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On optimum design of a vibrating plate with respect to its thickness and eigen-frequencies

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The eigenvalue optimization problem for anisotropic plates has been dealt with. The variable thickness of a plate plays the role of a design variable. The state problem arises considering free vibrations of a plate. The demand of the lowest first eigenfrequency means the maximal first eigenvalue of the elliptic eigenvalue problem. The continuity and differentiability properties of the first eigenvalue have been examined. The existence theorem for the optimization problem has been stated and verified. The finite elements approximation has been analyzed. The shifted penalization and the method of nonsmooth optimization can be used in order to obtain numerical results.

INTRODUCTION

The optimal design of a construction depends on the analysis of the link between design variables (the geometrical form of the construction, material cofficients) and state variables determined by the laws and rules of mechanics. Mathematically, the response of the construction to the design variables is modelled by ordinary and partial differential equations or variational inequalities. In the case of free vibrations of constructions the state problem is the eigenvalue problem. The variable thickness of the plate appeared as a design variable in the papers [1, 2, 5, 8]. One of the main problems in this case is the choise of a suitable admissible set of thickness-functions in order to obtain the existence of the optimal thickness as well as the best possible numerical approximation. The kind of differentiability of the eigenvalues plays a crucial role in the sensitivity and numerical analysis of the problem. These questions have been investigated in papers [4] and [9].

The numerical analysis has two basic tasks. Firstly, to approximate the originally infinite dimensional problem by a finite dimensional one with respect to both the design and state variable. Secondly, to choose suitable numerical algorithms involving all the prescribed constraints.

1. STATE PROBLEM FORMULATION

Let us assume free vibrations of a thin anisotropic plate. In the case of Kirchhoff model ([6, 10]), the deflection function $y \equiv y(t, x_1, x_2)$ is a solution of the hyperbolic equation

$$\rho(x)e(x)\partial_{tt}y - \tfrac{1}{12}\mathrm{div}(\rho(x)e^3(x)\mathrm{grad}\partial_{tt}y) + \partial_{ij}(e^3(x)A_{ijkl}\partial_{kl}y) = 0, \quad t \in \mathbb{R},\ x = (x_1, x_2) \in \Omega,$$

where $\rho(x)$ is the density, $e(x)$ is the variable thickness of the plate, Ω its middle plane and A_{ijkl} are material coefficients. The summation convention through the indices $i, j, k, l \in \{1, 2\}$ is considered. The third-order tensor A_{ijkl} is symmetric and positively definite, i.e.

$$A_{ijkl} = A_{jikl} = A_{klij}, \tag{1}$$

$$A_{ijkl}\tau_{ij}\tau_{kl} \geq \alpha\ \tau_{ij}\tau_{ij},\ \alpha > 0; \quad \text{for all } \boldsymbol{\tau} = (\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}) \in \mathbb{R}^4_{sym}. \tag{2}$$

Expressing the deflection-function y in an oscillating form

$$y(t, x) = c\ e^{i\omega t}w(x), \quad c \in \mathbb{C},\ \omega > 0;$$

we obtain, setting $\lambda = \omega^2$, the eigenvalue problem

$$\partial_{ij}(e^3(x)A_{ijkl}\partial_{kl}y) = \lambda[\rho(x)e(x)w(x) - \tfrac{1}{12}\mathrm{div}(\rho(x)e^3(x)\mathrm{grad}w(x))], \quad x \in \Omega \tag{3}$$

$$w\ =\ \partial_\nu w\ =\ 0 \quad \text{on } \Gamma_1 \tag{4}$$

$$w\ =\ M(w)\ = 0 \quad \text{on } \Gamma_2 \tag{5}$$

$$M(w)\ =\ T(w)\ = 0 \quad \text{on } \Gamma_3, \tag{6}$$

where $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3}$ $(\Gamma_i \cap \Gamma_j = \emptyset,\ i \neq j)$ is the Lipschitz continuous boundary of the region Ω (see [10]), Γ_1 is not a line segment, $\boldsymbol{\nu}(x) = (\nu_1(x), \nu_2(x))$ is the unit outward normal vector at the point $x \in \Gamma$,

$$M(w) = e^3(x)A_{ijkl}\partial_{ij}w\nu_k\nu_l$$

is the bending moment and

$$T(w) = -\partial_k[e^3(x)A_{ijkl}\partial_{ij}w\nu_l - \partial_\sigma[e^3(x)A_{ijkl}\partial_{ij}w\nu_k\sigma_l]$$

is the effective shear force of the plate, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2) = (-\nu_2, \nu_1)$ is the unit tangential vector with respect to Γ.

