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Implementation of the harmonic transformation method in shape optimization

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The paper describes an implementation of the variant of the speed method in shape optimization for plane elastic structures, based on harmonic transformations. It is coupled with special method for solving the singular elliptic problems resulting from geometric features like e.g. reentrant corners. Both approaches are based on the works of the author. The interactive system has been built, based on MATLAB environment, and the examples showing the robustness of the algorithms were solved.

1. INTRODUCTION

The subject of shape optimization for elastic structures has a very long and rich history and therefore has accumulated enormous literature, to vast to discuss it here. We shall cite here an excellent survey [9] and the papers [3, 4], where the problem is treated from the more engineering point of view, as a sample of, by no means exhaustive, references.

In this work we concentrate on the specific method proposed by the author [11], which may be called the harmonic transformation method. In addition, some algorithmic improvements have been developed, which improve the accuracy in cases, when the stresses have singularities, see [14, 13]. We have created a set of computer programs which may be used to solve quite general plane elastic shape optimization problems. They allow interactive design of triangulations for two dimensional domains, with consecutive mesh refinement and improvement. After defining in this way the plane domain together with its discretization, it may be used as an initial design for optimization. To this goal one may interactively set material properties (Lame coefficients), loading conditions, as well as the fixed part of the boundary. This is enough for solving the state equation, i.e. finding displacement, stresses and area of the domain. If the yield criterion is also set (of the Huber type), then one gets the initial values of three functionals, representing the area of the domain, maximal displacement and integral of the yield criterion. Their linear combination with user definable coefficients will serve as a goal functional, while other, also selectable combinations, define constrains.

There remains to designate the variable part of the boundary. It is done interactively. The user points to the nodes, which should be allowed to move, and shows on screen the proposed direction of movement. Then he shows the upper and lower limit of movement along the given direction.

Having thus defined the problem, he may now start the optimization process, and observe, how the shape gradually changes. While the program works, it is possible at any time to refine the mesh, to smooth the design, to see the history and compare the current result with the initial point. Also the mesh may be at any time corrected.

The originality of the work lies in a consistent concentration on the robustness and correctness of the approach, starting from the harmonic transformation method, which takes into account singularities, through the algorithmic improvements to the FEM calculations, and ending in a modified Pshenichny optimization algorithm.

In the paper we show some examples of solutions obtained in this way. As a platform for writing the programs we have chosen the MATLAB system of MathWorks Inc.. This allowed us to concentrate on essentials, and facilitated the design of the graphical user interface. As an additional bonus we have got transportability of the system, since it works on any hardware with Matlab installed (tried on Windows 95, SUN, Hewlett–Packard Apollo).

2. Problem formulation

Let there be given a connected two dimensional domain Ω with piecewise smooth boundary, satisfying for example the uniform cone condition [1] in order to exclude the degenerate cases. On such a domain we define the general elasticity problem

$$\begin{aligned} A^T \cdot D \cdot Au &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \Gamma_1 \\ B^T \cdot D \cdot Au &= h \quad \text{on } \Gamma_2. \end{aligned} \tag{1}$$

Here $u = (u_1, u_2)^T$ – denotes displacement, g – given displacement on a fixed part of the boundary, h – given traction on the loaded part of the boundary, and f – volume forces. In addition we introduce the following differential operator

$$A = \begin{bmatrix} \frac{\partial}{\partial x_1} & , & 0 \\ 0 & , & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} & , & \frac{\partial}{\partial x_1} \end{bmatrix},$$

and the matrix of material (Lame) constants

$$D = \begin{bmatrix} \lambda + 2\mu & , & \lambda & , & 0 \\ \lambda & , & \lambda + 2\mu & , & 0 \\ 0 & , & 0 & , & \mu \end{bmatrix},$$

as well as the matrix used in projecting stress on the boundary

$$B^T = \begin{bmatrix} n_1 & , & 0 & , & n_2 \\ 0 & , & n_2 & , & n_1 \end{bmatrix},$$

where $n = [n_1, n_2]^T$ – outward normal versor. In this notation stress $\sigma = [\sigma_{11}, \sigma_{22}, \sigma_{12}]^T$, strain $\varepsilon = [\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}]^T$ and the surface traction are given by

$$\varepsilon = A \cdot u, \quad \sigma = D \cdot \varepsilon, \quad t = B \cdot \sigma. \tag{2}$$

We shall also introduce the norms used for approximating the displacement in the given direction and yield criterion,

$$\|u\|_U^2 = u^T \cdot U \cdot u, \quad \|\sigma\|_S^2 = \sigma^T \cdot S \cdot \sigma. \tag{3}$$

For example, if we are interested in the displacement in the direction x_1 only, then

$$U = \begin{bmatrix} 1 & , & 0 \\ 0 & , & 0 \end{bmatrix},$$

while S corresponding to the Huber yield criterion has the form

$$S = \begin{bmatrix} 1 & , & -1/2 & , & 0 \\ -1/2 & , & 1 & , & 0 \\ 0 & , & 0 & , & 3 \end{bmatrix}.$$

