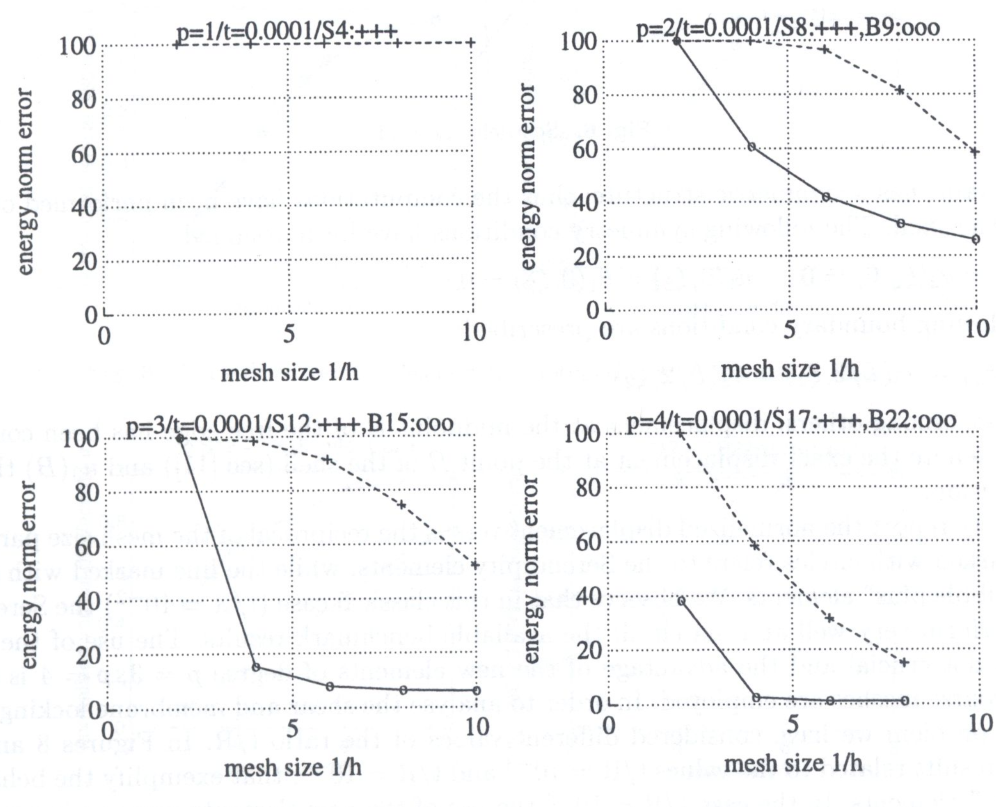

Fig. 5. Energy norm error *vs.* mesh size; $t = 0.0001$; p=1–4.

6. NUMERICAL SOLUTION OF A SHELL PROBLEM

The problem deals with a cylindrical shell known in the literature as *Scordelis–Lo roof*. The shell is described in Fig. 6 with the corresponding physical data. This typical shell is used in civil engineering using conventional processes proposed by Scordelis and Lo [14]. The shell is simply supported on rigid diaphgrams and is free on the other sides. The shell is loaded by its own weight. The barrel vault is a portion of cylindrical shell, with the region corresponding to the midsurface described as follows:

$$\Omega = \left\{(\xi_1, \xi_2) : -\frac{L}{2} < \xi_1 < \frac{L}{2}, -R\frac{2\pi}{9} < \xi_2 < R\frac{2\pi}{9}\right\}$$