In our further considerations we shall use the following spaces of the real functions (see [10] for details). All considered functions are defined on the Lipschitz region Ω or on its closure $\overline{\Omega} = \Omega \cup \Gamma$:

(1) $C(\overline{\Omega})$ — the space of all continuous functions $f : \overline{\Omega} \to \mathbb{R}$, $C(\overline{\Omega})$ is a Banach space with the norm

$$\|f\|_{C(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |f(x)|.$$

(2) $C^{0,1}(\overline{\Omega})$ — the space of all Lipschitz continuous functions $f : \overline{\Omega} \to \mathbb{R}$ characterized by constants $L_f \geq 0$ such that

$$|f(x) - f(y)| \leq L_f|x - y| \quad \text{for all } x,\ y \in \overline{\Omega}.$$

(3) $C^m(\overline{\Omega})$ — the space of all m-times continuously differentiable functions $f : \overline{\Omega} \to \mathbb{R}$.

(4) $C_0^\infty(\Omega)$ — the space of all infinitely times differentiable functions $f : \Omega \to \mathbb{R}$ with a compact support $K_f \subset \Omega$ (K_f is a bounded and closed subset of Ω such that $f(x) = 0$ for $x \notin K_f$).

(5) $L_2(\Omega)$ — the space of all measurable functions $f : \Omega \to \mathbb{R}$ such that $\int_\Omega f^2\,dx < \infty$. $L_2(\Omega)$ is a Hilbert space with a scalar product

$$(f,g)_0 = \int_\Omega fg\,dx, \quad f,\ g \in L_2(\Omega)$$

and the norm

$$\|f\|_0 = \int_\Omega f^2\,dx, \quad f \in L_2(\Omega).$$

(6) $H^m(\Omega)$, $m \in N$; — the Sobolev space of all functions $f \in L_2(\Omega)$ such that there exist all partial derivatives (in the sense of distributions) $D^k f \in L_2(\Omega)$, $|k| \le m$. It should be stressed that for the multiindex $k = (k_1, k_2)$, $k_1 \ge 0$, $k_2 \ge 0$; we set $|k| = k_1 + k_2$ and $D^k f = \dfrac{\partial^{k_1+k_2} f}{\partial x_1^{k_1} \partial x_2^{k_2}}$.

$H^m(\Omega)$ is a Hilbert space with a scalar product

$$(f,g)_m = \sum_{|k|\le m} (D^k f, D^k g)_0, \quad f,\ g \in H^m(\Omega),$$

and the norm

$$\|f\|_m = \sqrt{(f,f)_m}, \quad f \in H^m(\Omega).$$

We have identified $C^m(\overline{\Omega}) \subset H^m(\Omega)$ with classical derivatives as the distributive derivatives defined for the functions from $H^m(\Omega)$. The relations $H^k(\Omega) \subset\subset H^{k-1}(\Omega)$, $H^0(\Omega) = L_2(\Omega)$ hold simulataneously. It results that the space $H^k(\Omega)$ is imbedded compactly into the space $H^{k-1}(\Omega)$, i.e. each sequence of functions weakly convergent in $H^k(\Omega)$ is (strongly) convergent in $H^{k-1}(\Omega)$.

The space $H^2(\Omega)$ plays a crucial role in the theory of plates. Every function $f \in H^2(\Omega)$ is continuous on $\overline{\Omega}$ (after changing it in the zero-measure set when necessary). Moreover we have $H^2(\Omega) \subset\subset C(\overline{\Omega})$ which means that every sequence of functions weakly convergent in $H^2(\Omega)$ is uniformly convergent on $\overline{\Omega}$, this being equivalent with the convergence in the norm $\|.\|_{C(\overline{\Omega})}$.

In order to formulate the boundary conditions in a generalized (weak) form, for the functions from Sobolev spaces we can define their values on the boundary Γ of Ω in the form of traces. If $v \in H^1(\Omega)$, then $v|_\Gamma$ is a function from the space $L^2(\Gamma)$ such that there exists a sequence $\{v_n\} \subset H^1(\Omega)$ fulfilling the convergence $v_n \to v$ in $H^1(\Omega)$ and $v_n|_\Gamma \to v|_\Gamma$ in $L^2(\Gamma)$. It is possible to formulate $\dfrac{\partial v}{\partial \nu} \in L^2(\Gamma)$ for the functions $v \in H^2(\Omega)$ in a similar way.

