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A viscoelastic-plastic model for skeletal structural systems with clearances

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Dedicated to Professor Romuald Świtka on the occasion of his 65th birthday

The paper is concerned with the mathematical modelling and numerical solution of unilateral problems for viscoelastic-plastic structural systems. A new material model is proposed in which the viscoelastic and plastic strains are governed by different constitutive laws. The model is restricted to isothermal quasistatic deformation processes under conditions of geometric linearity. The mechanical problem is posed in the format of piecewise linear plasticity and the unilateral contact conditions are described by means of the clearance function. The linear viscoelastic laws are integrated by a creep approach method, which allows for jump-discontinuities in the history of stress. For the evolution of plastic strains an implicit method is used. The problem is formulated and solved as a sequence of nested (mixed) linear complementarity problems. The question of existence and uniqueness of a solution to the problem is discussed. A numerical algorithm based on the pivotal transformations is devised and its stability is shown numerically. Results of numerical experiments for several illustrative examples of a beam/foundation system subjected to non-proportional loading histories are presented. The results clearly demonstrate the impact of the history of loading and the unilateral constraints upon the current state of the structural system.

Key words: Viscoelasto-plasticity; Creep approach; Unilateral constraints; Finite element method; Linear complementarity problem.

1. Introduction

The work is a contribution to the mathematical modelling and numerical solution of inequality problems in mechanics of structural systems whose evolution in time is governed by the laws which may be written in a general form as

$$(\dot{q}(t), -\dot{\alpha}(t)) \in N_{\mathcal{K}}(q(t), \alpha(t)) \tag{1}$$

where $N_{\mathcal{K}}$ is the normal cone to a (convex) set $\mathcal{K}$ and q may be understood in our case as the plastic strain whilst α represents a set of other internal variables. As is well known the inclusion (1) is equivalent to a variational inequality defined on the tangent cone to the set $\mathcal{K}$, cf. [17, 32, 22] where further references can be found. In this work we are concerned with the skeletal structural systems which may exhibit both the instantaneous elastic-plastic response as well as the time-dependent viscoelastic-plastic one. An additional feature of the structural system is the presence of clearances between its constituents. Hence, the source of inequality relations describing our problem is the yield function and a clearance function due to Gawęcki [9] who extended the concept of locking materials introduced by Prager [35]. We use the matrix description of piecewise linear plasticity due to Maier [5, 28] and Borkowski [3]. Actually, in this finite dimensional framework of structural mechanics we formulate and solve our variational inequality corresponding to (1) as a complementarity problem. More precisely, the mechanical problems under consideration lead to some form of a mixed linear complementarity problem (LCP) which we call a *nested linear complementarity problem* (nLCP).