Now let us allow the domain Ω to vary, that is we shall consider a family of sets $\Omega_t \in \Pi_{ad}$, where Π_{ad} is for the moment unspecified. In this way we get also a family of boundary value problems

$$\begin{aligned} A^T \cdot D \cdot Au_t &= f \quad \text{in } \Omega_t \\ u_t &= g \quad \text{on } \Gamma_1^t \\ B^T \cdot D \cdot Au_t &= h \quad \text{on } \Gamma_2^t. \end{aligned} \tag{4}$$

For their solution we define functionals

$$J_v(\Omega_t) = \int_{\Omega_t} d\Omega, \tag{5}$$

$$J_u(\Omega_t) = \left(\int_{\Omega_t} ||u_t||_U^{2p} \, d\Omega \right)^{1/2p}, \tag{6}$$

$$J_\sigma(\Omega_t) = \int_{\Omega_t} ||\sigma_t||_S^2 \, d\Omega. \tag{7}$$

They represent the area of the domain, the approximation of the maximal displacement and the integral of the yield criterion respectively. From these functionals we create a vector

$$\mathbf{J}(\Omega_t) = [J_v, J_u, J_\sigma]^T.$$

Now we are ready to define the optimization problem

$$\min_{\Omega_t \in \Pi_{ad}} \quad c^T \cdot \mathbf{J}(\Omega_t), \tag{8}$$

subject to the constraints

$$R \cdot \mathbf{J}(\Omega_t) \leq r, \tag{9}$$

where R – 3×3 real matrix, $r \in \mathbb{R}^3$ – constant vector and $c \in \mathbb{R}^3$ – vector of prices (goal coefficients). The family Π_{ad} we will specify after discretization.

3. SENSITIVITY

The first step in calculating the sensitivities is to parameterize the family Π_{ad}. We shall use here the variant of the speed method [9, 11]. Let Ω_t, $t \in (-\epsilon, \epsilon)$ has the form

$$\Omega_t = \Phi_t(\Omega),$$

where Ω is a fixed reference domain, and the mapping Φ is expressed as

$$\begin{aligned} \Phi_t(x) &= x + t \cdot s(x), \\ s(x) &= [s_1(x), s_2(x)]^T. \end{aligned} \tag{10}$$

Here s is a C^2 – regular vector field given on a certain bigger domain containing all Ω_t. Next we define material derivative of the function $u_t = u(\Omega_t; x)$ as

$$\dot{u}(\Omega; x) = \frac{d}{dt} u(\Omega_t; \Phi_t(x))|_{t=0}. \tag{11}$$

The derivatives of the domain functionals are defined as

$$J(\Omega_t) = \int_{\Omega_t} F(u(\Omega_t; \Phi_t(x))d\Omega \implies \dot{J}(\Omega) = \frac{d}{dt} J(\Omega_t)|_{t=0}. \tag{12}$$

In further derivations we shall need also the formulae for the transport of differential forms, see e.g. [9].

$$\begin{aligned}
\dot{\overline{D\Phi_t}} &= Ds, \\
\dot{\overline{det(D\Phi_t)}} &= \nabla \cdot s, \\
\dot{\overline{n_t \circ \Phi_t}} &= (n^t \cdot Ds \cdot n)n - Ds \cdot n, \\
\dot{\overline{\nabla u_t \circ \Phi_t}} &= -Ds \cdot \nabla u + \nabla \dot{u}, \\
\dot{\overline{f \circ \Phi_t}} &= \nabla f \cdot s, \\
\dot{\overline{S(\Phi_t)}} &= \nabla \cdot s - n^T \cdot Ds \cdot n.
\end{aligned}$$

Here $S(\Phi_t)$ denotes the differential form for the arc length, and Ds is a derivative matrix for s.

The elasticity problem (4) may be formulated in a more convenient weak form: find $u_t \in U_t(g)$ such that

$$-\int_{\Omega_t} (Au_t)^T \cdot D \cdot A\phi \, d\Omega + \int_{\Gamma_2^t} h^T \cdot \phi \, dS = \int_{\Omega_t} f^T \cdot \phi \, d\Omega,$$

for any $\phi \in U_t(0)$, where

$$U_t(g) = \{\phi \in [W_2^1(\Omega_t)]^2 \mid \phi = g \text{ on } \Gamma_1^t\}.$$

After transformation of variables this weak formulation may be rewritten in a reference domain Ω as

$$- \int_{\Omega} (\tilde{\nabla}\tilde{\phi})^T \cdot Q_t^T \cdot N^T \cdot D \cdot N \cdot Q_t \cdot (\tilde{\nabla}\tilde{\phi}) \cdot det(D\Phi_t) \, d\Omega$$

$$+ \int_{\Gamma_2} (h \circ \Phi_t) \cdot \tilde{\phi} \cdot S(\Phi_t) \, dS = \int_{\Omega} (f \circ \Phi_t) \cdot \tilde{\phi} \cdot det(D\Phi_t) \, d\Omega, \tag{13}$$

where $\phi \in U_0(0)$.