Finally we introduce the spaces of functions fulfilling the zero boundary conditions on Γ. We set

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \ :\ v|_\Gamma = 0\}$$

and

$$H_0^2(\Omega) = \{v \in H^2(\Omega) \ :\ v|_\Gamma = \left.\frac{\partial v}{\partial \nu}\right|_\Gamma = 0\}.$$

Let us come back to the eigenvalue problem (3)–(6). We set

$$V = \{v \in H^2(\Omega) \ :\ v|_{\Gamma_1 \cup \Gamma_2} = 0,\ \left.\frac{\partial v}{\partial \nu}\right|_{\Gamma_1} = 0\}.$$

The functions from V fulfil the geometrical boundary conditions (4), (5) in the sense of traces. It can be verified that V is a Hilbert space with a scalar product

$$(v,w) = \int_\Omega \sum_{i,j=1}^{2} \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j}\,dx, \quad v,\ w \in V;$$

and the norm

$$\|v\| = \Big(\int_\Omega \sum_{i,j=1}^{2} (\frac{\partial^2 v}{\partial x_i \partial x_j})^2 \, \mathrm{d}x\mathrm{d}y \Big)^{\frac{1}{2}}, \quad v \in V.$$

We denote by V^* the Banach space of all linear bounded functionals over V with a norm

$$|f|_* = \sup_{\|v\|=1} \langle |f, v\rangle|.$$

After multiplying the equation (3) by the functions $v \in V$ and integrating on Ω, we obtain the generalized (weak) form of the eigenvalue problem (3)–(6):

$$a(e; w, v) = \lambda(e) b(e; w, v) \quad \text{for all } v \in V, \ \lambda(e) \in \mathbb{R}, \ w \equiv w(e) \neq 0, \ w \in V; \tag{7}$$

with

$$a(e; w, v) = \int_\Omega \rho(x) e^3(x) A_{ijkl} \partial_{ij} w \partial_{kl} v \, \mathrm{d}x,$$

$$b(e; w, v) = \int_\Omega \rho(x) [\tfrac{1}{12} e^3(x) (\mathrm{grad} w \cdot \mathrm{grad} v) + e(x) w(x) v(x)] \, \mathrm{d}x,$$

Let us denote $U = C(\overline{\Omega})$. The bilinear forms $a(e; u, v)$, $b(e; u, v)$ can be represented for every $e \in U$ by the operators

$$A(e) : V \to V^*,$$

$$B(e) : H^1(\Omega) \to V^*,$$

$$V \subset\subset H^1(\Omega) : \langle A(e)w, v\rangle = a(e; w, v), \ \langle B(e)w, v\rangle = b(e; w, v); \ w, \ v \in V.$$

The operators $A(e)$, $B(e)$ are linear bounded symmetric and positively definite on V and $H^1(\Omega)$, respectively:

$$\langle A(e)w, v\rangle = \langle A(e)v, w\rangle,$$

$$\langle A(e)v, v\rangle \geq \alpha_0(e)\|v\|^2, \quad \alpha_0(e) > 0;$$

$$\|A(e)\|_* \leq \alpha_1(e)\|v\|, \quad v, w \in V;$$

$$\langle B(e)w, v\rangle = \langle B(e)v, w\rangle,$$

$$\langle B(e)v, v\rangle \geq \beta_0(e)\|v\|_1^2, \quad \beta_0(e) > 0;$$

$$\|B(e)\|_* \leq \beta_1(e)\|v\|_1, \quad v, w \in H^1(\Omega);$$

Moreover, the operator B(e) is compact on the Hilbert space V due to the compact imbedding $V \subset\subset H^1(\Omega)$.

Now, we can reformulate the generalized eigenvalue problem (7) in an operator form:

$$A(e)w(e) = \lambda(e)B(e)w(e), \quad \lambda(e) \in \mathbb{R}, \ 0 \neq w(e) \in V. \tag{8}$$

Using the inverse operator $A(e)^{-1} : V^* \to V$ the problem can be expressed as the eigenvalue problem for the linear compact operator in the Hilbert space and we can apply the spectral theory of compact operators in Hilbert spaces.