The aim of this paper is twofold: (i) to present a constitutive model accounting for both viscous (creep, stress relaxation) and plastic effects and, (ii) to develop a mathematical formulation of the model and the corresponding solution method suitable for application to engineering problems. In this paper our considerations are restricted to initially isotropic, non-aging viscoelastic plastic media undergoing small deformations under isothermal quasistatic conditions. In the next section we recall some definitions and theorems related to the complementarity problem and the variational inequality problem. The numerical algorithm we used for solving the nLCP is presented in Sec. 5. The algorithm is a direct method based on the idea of pivoting. In Sec. 3 we provide a new mathematical model for viscoelastic-plastic contact problems. The novelty of the proposed formulation consists in the idea of accounting for the viscoelastic and plastic material properties, and the numerical treatment of the obtained relations which govern the problem. The characteristic feature of the suggested macroscopic model is that the **material response inside the yield surface is viscoelastic**. The basic assumption is that the plastic strain do not influence the viscoelastic properties of the material, the customary assumption in elastoplasticity. Thus, the model is based on more than one micromechanism responsible for the development of stress in a solid body. First, if some measure of deformation is smaller than a threshold value (clearance function), then there is no stress in the material point. The reversible changes in the microstructure of the material are described by viscoelastic laws, whereas the third mechanism is activated when the stresses reach some threshold (yield function). Upon releasing the stress when the third mechanism is induced there is permanent (plastic) deformation. In brief, whilst the second mechanism accounts for viscous and rate effects of the material response, the role of the third mechanism is in limiting the stresses which the material can sustain and in accounting for instantaneous permanent deformation. Notice that, depending on the kind of the viscoelastic model used, the second mechanism may also contribute to the final permanent deformation. Owing to this particular succession of the mechanisms the model considered here does not directly fall into the very well-known viscoplasticity theory due to Perzyna [33], Duvaut-Lions [8], and into the models of viscoplasticity developed and discussed by Lemaitre and Chaboche [25], and Maugin [30]. It should be noted that the proposed model describes deformation processes at different time scales in which the viscoelastic response depends upon a characteristic time of the material, whereas plasticity has no time scale. After we had finished the present paper, quite recently it came to our notice[1] that models of elastic/creep/plastic behaviour, which are similar in some aspects to our model, were considered by Ponter. He derived work bounds for cyclically loaded structures and analyzed the influence of the time scales corresponding to the (dynamic) loading and material characteristics, see e.g. [34]. As concerns the important issue of parameter identification for the viscoelastic-plastic model, here we wish only to mention that it is possible to adopt the optimization strategy due to Mahnken and Stein [27]. In Sec. 4 we describe the time integration method we employed in the context of the nLCP formulation for the incremental problem. Finite element approximations and time integration schemes for viscoplastic, elastoplastic and viscoelastic problems have been studied in many works, see Zienkiewicz and Cormeau [44], Kleiber [18], Stein *et al.* [38] and Shaw *et al.* [36], for example. For the evolution of viscoelastic strains we adapted the method due to Świtka and Husiar [39] who integrated linear viscoelasticity laws using a polynomial approximation of stresses. In the framework of nonlinear viscoelasticity analysis by Argyris *et al.* [2], such a method is called the *creep approach*. Section 6 contains the results of our numerical experiments for some simple but illustrative examples of the proposed general approach. We have analyzed the behaviour of a beam/foundation system subjected to nonproportional loading histories under conditions of unilateral contact. The numerical results confirm effectiveness of the proposed formulation and the time integration method used. On the other hand, the results have revealed some very interesting features of this structural systems. In particular, the great influence of the entire history of loading on the current state of the system is demonstrated.

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2. VARIATIONAL INEQUALITY AND COMPLEMENTARITY PROBLEMS

We define first the variational inequality problem and the complementarity problem in an infinite dimensional setting, e.g. [13, 41, 16], and then pass to the finite dimensional case, e.g. [31, 6]. To this aim we introduce some concepts using traditional notations employed in the mathematical literature. To avoid a possible confusion the meaning of the quantities used in describing the mechanical problems is redefined at their first appearance in the following sections.

Let U be a real Hilbert space, $U^\star$ its topological dual and let $\mathcal{K} \subset U$ indicate a non-empty closed convex subset of U. We say that $\mathcal{K}$ is convex iff for any two elements $u, v \in \mathcal{K}$ and $\lambda \in [0,1]$, the element $w = \lambda v + (1-\lambda)u$ is in $\mathcal{K}$, i.e. $w \in \mathcal{K}$. Let $P : U \to U^\star$ be a continuous operator, wherein the symbol $\langle \cdot|\cdot \rangle$ denotes duality pairing in $U^\star \times U$. The variational inequality problem may be formulated as follows.

Find u such that

$$u \in \mathcal{K}, \qquad \langle P(u), v-u \rangle \geq 0 \qquad \forall v \in \mathcal{K}. \tag{2}$$

Moreover, let $\mathcal{C} \subset U$ be a non-empty closed convex cone in U, and let $\mathcal{C}^\star \subset U^\star$ stand for its polar cone. We say that $\mathcal{C}$ is a cone (more precisely, a convex cone with its vertex at the origin) iff for any $u, v \in \mathcal{C}$ the element $u + v \in \mathcal{C}$ and for any $u \in \mathcal{C}$ the element $\lambda u \in \mathcal{C}, \forall \lambda \geq 0$. The (generalized) complementarity problem may be formulated as follows.