We have used here some additional notation

$$\tilde{u} = u_t \circ \Phi_t, \quad \tilde{\phi} = \phi \circ \Phi_t,$$

$$\tilde{\nabla} u = \text{diag}\{\nabla u_1, \nabla u_2\},$$

$$Q_t = \begin{bmatrix} D\Phi_t^{-T} & , & 0 \\ 0 & , & D\Phi_t^{-T} \end{bmatrix},$$

$$N = \begin{bmatrix} 1 & , & 0 & , & 0 & , & 0 \\ 0 & , & 0 & , & 0 & , & 1 \\ 0 & , & 1 & , & 1 & , & 0 \end{bmatrix},$$

so that $Au = N \cdot \tilde{\nabla} u$.

Now we shall take the material derivative of the weak formulation (13). Denoting

$$\dot{Q} = \begin{bmatrix} -Ds^T & , & 0 \\ 0 & , & -Ds^T \end{bmatrix}$$

and

$$\begin{aligned} D(s) &= (\nabla \cdot s) \cdot N^T D N + \dot{Q} \cdot N^T D N + N^T D N \cdot \dot{Q}, \\ S(s) &= \nabla \cdot s - n^T \cdot Ds \cdot n, \end{aligned}$$

and keeping in mind transport formulae, we get [11] the weak equation for $\dot{u}$: find $\dot{u} \in U_0(Dg \cdot s)$ such that

$$- \int \Omega[(A\phi)^T \cdot D \cdot A\dot{u} + (\tilde{\nabla}\phi)^T \cdot D(s) \cdot \tilde{\nabla} u] \, d\Omega$$

$$+ \int_{\Gamma_2} [(Dh \cdot s) + S(s) h]^T \cdot \phi \, dS = \int_{\Omega} [Df \cdot s + (\nabla \cdot s) f]^T \cdot \phi \, d\Omega, \tag{14}$$

for all $\phi \in U_0(0)$.

Obtained in the same way expression for $J(\Omega_t)$ reads

$$\dot{J} = \int_{\Omega} (D_u F \cdot \dot{u} + F \cdot (\nabla \cdot s)) \, d\Omega. \tag{15}$$

Now we introduce the adjoint equation: find $w \in U_0(0)$ such that

$$- \int_{\Omega} (A\phi)^T \cdot D \cdot Aw \, d\Omega = \int_{\Omega} D_u F \cdot \phi \, d\Omega, \tag{16}$$

for all $\phi \in U_0(0)$. Then by standard argument it is possible to get

$$\begin{aligned} \dot{J} = & \int_{\Omega} [F(u) \cdot (\nabla \cdot s) + (\tilde{\nabla} w)^T \cdot D(s) \cdot \tilde{\nabla} u] d\Omega \\ & + \int_{\Omega} \left[(f^T \cdot w)(\nabla \cdot s) + (Df \cdot s)^T \cdot w \right] d\Omega \\ & - \int_{\Gamma_2} [(Dh \cdot s)^T \cdot w + S(s) \cdot (h^T \cdot w)] dS \\ & + \int_{\Gamma_1} (Dg \cdot s)^T \cdot B^T \cdot D \cdot Aw \, dS. \end{aligned} \tag{17}$$

Now we shall make the additional assumptions, *purely for the simplicity sake.*

1. The loaded part of the boundary does not change.
2. The Dirichlet data on Γ_1 are null, i.e. at this part of the boundary the structure is fixed (but Γ_1^t *may* change).
3. The volume forces are absent. This assumption is made purely for the convenience.

Then the formula (17) takes on a simple form:

$$\dot{J} = \int_{\Omega} \left[F(u) \cdot (\nabla \cdot s) + (\tilde{\nabla} w)^T \cdot D(s) \cdot \tilde{\nabla} u \right] d\Omega, \tag{18}$$

which will be used in later parts of the paper. The above assumptions are not as restrictive as they look like, and are satisfied in most practical problems.

4. Harmonic transformation method

The expression (18) is linear with respect to the C^2 – regular field s. However, it is difficult to construct such a field in a way, which gives accurate control over the movements of the boundary $\partial\Omega$. But precisely these movements, not the global field s, are of primary interest.

In order to overcome these difficulties, we construct the field in a special way, using harmonic functions. Let us recall, that the boundary of Ω consists of finite number of smooth arcs, separated by vertices: $\gamma_1, V_1, \gamma_2, V_2, \ldots$, where γ_i – arcs, V_i – vertices. Let a vector field $d = [d_1, d_2]^T$ be defined on the variable part of the boundary, constituting the desired movement. Furthermore, we assume:

- d is $C^{1,1}$ on each γ_i,
- $d_{\gamma_i}(V_i) = d_{\gamma_{i+1}}(V_i)$, compatibility conditions.

Then we construct a vector field $v = [v_1, v_2]^T$ using the equations

$$\Delta v_i = 0 \quad \text{in } \Omega, \qquad v_i = d_i \quad \text{on } \Gamma, \quad i = 1, 2. \tag{19}$$

The functions v_1, v_2 may be extended for our class of admissible domains on larger sets containing Ω. It has been proved in [11], that such a field v may be substituted into (18), thus giving way to defining the material derivative of the functional

$$\dot{J} = \dot{J}(v).$$

The questions of regularity concerning both u and v will be discussed in the next section.