Theorem 1 *There exist sequences $\{\lambda_n(e)\}$ of eigenvalues and $\{w_n(e)\}$ of eigenfunctions solving the problem (7) and satisfyng the relations*

$$0 < \lambda_1(e) \le \lambda_2(e) \le \dots \le \lambda_n \le \dots \tag{9}$$

$$\lim_{n\to\infty} \lambda_n(e) = \infty \tag{10}$$

$$\langle A(e)w_j(e), w_k(e)\rangle = \langle B(e)w_j(e), w_k(e)\rangle = 0 \quad \text{for all } j \ne k \tag{11}$$

$$\lambda_1(e) = \min_{v\in K(e)} \langle A(e)v, v\rangle = \langle A(e)w_1(e), w_1(e)\rangle, \tag{12}$$

$$K(e) = \{v \in V : \langle B(e)v, v\rangle = 1\}.$$

The set $\{w_n(e)\}$ is the basis of the Hilbert space V and every eigenvalue $\lambda_n(e)$ has a finite multiplicity, i.e. the space of its eigen-functions is a finite-dimensional one.

2. The optimal design problem

Before formulating the optimal design problem with the eigenvalue problem (3)–(6) or (8) in the role of state problem and the thickness-function e as the design variable, we recall the result of the continuity analysis in the case of eigenvalues depending on the operator coefficients ([4]).

Let us take into account that the set $U = C(\overline{\Omega})$ is a Banach space with the norm

$$\|e\|_U = \max_{x\in\overline{\Omega}} |e(x)| \tag{13}$$

Theorem 2 (Continuity of the first eigenvalue). *Let $e_n \in U$ be the sequence of thickness-functions of the plate and*

$$\lim_{n\to\infty} e_n = e^* \quad \text{in } U\,. \tag{14}$$

Then

$$\lim_{n\to\infty} \lambda_1(e_n) = \lambda_1(e^*) \quad \text{in } \mathbb{R}\,. \tag{15}$$

In order to achieve the existence result for the following optimal design problem, we restrict the set of admissible thickness $e \in U$. We introduce the admissible set

$$U_{\text{ad}} = \{e \in C^{0,1}(\overline{\Omega}) :\ 0 < e_{\min} \le \|e\|_U \le e_{\max},\ |\partial_i e| \le C_i,\ i = 1,2;\ \int_\Omega \rho(x)e(x)\,\mathrm{d}x = C_3\}.$$

The set U_{ad} is convex and compact in the Banach space U.

One of the basic control problems for eigenvalues is to determine the control parameters in such a way that the first eigenvalue is maximum that corresponds to the minimum possible first eigen-frequency of the construction. We are looking for $e^* \in U_{\text{ad}}$ fulfilling

$$\lambda_1(e^*) = \max_{e\in U_{\text{ad}}} \lambda_1(e)\,. \tag{16}$$

Setting the cost functional in the form

$$J(e) = \lambda_1(e)^{-1}, \quad e \in U_{\text{ad}}\,,$$

we obtain the equivalent

Optimal Design Problem P. To find $e^* \in U_{\text{ad}}$ such that

$$J(e^*) = \min_{e\in U_{\text{ad}}} J(e) \tag{17}$$

Using the method of compactness, it is possible to obtain the existence result in the same way as in [1, 2, 5].

Theorem 3 *Let $\lambda_1(e)$ be the first eigenvalue of the problem (8). Then there exists a solution of the (maximum) problem (16) or (minimum) problem (17).*

3. On the finite elements approximation

We shall assume that the domain Ω is polygonal and divided into a regular partition of rectangulars $Q_i \in \mathcal{T}_h,\ h \in (0, h_0)$:

$$\overline{\Omega} = \bigcup_{i=1}^{I(h)} \overline{Q}_i, \quad Q_i \cap Q_j = \emptyset,\ i \neq j,\ h = \operatorname{diam} Q_i.$$

We assume that $\mathcal{T}_h$ is consistent with the partition of the boundary $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3}$. Further we set $P_k(Q)$ the spaces of bilinear ($k = 1$) or bicubic ($k = 3$) polynomials on Q. The admissible sets U_{ad} and V are approximated by

$$U_{\text{ad}}^h = \{e \in U_{\text{ad}} :\ e|_{Q_i} \in P_1(Q_i),\ 1 \leq i \leq I(h)\},$$

$$V_h = \{v \in V :\ v|_{Q_i} \in P_3(Q_i),\ 1 \leq i \leq I(h)\}.$$

We formulate the

Optimal Design Problem P_h. To find a thickness-function $e_h^* \in U_{\text{ad}}^h$ such that

$$J_h(e_h^*) = \min_{e \in U_{\text{ad}}^h} J_h(e), \tag{18}$$

$$J_h(e) = \lambda_1^h(e)^{-1}, \quad e \in U_{\text{ad}}^h, \tag{19}$$

$$\lambda_1^h(e) = \min_{v \in K_h(e)} \langle A(e)v, v\rangle = \langle A(e)w_h^1(e), w_h^1(e)\rangle, \tag{20}$$

$$K_h(e) = \{v \in V_h :\ \langle B(e)v, v\rangle = 1\}.$$

$\lambda_1^h(e)$ is the first approximated eigenvalue obtained by the Raleigh–Ritz method.