Find u such that

$$u \in \mathcal{C}, \qquad P(u) \in \mathcal{C}^\star, \qquad \langle P(u), u \rangle = 0. \tag{3}$$

The following useful lemma can be proved [15, 24].

Lemma 1. (equivalence)
The complementarity problem (3) and the variational inequality (2), both defined on the same cone $\mathcal{C}$, are equivalent.

The complementarity problems first appeared in optimization theory and mathematical programming as the necessary conditions of optimality (Karush-Kuhn-Tucker conditions), and usually were formulated in the finite dimensional context, i.e. $U \equiv \mathcal{R}^n$, where $\mathcal{R}^n$ stands for the Euclidean space of real ordered n-tuples. In particular, any quadratic programming problem may be solved as a linear complementarity problem. Restricting our further consideration to the linear case we let P be now an affine mapping of the form $P(\mathbf{x}) \equiv \mathbf{Ax} + \mathbf{b}$, where $\mathbf{A} \in \mathcal{R}^{n \times n}$ is a square matrix and $\mathbf{x} \in \mathcal{R}^n$, $\boldsymbol{b} \in \mathcal{R}^n$ are n-dimensional vectors. Thus, the linear complementarity problem (LCP) is to find the vector $\boldsymbol{x} \in \mathcal{R}^n$ such that

$$\begin{aligned} \mathbf{x} &\geq \mathbf{0} \\ \mathbf{Ax} + \mathbf{b} &\geq \mathbf{0} \\ \mathbf{x}^{\mathrm{T}}(\mathbf{Ax} + \mathbf{b}) &= 0 \end{aligned} \tag{4}$$

By introducing a slack variable $\mathbf{y} \equiv \mathbf{Ax} + \mathbf{b}$ we can rewrite the system (4) in the following standard form which is convenient when solving the problem by a direct method:

$$\begin{aligned} &\mathbf{Dx} + \mathbf{y} = \mathbf{b}, \\ &\mathbf{x} \geq \mathbf{0}, \qquad \mathbf{y} \geq \mathbf{0}, \qquad \mathbf{x}^{\mathrm{T}}\mathbf{y} = 0, \end{aligned} \tag{5}$$

wherein $\mathbf{D} = -\mathbf{A}$. When matrix $\mathbf{A}$ is symmetric, the problems (4) and (5) are called *symmetric*, otherwise they are *nonsymmetric*.

In the problems we consider in Sec. 4 the unknown vector consists of three subvectors, $\mathbf{x} = \text{col}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, and the matrix $\mathbf{D}$ has the structure,

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13} \\ \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23} \\ \mathbf{D}_{31} & \mathbf{D}_{32} & \mathbf{D}_{33} \end{bmatrix} \tag{6}$$

where the submatrices $\mathbf{D}_{ij} \in \mathcal{R}^{n_i \times n_j}$, with $i, j = 1, 2, 3$ and $n_i, n_j \in \mathcal{N}$ being a positive integers, cf. (31), (33). The critical property of those problems is that the first unknown component vector $\mathbf{x}_1$ is unrestricted in sign. Accounting for this we arrive at the form of LCP which we call a *nested linear complementarity problem* (nLCP):

$$\begin{aligned} &\mathbf{Dx} + \mathbf{y} = \mathbf{b}, \\ &\mathbf{x}_2 \geq \mathbf{0}, \quad \mathbf{x}_3 \geq \mathbf{0}, \quad \mathbf{y}_1 = \mathbf{0}, \quad \mathbf{y}_2 \geq \mathbf{0}, \quad \mathbf{y}_3 \geq \mathbf{0}, \quad \mathbf{x}^{\mathrm{T}}\mathbf{y} = 0. \end{aligned} \tag{7}$$