As a result of (19), the vector field v depends linearly on its boundary conditions d. Therefore also $\dot{J}$ is a linear function of d, but this dependence is implicit. We may make it explicit by the following construction. Assume, that the boundary Γ consists of N_0 arcs Γ_j, $j = 1, \ldots, N_0$. Let also every arc have a parametrization

$$\eta_j : [j-1, j] \longrightarrow \Gamma_j, \quad j = 1, \ldots, N_0.$$

By patching together all functions η_j one gets a parametrization

$$\eta : [0, N_0] \longrightarrow \Gamma.$$

Now let us use basis functions (for example splines) $\{\psi_k\}$, defined on $[0, N_0]$ for approximating η. This leads to formulae

$$d_1 \simeq \sum_k d_{1k}\psi_k,$$

$$d_2 \simeq \sum_k d_{2k}\psi_k.$$

The coefficients $\{d_{1k}, d_{2k}\}$ should satisfy some additional conditions, so that the changed domain remains admissible. For such basis we formulate boundary value problems

$$\Delta v^k = 0 \quad \text{in } \Omega, \qquad v^k = \psi_k \quad \text{on } \Gamma.$$

As a result, after introducing notation

$$S_{1k} = \dot{J}([v^k, 0]^T), \qquad S_{2k} = \dot{J}([0, v^k]^T),$$

one gets

$$\dot{J} = \sum_{i=1}^{2} \sum_k S_{ik} d_{ik}. \tag{20}$$

In this way we have achieved the goal of expressing the derivative of J as a linear function of boundary variations. Furthermore, let us notice, that the field v is defined over the whole Ω. This means, that as an additional bonus we have obtained the method for moving the triangulation during the optimization process. And these movements are the smoothest possible over the whole domain, what is important from the numerical point of view.

5. Singularities – the special approach

Let us now return to the questions of regularity concerning u and v. As it is well known [5, 10], and has been discussed in detail in [11] for this specific case, both functions may have gradient singularities. They are of the form:

$$\begin{aligned} u &\longleftrightarrow r^{1/4+\delta}, \\ v &\longleftrightarrow r^{1/2+\delta}. \end{aligned} \tag{21}$$

The value of $\delta > 0$ depends on the size of reentrant corners (for both v and u) and types of boundary conditions on the sides of these corners (for u). However, as it has been pointed out in [11], the possibility of substituting $s := v$ in (18) depends crucially on the positivity of δ. Hence also in numerical approximations the rate of convergence is very sensitive with respect to δ. It would be very beneficial to remove this dependence, and make the convergence the same, as in the case when u, v belong to $W_2^2(\Omega)$ Sobolev space.

To this goal we propose the infinite element method, as developed in [6, 13, 12]. All reasoning will now concern the discretized problem. Let the domain Ω be triangulated, and consider the polygon P consisting of all triangles having one point (say (0,0)) in common. Such a domain is star shaped, with the centre inside or on the perimeter, see Fig.1.

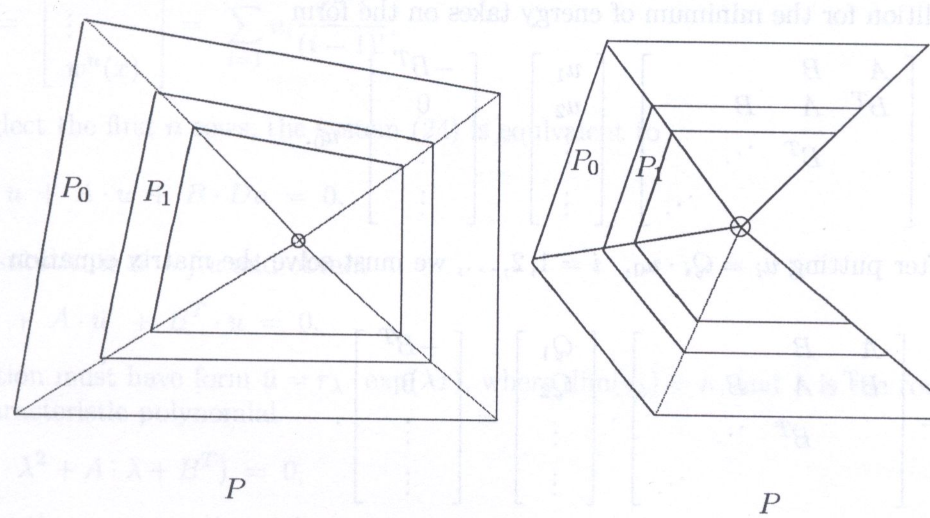

Fig. 1. The star-shaped domains with infinite discretizations

Now we construct from the outer boundary Γ_0 the consecutive cuts of these domains by similarity transformations, $\Gamma_i = r^i \cdot \Gamma_0$, where $0 < r < 1$. Between Γ_i and Γ_{i+1} lay the ring–like parts of P, denoted by P_i. These rings are also similar, and sum up to the whole P.