The existence of a solution $e_h^* \in U_{\text{ad}}^h$ of the Problem P_h can be established in the same way as that of a solution e^* of the original Problem P. The following theorem describes the convergence of the method.

Theorem 4 *Let $\{e_{h_n}^*\} \subset U_{\text{ad}}^h$ be an arbitrary sequence of solutions of the Problem P_{h_n}, $h_n \to 0$. Then there exists a subsequence (again denoted by $e_{h_n}^*$) such that*

$$e_{h_n}^* \to e^* \quad \text{in } U \text{ (uniformly on } \overline{\Omega}), \tag{21}$$

$$\lambda_1^{h_n}(e_{h_n}^*) \to \lambda_1(e^*) \quad \text{in } \mathbb{R}, \tag{22}$$

$$w_{h_n}^1(e_{h_n}^*) \rightharpoonup w_1(e^*) \quad \text{in } V, \tag{23}$$

where $[e^*, w_1(e^*)] \in U_{\text{ad}} \times V$ is a solution of the Optimal Design Problem P.

Proof: The set U_{ad} is compact in U and hence there exists a subsequence $\{e_{h_n}^*\}$ such that (21) holds.

We shall verify that e^* is a minimizing element of the Problem P. The approximative eigenvalue $\lambda_1^{h_n}(e_{h_n}^*)$ and the correspondent eigenfunction $w_{h_n}^1 \equiv w_{h_n}^1(e_{h_n}^*)$ fulfil the relations

$$\langle A(e_{h_n}^*)w_{h_n}^1, v\rangle = \lambda_1(e_{h_n}^*)\langle B(e_{h_n}^*)w_{h_n}^1, v\rangle \quad \text{for every } v \in V_{h_n}, \tag{24}$$

$$\langle B(e_{h_n}^*)w_{h_n}^1, w_{h_n}^1\rangle = 1. \tag{25}$$

The last relations as well as the convergence of the Raleigh–Ritz method and the continuity of the operators $e \to A(e)$, $e \to B(e)$ imply the convergence of the subsequences:

$$\lambda_1(e^*_{h_n}) \to \lambda^*, \quad w^1_{h_n} \rightharpoonup w^* \quad \text{in } V, \quad w^1_{h_n} \to w^* \quad \text{in } H^1(\Omega)$$

and further

$$A(e^*)w^* = \lambda^* B(e^*)w^*, \quad \langle B(e^*)w^*, w^*\rangle = 1$$

which implies the relations (22), (23). Let $\eta \in U_{\mathrm{ad}}$ be an arbitrary function. Then due to interpolation properties of finite elements, there exists a sequence $\{\eta_n\} \subset U^{h_n}_{\mathrm{ad}}$ such that

$$\lim_{n\to\infty} \eta_n = \eta \quad \text{in } U = C(\overline{\Omega}).$$

One gets the relations

$$J_{h_n}(e^*_{h_n}) \le J_{h_n}(\eta_n), \quad n = 1,\ 2, \dots \tag{26}$$

Due to the continuity of the relation $e \to \lambda_1(e)$ and the convergence of the Raleigh–Ritz method, we obtain, after taking the limit in (26), the inequality

$$J(e^*) \le J(\eta) \quad \text{for every } \eta \in U_{\mathrm{ad}}$$

and hence e^* is a solution of the Optimal Design Problem P what completes the proof. □

4. Numerical realization of the finite elements method

Thickness-functions $e \in U_{\mathrm{ad}}$ and deflections $w \in V$ of the plate are approximated by the functions $e_h \in U^h_{\mathrm{ad}}$ and $w_h \in V_h$:

$$e_h(x_1, x_2) = \sum_{i=1}^{N} q_i \Psi_i(x_1, x_2), \quad N \equiv N(h)$$

$$w_h(x_1, x_2) = \sum_{i=1}^{M} r_i \Phi_i(x_1, x_2), \quad M \equiv M(h),$$

where $\{\Psi_i\}_{i=1}^N$ and $\{\Phi_i\}_{i=1}^M$ are basic functions of the spaces U_h and V_h, respectively. We denote

$$U_h = \{e \in C^{0,1}(\overline{\Omega}) : \ e|_{Q_i} \in P_1(Q_i),\ 1 \le i \le I(h)\}.$$