Another characteristic feature of the systems (31), (33) is that $\mathbf{D}$ is *bisymmetric*, i.e. its components satisfy the conditions: $\mathbf{D}_{ii}$ $(i = 1, 2, 3)$ are symmetric, and $\mathbf{D}_{ij} = -\mathbf{D}_{ji}^{\mathrm{T}}$, with $i, j = 1, 2, 3$, $i \neq j$. A similar modification is pertinent to the nLCP expressed in matrix $\mathbf{A}$, so we have

$$\begin{aligned} &\mathbf{Ax} + \mathbf{b} \geq \mathbf{0}, \\ &\mathbf{x}_2 \geq \mathbf{0}, \quad \mathbf{x}_3 \geq \mathbf{0}, \qquad \mathbf{x}^{\mathrm{T}}(\mathbf{Ax} + \mathbf{b}) = 0. \end{aligned} \tag{8}$$

The question of existence and uniqueness of a solution to the linear complementarity problem is dealt with in the following theorem proved in Cottle *et al.* [6]. We formulate the theorem in terms of the properties of matrix $\mathbf{A}$. Before stating the theorem we recall that a matrix $\mathbf{A} \in \mathcal{R}^{n \times n}$ is said to be *positive semi-definite* if $\mathbf{x}^{\mathrm{T}}\mathbf{Ax} \geq 0, \forall \mathbf{x} \in \mathcal{R}^n$. It is *positive definite* if $\mathbf{x}^{\mathrm{T}}\mathbf{Ax} > 0$, $\forall \mathbf{x} \in \mathcal{R}^n, \mathbf{x} \neq \mathbf{0}$.

Lemma 2. (existence and uniqueness)

- *Let $\mathbf{A} \in \mathcal{R}^{n \times n}$ be positive definite, then the linear complementarity problem (4) (and (7)) has a unique solution for any $\mathbf{b} \in \mathcal{R}^n$.*
- *Let $\mathbf{A} \in \mathcal{R}^{n \times n}$ be positive semi-definite and $\mathbf{b} \in \mathcal{R}^n$. Let $\mathbf{x}^\star$ be a solution of the linear complementarity problem (4) or (7). If $\mathbf{x}^\star$ is not degenerate (i.e. there is no index $0 \leq i \leq n$ that both x_i and y_i are equal to zero) and the corresponding matrix $\mathbf{A}^\star \subset \mathbf{A}$ ($\mathbf{A}^\star$ is composed of these rows $A_{k.}$ and columns $A_{.k}$ for which $x_k > 0$) is nonsingular, then $\mathbf{x}^\star$ is the unique solution of the problem.*

3. Constitutive relations

In this section we present the fundamental relations which define our model of viscoelastic-plastic behaviour. The description is purely phenomenological and is based on the theory of reological models that makes use of an adequate combination of a few basic elements (spring, dashpot, and slide) which idealize elastic, viscous and plastic behaviour. On developing the model, we have in mind mainly a soil foundation which is a medium whose behaviour is quite complex and contains to some extent all reological characteristics, including both time-depending and plastic phenomena, cf. Vyalov [43]. Furthermore, our purpose is to develope a structural mechanics model which is a simple description of the complicated processes which occur in soils, cf. Szefer [40]. Thus, confining to our one-dimensional model of skeletal systems we consider here the one-dimensional case. (The viscoelastic relations for 3D and 2D cases are given in [20].) The total strain is additively decomposed into a clearance part ϵ^l, a viscoelastic part ϵ^{ve} and a plastic part ϵ^p, see Fig. 1,

$$\epsilon = \epsilon^l + \epsilon^{ve} + \epsilon^p, \tag{9}$$

or in incremental form

$$\dot{\epsilon} = \dot{\epsilon}^l + \dot{\epsilon}^{ve} + \dot{\epsilon}^p. \tag{10}$$