Our goal is to solve the elasticity and Laplace equations in such polygons using the finite element method. Let the parts between cross–sections k and $k+1$ be triangulated and linear shape functions used for approximations. We shall denote by u_k the vector of all nodal values of the solution corresponding to the k–th cross–section (be it Laplace or elasticity case). Now the energy of the whole body after discretization can be written as

$$E = \sum_{k=0}^{\infty} E_k(u_k, u_{k+1}), \tag{22}$$

where E_k denotes the energy of the k–th ring. Let us concentrate on E_0. By eliminating internal nodes between sections Γ_0 and Γ_1 we get

$$E_0 = \frac{1}{2}[u_0^T, u_1^T] \cdot M \cdot \begin{bmatrix} u_0 \\ u_1 \end{bmatrix},$$

where the $2n \times 2n$ symmetric stiffness matrix M $(n = \dim(u_k))$ has the form

$$M = \begin{bmatrix} A_1 & , & B \\ B^T & , & A_2 \end{bmatrix},$$

with symmetric, positive definite A_1, A_2.

The crucial observation and the basis of the whole approach consists in the fact, that for linear elements the matrix M is the same for all rings, i.e. E_k has the exactly the same form for all k. Hence the energy of the whole P may be written as

$$E = \frac{1}{2} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \end{bmatrix}^T \begin{bmatrix} A_1 & B & & \\ B^T & A & B & \\ & B^T & \ddots & \\ & & & \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \end{bmatrix}, \tag{23}$$

where $A = A_1 + A_2$.

Now we shall try to solve elasticity equation in P imposing boundary conditions on u_0. The necessary condition for the minimum of energy takes on the form

$$M_\infty \cdot u_\infty = \begin{bmatrix} A & B & & \\ B^T & A & B & \\ & B^T & \ddots & \\ & & & \ddots \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} -B^T \\ 0 \\ \vdots \\ \vdots \end{bmatrix} \cdot u_0.$$

As a result, after putting $u_i = Q_i \cdot u_0, \quad i = 1, 2, \ldots$, we must solve the matrix equation of infinite order

$$M_\infty \cdot Q_\infty = \begin{bmatrix} A & B & & \\ B^T & A & B & \\ & B^T & \ddots & \\ & & & \ddots \end{bmatrix} \cdot \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} -B^T \\ 0 \\ \vdots \\ \vdots \end{bmatrix}. \tag{24}$$

It is well known [2] that such systems may have infinitely many solutions. Therefore we impose the physical condition, that the consecutive energy terms diminish, or that the necessary condition gives minimum, not a saddle point or maximum for elastic energy. As it will turn out, this makes the solution unique.

To this goal we embed the problem into the framework of operations on infinite series [8]. Let us establish the correspondence between infinite vector $f_\infty = [f_1, f_2, \ldots]^T$ and the formal power series

$$f(x) = \sum_{i=1}^{\infty} f_i \frac{x^{i-1}}{(i-1)!}.$$

Differentiating this series gives

$$Df(x) = \sum_{i=2}^{\infty} f_i \frac{x^{i-2}}{(i-2)!},$$

or in vector representation,

$$f_\infty = [f_1, f_2, \dots]^T, \qquad Df_\infty = [f_2, f_3, \dots]^T.$$

This allows the differentiation to be represented as multiplication by the matrix

$$Df_\infty = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \end{bmatrix} \cdot f_\infty,$$

and similarly for integration

$$\int f_\infty = \begin{bmatrix} 0 & 0 & & \\ 1 & 0 & & \\ & \ddots & \ddots & \end{bmatrix} f_\infty.$$

Now M_∞ has a block structure, and therefore in order to use this representation we must introduce the whole vector of functions

$$w^k(x) = \sum_{j=1}^{\infty} w_j^k \frac{x^{j-1}}{(j-1)!},$$

and write $u_i = [w_i^1, \dots, w_i^n]$, so that

$$u(x) = \begin{bmatrix} w^1(x) \\ \vdots \\ w^n(x) \end{bmatrix} = \sum_{i=1}^{\infty} u_i \frac{x^{i-1}}{(i-1)!}.$$

If we neglect the first n rows, the system (24) is equivalent to

$$B^T \int u + A \cdot u + B \cdot Du = 0.$$

Now we substitute $\bar{u} = \int u$ and obtain

$$B \cdot \bar{u}'' + A \cdot \bar{u}' + B^T \cdot \bar{u} = 0.$$

The solution must have form $\bar{u} = r_\lambda \cdot \exp(\lambda x)$, where $\dim(r_\lambda) = n$, and λ is the root of the $2n$–th order characteristic polynomial

$$\det(B \cdot \lambda^2 + A \cdot \lambda + B^T) = 0, \tag{25}$$

and r_λ are the corresponding right eigenvectors

$$(B \cdot \lambda^2 + A \cdot \lambda + B^T) \cdot r_\lambda = 0.$$

In general, (25) has $2n$ roots. However, from the symmetry of the equation it follows, that they occur in pairs $(\lambda_i, 1/\lambda_i)$, $i = 1, \dots, n$. Let us eliminate at this point the roots greater than 1, and consider the rest. The root $\lambda =$ appears with multiplicity 2 for the Laplace equation (it corresponds to the constant solution) and the eigenvector $r_1 = [1, \dots, 1]^T$. For the elasticity equation the root $\lambda = 1$ has in $\mathbb{R}^2$ the multiplicity 4 and the eigenvectors e_1, e_2 corresponding to the rigid translation in the directions of the first and second axis.