We set further

$$\mathbf{q} = [q_1, q_2, \dots, q_N] \in \mathbb{R}^N, \quad \mathbf{r} = [r_1, r_2, \dots, r_M] \in \mathbb{R}^M$$

and introduce the matrices

$$\mathcal{A}_h(\mathbf{q}) = \{a(e_h; \Phi_i, \Phi_j)\}_{i,j=1}^M, \quad \mathcal{B}_h(\mathbf{q}) = \{b(e_h; \Phi_i, \Phi_j)\}_{i,j=1}^M,$$

and the vector $\mathbf{d}^h$ with the coordinates

$$d_i^h = \int_\Omega \Psi_i \, \mathrm{d}x, \quad i = 1, \dots, N.$$

The approximating admissible set U^h_{ad} can be expressed in the form

$$U^h_{\text{ad}} = \{\mathbf{q} \in \mathbb{R}^N : \ g_i = 1 - \frac{q_i}{e_{\min}} \le 0, \quad i = 1, 2, ..., N;$$

$$g_i = \frac{q_{i-N}}{e_{\max}} - 1 \le 0, \quad i = N+1, ..., 2N;$$

$$g_i = q_{L(i-2N)} - q_{\hat{L}(i-2N)} - C_1 h_1 \le 0, \quad i = 2N+1, ..., 2N+N_0;$$

$$g_i = q_{\hat{L}(i-2N-N_0)} - q_{L(i-2N-N_0)} - C_1 h_1 \le 0, \quad i = 2N+N_0+1, ..., 2(N+N_0);$$

$$g_i = q_{K(i-2N-N_0)} - q_{\hat{K}(i-2N-2N_0)} - C_2 h_2 \le 0, \quad i = 2N+2N_0+1, ..., 2N+N_0+N_1;$$

$$g_i = q_{\hat{K}(i-2N-N_0-N_1)} - q_{K(i-2N-2N_0-N_1)} - C_2 h_2 \le 0,$$

$$i = 2N+2N_0+N_1+1, ..., 2(N+N_0+N_1);$$

$$g_i = \sum_{j=1}^{N} d_j(h) q_j - C_3 = 0, \quad i = 2(N+N_0+N_1)+1.$$

$q_{L(1)}, ..., q_{L(N_0)} \in \{q_1, ..., q_N\}$ are values of the function e_h in all such nodal points (x_i, y_j) that $(x_i - h_1, y_j) \in \overline{\Omega}_h$. If $q_{L(j)} = e_h(x_i, y_j)$, then $q_{\hat{L}(j)} = e_h(x_i - h_1, y_j)$.

Similarly $q_{K(1)}, ..., q_{K(N_1)} \in \{q_1, ..., q_N\}$ are values of the function e_h in all such nodal points (x_i, y_j) that $(x_i, y_j - h_2 \in \overline{\Omega}_h$. If $q_{K(j)} = e_h(x_i, y_j)$, then $q_{\hat{K}(j)} = e_h(x_i, y_j - h_2)$, where h_1 and h_2 are lengths of the sides of rectangulars of the partition in the directions Ox_1 and Ox_2 respectively.

We formulate the goal function in the form

$$\mathcal{L}_h(\mathbf{q}) = \lambda^1_h(\mathbf{q}) = \inf\{\mathcal{A}_h(\mathbf{q})\mathbf{r}, \mathbf{r})_{\mathbb{R}^M} : \ \mathbf{r} \in \mathbb{R}^M, \ \mathcal{B}_h(\mathbf{q})\mathbf{r}, \mathbf{r})_{\mathbb{R}^M} = 1\} \tag{27}$$

Then the corresponding finite dimensional optimal control problem can be formulated as follows:
Problem $\mathcal{P}^N_h$. To find $q^* \in \mathbb{R}^N$:

$$\mathcal{L}_h(\mathbf{q}^*) = \max_{\mathbf{q} \in U^h_{\text{ad}}} \mathcal{L}_h(\mathbf{q}), \tag{28}$$

$$[\mathcal{A}_h(\mathbf{q}) - \lambda^1_h \mathcal{B}_h(\mathbf{q})]\mathbf{r} = \bar{0}, \quad \lambda_h \in \mathbb{R}, \quad \mathbf{r} \in \mathbb{R}^M, \quad \mathbf{r} \neq \bar{0}. \tag{29}$$