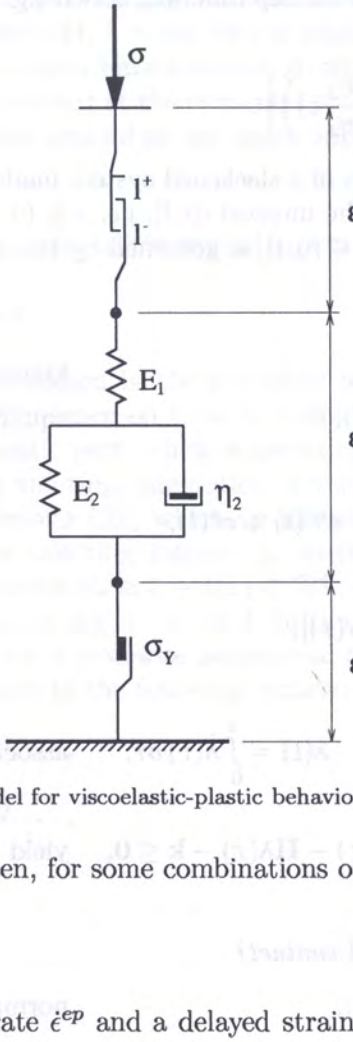

Fig. 1. Conceptual model for viscoelastic-plastic behaviour, with clearance strain.

The rate form can also be written, for some combinations of the mechanical basic elements, as the following decomposition

$$\dot{\epsilon} = \dot{\epsilon}^l + \dot{\epsilon}^{ep} + \dot{\epsilon}^v \tag{11}$$

in which an instantaneous strain rate $\dot{\epsilon}^{ep}$ and a delayed strain rate $\dot{\epsilon}^v$ are separated. In fact, the split (11) together with $\dot{\epsilon}^v = 0$ will be employed later at the times at which stress σ suffers a discontinuity of first order.

For the plastic strain rate an associative flow rule is used, whilst the evolution of the viscoelastic strain is assumed to be governed by the following linear differential equation

$$a_0\sigma + a_1\dot{\sigma} + a_2\ddot{\sigma} = b_0\epsilon^{ve} + b_1\dot{\epsilon}^{ve} + b_2\ddot{\epsilon}^{ve} \tag{12}$$

wherein a_i, b_i $(i = 1, 2, 3)$ are viscoelastic material constants. In the particular case of the standard three-parameter solid shown in Fig. 1, Eq. (12) simplifies to the equation of first order

$$\frac{E_1 + E_2}{\eta_2}\sigma + 1.0\,\dot{\sigma} = \frac{E_1 E_2}{\eta_2}\epsilon^{ve} + E_1\dot{\epsilon}^{ve}. \tag{13}$$

Let us assume that the stress history is known, then we can solve Eq. (12) for strains and express the result at a time t in the form of linear hereditary constitutive relation

$$\epsilon^{ve}(t) = \int_{t_0}^{t} J(t-\tau)\frac{\partial\sigma(\tau)}{\partial\tau}\,d\tau \equiv \mathcal{F}^t_{\tau=t_0}[\sigma(\tau)] \tag{14}$$

which emphasizes the fact that the strain at a current moment t depends upon the entire history of stress from a time $t = t_0$ to the time t. Equivalence conditions of the differential operator law (12) and the creep integral law (14) are discussed at length by Gurtin and Sternberg [14]. Usually one takes $t_0 = 0$ and zero initial condition which should be defined at the left-hand side of time t, i.e. $t = t_0^- \equiv t_0 - 0$. In (14) $J(s)$ denotes a creep function, which e.g. for the viscoelastic three-parameter solid shown in Fig. 1 is

$$J(s) = \frac{1}{E_1}\left[1 + \frac{E_1}{E_2}\left(1 - \exp(-\frac{E_2}{\eta_2}s)\right)\right]. \tag{15}$$

Suppose the deformation process of a slackened system made of the viscoelastic-plastic material has taken place on times τ along the interval $(0,t]$, i.e. $\tau \in (0,t]$. Then the state of the system at the present instant $t \in \mathcal{R}^+$, with $\tau \in (0,t]$, is governed by the following relations.