Now we select n eigenvalues $\lambda_i \le 1$ and their eigenvectors. The solution constructed from them has the form

$$\bar{u} = c_{1,1} r_{\lambda_1} \exp(\lambda_1 x) + \ldots + c_{n,1} r_{\lambda_n} \exp(\lambda_n x). \tag{26}$$

The constants $c_{1,1}, \ldots, c_{n,1}$ are chosen in such a way, that the first n rows in (24) are satisfied. However, the equation has n right–hand sides. Hence double subscript: $c_{p,q}$ is responsible for the p–th constant corresponding to the q–th column on the right.

Let us now denote

$$R_\lambda = [r_{\lambda_1}, \ldots, r_{\lambda_n}], \qquad C = [c_{j,k}]_{j,k=1,\ldots,n},$$

$$\Lambda = \mathrm{diag}[\lambda_1, \ldots, \lambda_n].$$

If we return to (26) and replace exponentials by their power series, and then recall correspondence between vectors and series, we get the solution to the infinite matrix equation (24) in the form

$$Q_1 = R_\lambda \cdot \Lambda \cdot C, \quad Q_2 = R_\lambda \cdot \Lambda^2 \cdot C, \ldots \; Q_n = R_\lambda \cdot \Lambda^n \cdot C, \ldots$$

The choice of the matrix C results from the requirement

$$AQ_1 + BQ_2 = 0,$$

$$B^T Q_1 + AQ_2 + BQ_3 = 0,$$

which leads to

$$(AR_\lambda \Lambda + BR_\lambda \Lambda^2)(I - CR_\lambda)\Lambda C = 0,$$

and, as a consequence $C = R_\lambda^{-1}$. This gives fundamental relations

$$Q_1 = Q = R_\lambda \cdot \Lambda \cdot R_\lambda^{-1}, \ldots, Q_i = Q^i, \quad i = 1, 2, \ldots. \tag{27}$$

Let us now return to the expression (22) for the energy. Taking into account relation $u_k = Q_k \cdot u_0$, we get

$$E(u_k, u_{k+1}) = \frac{1}{2} u_0^T Q^{kT} \left[A_1 + Q^T B^T + BQ + Q^T A_2 Q \right] Q^k u_0.$$

The energy over the whole polygon P has the form

$$E = \frac{1}{2} u_0^T \cdot S \cdot u_0,$$

where

$$S = \sum_{k=0}^{\infty} Q^{kT} R Q^k, \qquad R = A_1 + Q^T B^T + BQ + Q^T A_2 Q.$$

In fact, the series for S may be very nicely summed, giving a short result

$$S = A_1 + BQ. \tag{28}$$

The convergence of the series requires comment. It follows from the fact, that $\lambda = 1$ does not contribute to energy (constant solution or rigid motion), and therefore only $\lambda < 1$ count.

In this way we get the stiffness matrix for the superelement P. When the centre coincides with the singularity, it is suitable for representing singular solutions. In [13] it has been shown, that indeed it raises the convergence rate to the highest possible level. And the application is very simple: replace some triangles in sensitive points by superelements. Observe, that the dimensionality of the system does not increase, but in fact decreases by 1 with every superelement. The "star centre" disappears from the equations, but the value of the solution in this point can be retrieved with ease.

6. Sequential optimization algorithm

At this point we know, how to compute the sensitivity of the functional, how to treat special points (reentrant corners, change of boundary conditions) in the domain, and may begin optimization. As our main tool we have chosen here the sequential quadratic programming algorithm, in the form first proposed by Pshenichny. Take the problem

$$\min f_0(x)$$
$$\text{s.t.} \quad g(x) \leq 0.$$

The functions are linearized around current point x_k and written in the form

$$\min \{c^T h + \frac{1}{2}||h||^2\} \tag{29}$$
$$\text{s.t.} \quad Ah \leq -g(x_k),$$

where $x = x_k + h, \quad A = Dg(x_k)$. Observe, that the Hessian is replaced by I. After the direction h is obtained, the step is subdivided by the powers of 2 until the success criterion is satisfied $(x_{k+1} = x_k + 2^{-l}h)$,

$$f_0(x_{k+1}) + NG(x_{k+1}) \leq f_0(x_k) + NG(x_k) - \eta||h||^2,$$

where $\eta \in (0,1)$ is an algorithm parameter,

$$G(x_k) = \max[\{g_i(x_k)\}, 0],$$

and the constant N must at any iteration be greater than the sum of dual variables for (29). The stop criterion is $||h||^2 \leq \epsilon$.

It turns out, that this algorithm is reasonably simple and quite robust. Some properties of our implementation are discussed in [11].