The method of shifted penalization ([3, 7]) can be used solving $\mathcal{P}^N_h$. The shifted penalized cost functional has the form

$$\mathcal{F}(\mathbf{q}, \mathbf{p}, \kappa) = \mathcal{L}_h(\mathbf{q}) - \tfrac{1}{2}\kappa \mathcal{K}(\mathbf{q}, \mathbf{p}), \tag{30}$$

where

$$\mathcal{K}(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{2(N+N_0+N_1)} [\max(0, g_i - p_i)]^2 + (g_{2(N+N_0+N_1)+1} - p_{2(N+N_0+N_1)+1})^2$$

with a penalization coefficient $\kappa > 0$ and a fixed vector $\mathbf{p} \in \mathbb{R}^{2(N+N_0+N^1)+1}$.

Instead of Problem $\mathcal{P}^N_h$ we solve the penalized

Problem $\mathcal{P}^N_{h,\kappa}$:

$$\max\{\mathcal{F}(\mathbf{q}, \mathbf{p}, \kappa) : \mathbf{q} \in \mathbb{R}^N\} \tag{31}$$

with the constraint (29). Following steps describe the solving of $\mathcal{P}^N_{h,\kappa}$:

Step 0.

$p = p^0, \quad \kappa = \kappa_0, \quad l_0 \in (0,1), \quad \beta \in (0,1), \quad \varepsilon \in (0,1)$ fixed.

We set $k = 0$.

Step 1.

$$\mathcal{H}(\mathbf{q}^*) = \max\{\mathcal{H}(\mathbf{q}) : \mathbf{q} \in \mathbb{R}^N\}, \ \mathcal{H}(\mathbf{q}) = \mathcal{F}(\mathbf{q}, \mathbf{p}_k, \kappa_k) \tag{32}$$

Step 2.

If $|g_i| \leq l_k, \ i = 1, 2, ..., 2(N + N_0 + N_1) + 1$

then $\mathbf{p}_{k+1} = \beta \mathbf{p}_k, \ l_{k+1} = \beta l_k, \ \kappa_{k+1} = \kappa_k,$

else $\mathbf{p}_{k+1} = \mathbf{p}_k, \ l_{k+1} = l_k, \ \kappa_{k+1} = \varepsilon^{-1}\kappa_k.$

$k := k + 1$ and go to 2.

In the same way as in [3], the convergence of the method can be verified.

Let $\partial\mathcal{H}(\mathbf{q})$ be the subgradient of the functional $\mathcal{H}$ at $\mathbf{q} \in \mathbb{R}^N$. Lemarechal algorithm ([7]) is to be used solving the problem (32). It can be described in a following way:

Let $\mathbf{q}_0 \in \mathbb{R}^N, \ \nu > 0, \ \rho > 0, \ \xi_0 \in \partial\mathcal{H}(\mathbf{q}_0), \ i = n = 0.$

Step 1.

$t_n = P_n(\mathbf{0})$ — projection of $\mathbf{0} \in \mathbb{R}^N$ on the convex hull of $\{\xi_{m_i}, ..., \xi_n\}$. If $|t_n| \leq \rho$ then terminate.

Step 2.

$\tau_n = \arg\max\{\mathcal{H}(\mathbf{q}_n + \tau t_n) : \tau \geq 0\}, \ \mathbf{q}_{n+1} = \mathbf{q}_n + \tau_n t_n.$

Step 3.

$\xi_{n+1} \in \partial\mathcal{H}(q_n + 1) : \ \langle \xi_{n+1}, t_n \rangle_{\mathbb{R}^N} = 0, \ n := n + 1.$

Step 4.

If $\langle x_n, \mathbf{q}_n - \mathbf{q}_{m_i} \rangle_{\mathbb{R}}^N \leq \nu$, go to *Step 1*, else

Step 5.

Set $i := i + 1, \ m_i = n$, go to *Step 1*.

Remark 1: Using the G-convergence J. Myslinski and A. Sokolowski solved in [9] the problem of maximizing the first eigenvalue on less smooth set of admissible functions. The admissible thicknesses are approximated by partly constant functions. The resulting method is convergent only considering the thickness-functions involved in stiffness matrices. Also the resulting optimal control is not a thickness-function but the stiffness matrix.