$$\begin{array}{rrcll}
 & \multicolumn{4}{l}{\textit{General relations}} \\
1) & \boldsymbol{\epsilon}(t) & = & \mathbf{C}\,\mathbf{u}(t), & \text{kinematic compatibility,} \\
2) & \mathbf{p}(t) & = & \mathbf{C}^{\mathrm{T}}\boldsymbol{\sigma}(t), & \text{equilibrium,} \\
 & \multicolumn{4}{l}{\textit{Strain decomposition}} \\
3) & \boldsymbol{\epsilon}(t) & = & \boldsymbol{\epsilon}^{l}(t) + \boldsymbol{\epsilon}^{ve}(t) + \boldsymbol{\epsilon}^{p}(t), & \\
 & \multicolumn{4}{l}{\textit{Linear viscoelasticity}} \\
4) & \boldsymbol{\epsilon}^{ve}(t) & = & \mathcal{F}_{\tau=0}^{t}[\boldsymbol{\sigma}(\tau)], & \\
 & \multicolumn{4}{l}{\textit{Plasticity}} \\
5) & \boldsymbol{\epsilon}^{p}(t) & = & \mathbf{N}\boldsymbol{\lambda}(t), \;\; \boldsymbol{\lambda}(t) = \int_0^t \dot{\boldsymbol{\lambda}}(\tau)\,d\tau, & \text{associative flow rule,} \\
6) & \dot{\boldsymbol{\lambda}}(\tau) & \geq & \mathbf{0}, & \\
7) & \mathbf{f}(\tau) & = & \mathbf{N}^{\mathrm{T}}\boldsymbol{\sigma}(\tau) - \mathbf{H}\boldsymbol{\lambda}(\tau) - \mathbf{k} \leq \mathbf{0}, & \text{yield function,} \\
8) & \dot{\boldsymbol{\lambda}}^{\mathrm{T}}(\tau)\cdot\mathbf{f}(\tau) & = & 0, & \\
 & \multicolumn{4}{l}{\textit{Slackening (unilateral contact)}} \\
9) & \boldsymbol{\sigma}(\tau) & = & \mathbf{M}\boldsymbol{\psi}(\tau), & \text{normality law,} \\
10) & \boldsymbol{\psi}(\tau) & \geq & \mathbf{0}, & \\
11) & \mathbf{g}(\tau) & = & \mathbf{M}^{\mathrm{T}}\boldsymbol{\epsilon}^{l}(\tau) - \mathbf{l} \leq \mathbf{0}, & \text{clearance function,} \\
12) & \boldsymbol{\psi}^{\mathrm{T}}(\tau)\cdot\mathbf{g}(\tau) & = & 0. &
\end{array} \tag{16}$$

The manner in which the time dependence of the problem quantities is displayed in the relations (16) is just to indicate which of the quantities are time-dependent and where the history of the deformation process is involved; of course, the conditions of equilibrium $(16)_2$ and of geometric compatibility $(16)_1$ must be satisfied at each instant $\tau \in (0,t]$. Furthermore it should be mentioned that relations (16) represent the finite-dimensional model of the whole structure/foundation system obtained by means of the finite element method, which consists of the structure's elements and those modelling its foundation. In fact, the similar relations as (16) are valid at the elemental level, the system (16) being obtained by the usual assembly process. Although the aspects of finite dimensional approximation are very important, in this paper we treat (16) as granted and wish to focus our attention rather on the time approximation of the problem at hand. Yet, let us recall merely the notations used in (16). The geometric compatibility condition $(16)_1$ states a relation between the vector of nodal displacements $\mathbf{u}(t)$ and the vector of generalized strains, with $\mathbf{C}$ being called the kinematic compatibility matrix whose transpose $\mathbf{C}^{\mathrm{T}}$ defines the equilibrium condition $(16)_2$.