7. Implementation – general description

Before description of the final implementation we must describe the family of admissible domains Π_{ad}. Let us begin with the discretized domain Ω^h. In the first step we select the boundary points that are allowed to move, say $\{p_1, p_2, ..., p_k\}$. Then for every point we define the direction of movement, represented by the versor a_i, $i = 1, .., k$, $||a_i|| = 1$, and the range of movement $\underline{t}_i, \bar{t}_i$, so that the position of p_i is

$$p_i(t) = p_i + t_i \cdot a_i, \quad \underline{t}_i \leq t_i \leq \bar{t}_i. \tag{30}$$

Hence the domains $\Omega^h(t)$ are defined by the vector of variables $t = [t_1, \ldots, t_k]^T$ and Π_{ad} by the inequalities $\underline{t} \leq t \leq \bar{t}$. We see, that the movement of the boundary is represented by simple linear finite elements (1–dimensional) and no attempt is made to use spline representation. This is done intentionally, in order to test the robustness of the method unmasked by the additional smoothness of splines.

With respect to the Matlab realization, let us mention here only, that it consists of two parts. MESHER serves as an interactive triangulation construction tool. SHAPER, using defined already discretization, allows to define on screen all the data needed to define in optimization and to perform consecutive optimization steps, controlling interactively the process. It is also possible to refine, (double) the discretization at any time during the optimization, as well as view the history of the goal functional and functional constraints. Among many other options are: showing the yield function distribution and the solution of the pure elasticity problem, remeshing in order to improve triangulation.

8. EXAMPLES OF RESULTS

We shall describe below three examples, illustrating the main features of the method and its implementation.

Nonsimply connected domain. The initial problem consisted of a rectangular beam fixed at both left and right sides and containing a rectangular hole in the middle. The vertical, downward directed load was uniformly distributed on the upper edge. For the initial domain (see Fig.2) we have computed J_v^0, J_u^0, J_σ^0 and then formulated the design problem as ($A = I$, $p = 4$, S – Huber criterion):

$$\begin{aligned} &\min_{\Omega} \quad J_u(\Omega), \\ \text{s.t.}\quad &J_v(\Omega) \leq J_v^0, \\ &J_u(\Omega) \leq J_u^0, \\ &J_\sigma(\Omega) \leq J_\sigma^0. \end{aligned}$$

As a design parameter (variable part of the boundary) we have chosen lower edge and the shape of the hole. In fact, the goal was to find *the stiffest structure of a given volume.*

The results are shown in Fig. 2. The gain in terms of J_u was 20%. It turns out, that only the volume constraint was active. Such a formulation of the design goal leads, as we have found out also in other cases, to a good distribution of stresses.

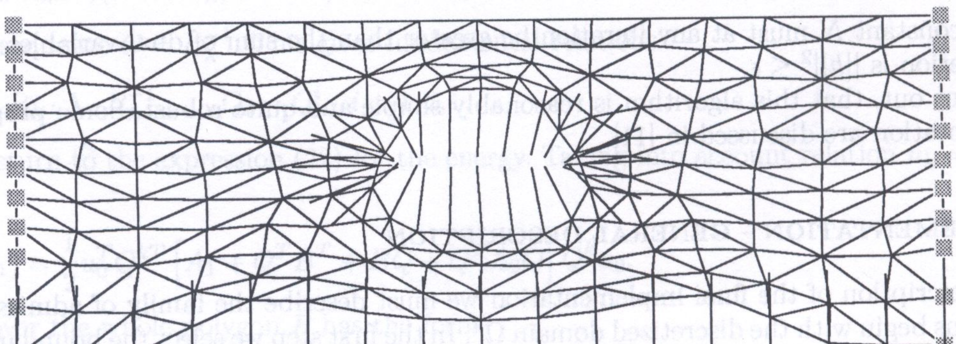

Fig. 2. Simultaneous design of the inner and outer boundary

Implant. The peculiarity of this example lies in the fact, that the fixed part of the boundary could vary. Therefore the shape of insertion in the rigid support changed. This time the problem was (the same A and S):

$$\begin{aligned} &\min_{\Omega} \quad J_\sigma(\Omega), \\ \text{s.t.}\quad &J_v(\Omega) \leq J_v^0, \\ &J_u(\Omega) \leq J_u^0 + \text{big}, \\ &J_\sigma(\Omega) \leq J_\sigma^0. \end{aligned}$$

The results are shown in Fig. 3, and here one can also see which part of the boundary was variable. The gain was about 17%. Worth noting is the way the support condition has changed, in order to eliminate singularity.