$R = 25$ ft

$L = 50$ ft

$\nu = 0$

$E = 4.32 \times 10^8$ lb/ft^2

$t = 0.25$ ft

$\theta_0 = \frac{2}{9}\pi$ rad

load $= 90$ lb/ft^2

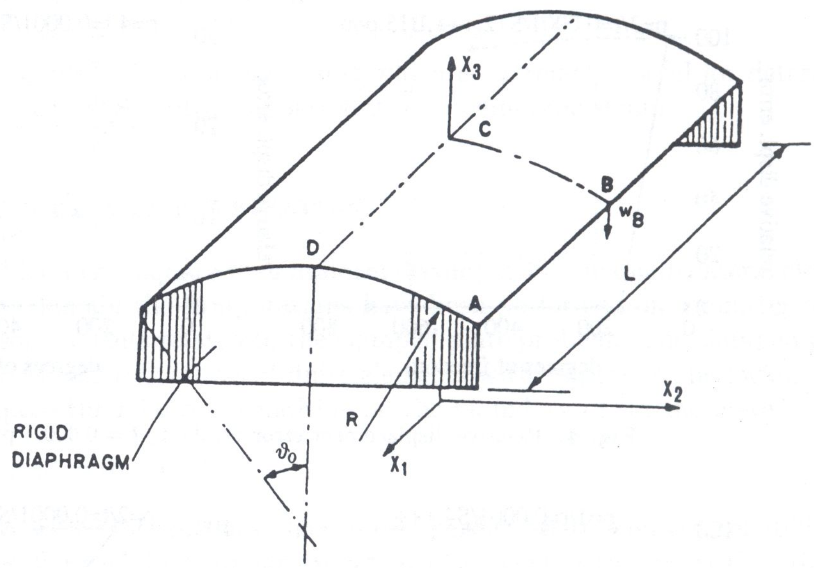

Fig. 6. Scordelis–Lo roof

The barrel vault has a symmetric structure; thus the computations have been performed only on a quarter of the shell. The following symmetry conditions have been assumed:

$$u_2(\xi_1, 0) = \theta_2(\xi_1, 0) = 0\,, \quad u_1(0, \xi_2) = \theta_1(0, \xi_2) = 0$$

and the following boundary conditions are prescribed:

$$u_2(L/2, \xi_2) = u_3(L/2, \xi_2) = \theta_2(L/2, \xi_2) = 0 \tag{1}$$

For each test, among others, displacement at the midpoint B of the free edge has been computed. Let $u_3^{\text{ex}}(B)$ denote the exact displacement at the point B of the shell (see [11]) and $u_3(B)$ the finite element solution.

In Fig. 7 we report the normalized displacement versus the reciprocal of the mesh size parameter. The line marked with circles refers to the Serendipity elements, while the line marked with stars to the new "bubble plus" elements. We observe that in this classical case ($t/R = 10^{-2}$) the Serendipity elements perform very well and match all the available benchmark results. The use of the bubble functions is not crucial and the advantage of the new elements of degree $p = 3$, $p = 4$ is evident only when coarse meshes are employed. In order to analyse the shear and membrane locking arising in this test problem we have considered different values of the ratio t/R. In Figures 8 and 9 we present the results related to the values $t/R = 10^{-3}$ and $t/R = 10^{-4}$, that exemplify the behavior of "bubble plus" elements. In the case $t/R = 10^{-3}$ the use of the new elements gives an improvement of the rate of convergence when $p = 3$, while for the degree $p = 2$ the added bubble is not able to improve the convergence. In the case $t/R = 10^{-4}$ the performance of "bubble plus" elements is substantially improved with respect to the Serendipity elements of degree $p = 3$, $p = 4$.