5. Numerical results

We have chosen for simplicity an isotropic plate. The approximating Optimal Design Problem $\mathcal{P}_h^N$ in (28), (29) has been solved for a rectangular plate with a middle plane $\Omega = (-a, a) \times (-b, b)$ and for the following boundary conditions on the sides parallel to the axis o_x and o_y:

1. Simply supported, simply supported.
2. Simply supported, clamped.
3. Free, clamped.
4. Clamped, clamped.

The geometrical and mechanical data are:

$a = 3\text{m}, \ b = 2\text{m}, \ e_{\min} = 0.1\text{m}, \ e_{\max} = 0.25\text{m}.$

The initial constant thickness: $e_0(x) = 0.2m$.

The coefficients of admissibility conditions, penalization and convergence are set:

$p_i^0 = 0.20, \ l_i^0 = 0.35, \ \beta = 0.30, \ \mathcal{X}_0 = 10^5, \ \varepsilon = 0.25.$

The computing was finished if $|g_i| \leq 0.001$. The achieved results are displayed in Tables 1-4 and compared with the results in [8] where, using the G-convergence approach, the constant thickness distributions on the individual elements were considered, whilst in this paper the thickness was approximated by bilinear polynomials. The symmetry of the boundary conditions enabled to calculate only with one quadrant of the rectangular region Ω.

The analysis of the numerical results:

1. The material is concentrated at places near the clamped part of the boundary and the center where the curvature function attains its extreme value for a fixed eigenfunction.
2. The extreme increasing of the eigenfrequency is
   (a) 22.62% in the case of plate clamped on the part $x = 0, \ x = a$ of the boundary and free on the part $y = 0, \ y = b$.
   (b) 18.34% for a wholy clamped plate. The difference is influenced also by the value of the maximal possible thickness $e_{\max}$. The value of the first frequency is greater for greater values of $e_{\max}$.
3. It is confirmed that the variable thickness of the elements of the region is more suitable then the constant case. On the contrary the method of gradient projection used in [8] is more effective from the computational point view than the method of shifted penalization.

Table 1. Optimal thickness of a simply supported plate

Boundary conditions: simply supported, simply supported					
The least eigen-frequency		Constant thickness			26.88 Hz
		Optimal thickness			29.64 Hz
		Optimal thickness [8]			29.94 Hz
$x \setminus y$	0.00	0.75	1.50	2.25	3.00
2.00	0.1103	0.1272	0.2502	0.2502	0.2502
1.50	0.1313	0.1169	0.2041	0.2502	0.2502
1.00	0.2502	0.1724	0.1133	0.2502	0.2502
0.50	0.2502	0.2502	0.2049	0.1467	0.2497
0.00	0.2502	0.2502	0.2502	0.1049	0.1234

Table 2. Optimal thickness of a simply supported and clamped plate

Boundary cond.		simply supported, clamped			
The least eigen-frequency		Constant thickness			32.60 Hz
		Optimal thickness			36.10 Hz
		Optimal thickness [8]			36.70 Hz
$x \setminus y$	0.00	0.75	1.50	2.25	3.00
2.00	0.2502	0.2502	0.1123	0.2502	0.2502
1.50	0.2502	0.2502	0.1119	0.1917	0.2502
1.00	0.2502	0.2502	0.1498	0.1214	0.2491
0.50	0.1533	0.2502	0.2502	0.1357	0.1390
0.00	0.1183	0.2407	0.2502	0.2129	0.1185

Table 3. Optimal thickness of a free and clamped plate

Boundary cond.:		free boundary, clamped boundary			
The least eigen-frequency		Constant thickness			18.83 Hz
		Optimal thickness			23.09 Hz
		Optimal thickness [8]			23.37 Hz
$x \setminus y$	0.00	0.75	1.50	2.25	3.00
2.00	0.2501	0.2502	0.2098	0.1429	0.2207
1.50	0.2502	0.2502	0.2074	0.1112	0.1433
1.00	0.2502	0.2502	0.2083	0.1117	0.1378
0.50	0.2502	0.2502	0.2114	0.1280	0.1739
0.00	0.2501	0.2502	0.2339	0.1705	0.2358

Table 4. Optimal thickness of a clamped plate

Boundary cond.:		clamped boundary			
The least eigen-frequency		Constant thickness			49.10 Hz
		Optimal thickness			56.92 Hz
		Optimal thickness [8]			59.05 Hz
$x \setminus y$	0.00	0.75	1.50	2.25	3.00
2.00	0.2502	0.2502	0.1423	0.1424	0.1613
1.50	0.2502	0.2502	0.1452	0.1412	0.1013
1.00	0.2048	0.1324	0.2291	0.2462	0.2435
0.50	0.1417	0.1841	0.2502	0.2502	0.2502
0.00	0.1033	0.2104	0.2502	0.2502	0.2502

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