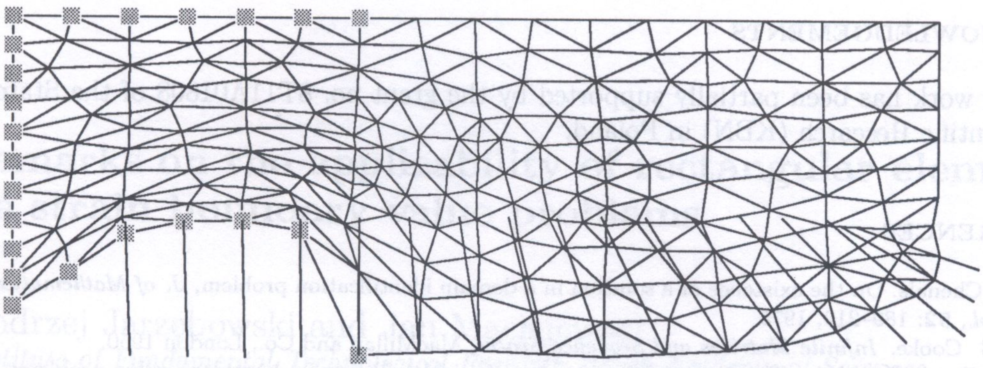

Fig. 3. Implant – design of the fixed part

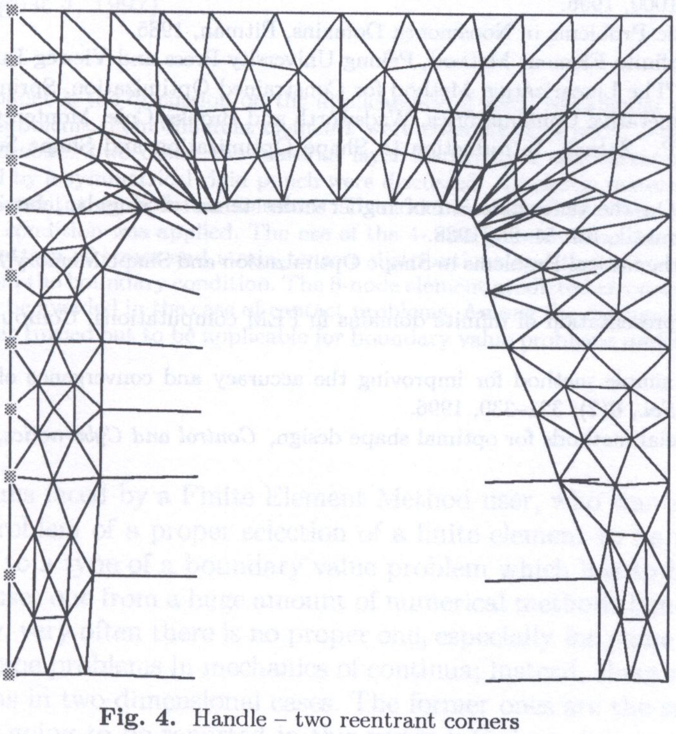

Fig. 4. Handle – two reentrant corners

Handle. This time the initial domain had C–like shape, see Fig. 4. From one side it was fixed, from the other it was pulled by the end. The problem (with again the same A and S) was

$$\begin{aligned} &\min_{\Omega} \quad J_\sigma(\Omega), \\ \text{s.t.}\quad &J_v(\Omega) \le J_v^0, \\ &J_u(\Omega) \le J_u^0, \\ &J_\sigma(\Omega) \le J_\sigma^0. \end{aligned}$$

The result was 50% better as an initial design.

Conclusions: All these examples contained singularities, so they illustrate also how the algorithm copes with such cases. It must be stressed, that no attempt of smoothing the obtained shapes has been made. Their appearance confirms the reliability of the proposed combination of algorithms. As a conclusion we may state, that the implementation of the shape optimization system as described above, which was meant as a technology demonstration, validated the claims concerning the robustness and generality formulated in the introduction.

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REFERENCES

[1] D. Chenais. On the existence of a solution in a domain identification problem, *J. of Mathematical Analysis and Appl.*, **52**: 189–219, 1975.
[2] R.G Cooke. *Infinite Matrices and Sequence Spaces*, MacMillan and Co., London 1950.
[3] K. Dems, Z. Mróz. Shape sensitivity in mixed Dirichlet–Neumann boundary value problems and associated class of path independent integrals, *Eur. J. Mech., A/Solids*, **14**(2): 169–203, 1995.
[4] K. Dems. Shape and materials parameters sensitivity analysis in quasi-harmonic problems, *Control and Cybernetics*, **25**(5); 969–1000, 1996.
[5] P. Grisvard. Elliptic Problems in Nonsmooth Domains, Pitman, 1985.
[6] Ying Luang-an. Infinite Element Method, Peking University Press and Vieweg Publishing, 1995.
[7] B.N. Pshenichnyj. The Linearization Method for Constrained Optimization, Springer Verlag 1994.
[8] R.P. Stanley. Enumerative Combinatorics, Wadsworth and Brooks/Cole, Monterey 1986.
[9] J. Sokołowski, J–P. Zolesio. Introduction to Shape Optimization and Shape Sensitivity Analysis, Springer Verlag, 1992.
[10] D. Vasilopoulos. On the determination of higher order terms of singular elastic stress fields near corners, *Numerische Mathematik*, **53**: 51–95, 1988.
[11] A. Żochowski. Mathematical Problems in Shape Optimization and Shape Memory Materials, Peter Lang Verlag, 1992.
[12] A. Żochowski. Representation of infinite domains in FEM computations, *Computers and Structures*, **63**: (5), 889–897, 1977.
[13] A. Żochowski. A simple method for improving the accuracy and convergence of FEM computations, *Appl. Math. and Comp. Sci.*, **6**(2): 321–330, 1996.
[14] A. Żochowski. Special methods for optimal shape design, *Control and Cybernetics*, **25**(6): 1155–1163, 1996.