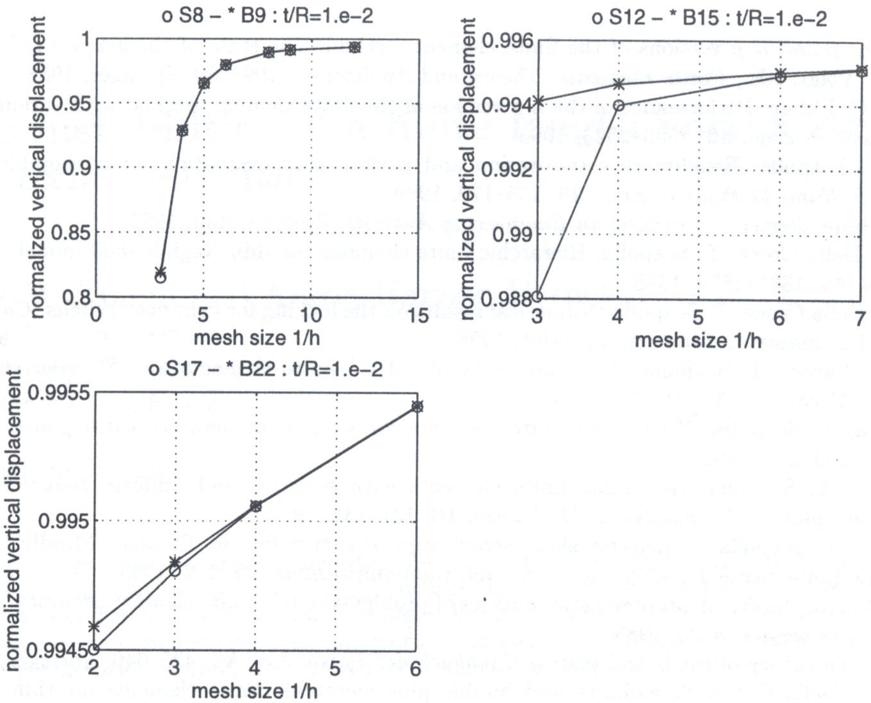

Fig. 7. Normalized vertical displacement *vs.* mesh size; $t/R = 0.01$; p=2–4.

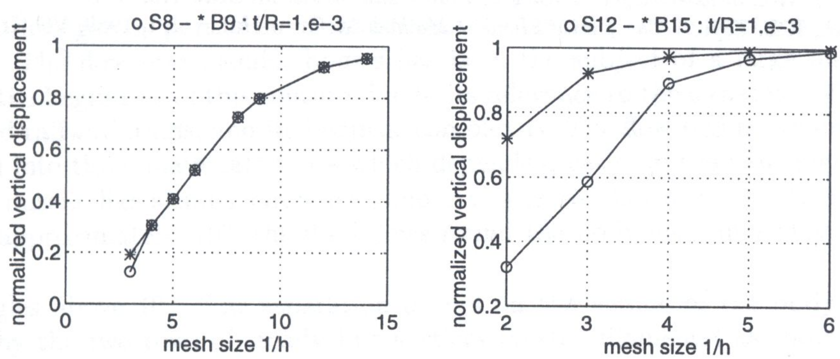

Fig. 8. Normalized vertical displacement *vs.* mesh size; $t/R = 0.001$; p=2–3.

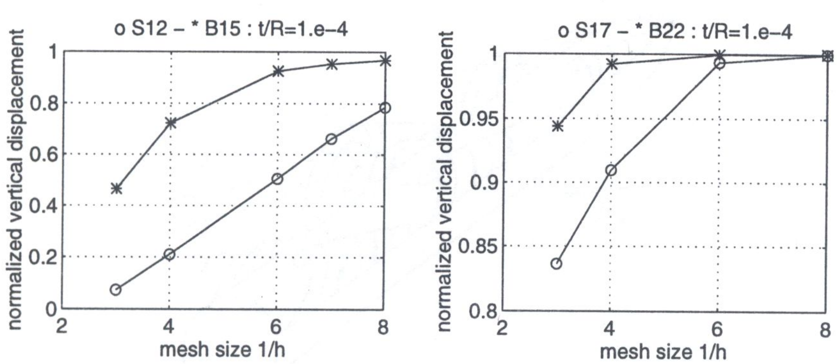

Fig. 9. Normalized vertical displacement *vs.* mesh size; $t/R = 0.0001$; p=3–4.

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