By $\mathbf{p}(t)$, $\boldsymbol{\sigma}(t)$ we denote the vector of external nodal forces and the vector of generalized stresses, respectively. Notice that the vectors $\mathbf{u}$, $\mathbf{p}$, and $\boldsymbol{\epsilon}, \boldsymbol{\sigma}$ should satisfy the compatibility condition of virtual work, $\mathbf{p}^{\mathrm{T}}\mathbf{u} = \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}$. The development of plasticity effects is governed by the associative flow rule $(16)_5$ and complementarity conditions $(16)_6 - (16)_8$, in which the piecewise linear yield function $(16)_7$ is defined by (constant) matrix $\mathbf{N}$ collecting outward normal vectors to all segments of yield function $\mathbf{f}$ and linear hardening matrix $\mathbf{H}$, $\dot{\boldsymbol{\lambda}}$ is the rate of plastic multiplier $\boldsymbol{\lambda}$. The slackening (or unilateral contact) relations $(16)_9 - (16)_{12}$ have a similar structure. Vectors $\mathbf{k}$ and $\mathbf{l}$ contain values of the plastic and clearance moduli defined at the elemental level. It should be stressed that entries and dimensions of the above matrices depend on the space approximations applied, cf. [4, 3, 11].

4. Time discretization and the nLCP formulation

4.1. Time discretization method

The viscoelastic-plastic model proposed in the preceding section governs an evolutionary deformation process in which the dependence on time is twofold. First, the viscoelastic part ϵ^{ve} is time-dependent, whereas for the plastic part, which is path-dependent, the time variable t merely labels the succession of events. For the time integration of the viscoelastic part of strains we will follow the method of Świtka and Husiar [39], which we already applied to unilateral viscoelastic contact problems [23, 21]. Now, by selecting instants t_τ on the time axis, i.e. $t_\tau \in \mathcal{R}^+$, we can divide it into a number of finite intervals $\vartheta_\tau \equiv t_\tau - t_{\tau-1} \subset \mathcal{R}^+$. In the sequel we indicate a function of t at instant $t = t_\tau$ by a subscript τ, e.g. $\epsilon_\tau \equiv \epsilon(t_\tau)$. In the method [39], the strain ϵ is solved from the differential equation (12) for a piecewise polynomial approximation of stress successively within each time interval, which leads to the following recurrent expressions

$$\epsilon_\tau^{ve} = \gamma_\tau \sigma_\tau + \tilde{\epsilon}_{\tau,\tau-1}, \qquad \tau = 1, 2, \dots, \tag{17}$$

$$\dot{\epsilon}_\tau^{ve} = \dot{\gamma}_\tau \sigma_\tau + \dot{\tilde{\epsilon}}_{\tau,\tau-1}, \qquad \tau = 1, 2, \dots, \tag{18}$$

with

$$\tilde{\epsilon}_{\tau,\tau-1} \equiv \mathbf{A}_\tau \mathbf{s}_{\tau-1}^{ve},$$

$$\dot{\tilde{\epsilon}}_{\tau,\tau-1} \equiv \dot{\mathbf{A}}_\tau \mathbf{s}_{\tau-1}^{ve}.$$

In the above relations, scalars γ_τ, $\dot{\gamma}_\tau$ and vectors $\mathbf{A}_\tau$, $\dot{\mathbf{A}}_\tau$ depend upon the viscoelastic properties of a material and the degree of approximation (linear, or quadratic), whereas $\mathbf{s}_{\tau-1}$ is a vector which describes the state of the material at a previous instant, accounting only for the viscoelastic strain part,

$$\begin{aligned}
\mathbf{s}_{\tau-1}^{ve} &= \mathrm{col}(\epsilon_{\tau-1}^{ve}, \dot{\epsilon}_{\tau-1}^{ve}, \sigma'_{\tau-1}, \dot{\sigma}'_{\tau-1}, \sigma_{\tau-1}, \dot{\sigma}_{\tau-1}),\\
\mathbf{A}_\tau &= \mathrm{row}(A_{1\tau}, A_{2\tau}, A_{3\tau}, A_{4\tau}, A_{5\tau}, A_{6\tau}),\\
\dot{\mathbf{A}}_\tau &= \mathrm{row}(\dot{A}_{1\tau}, \dot{A}_{2\tau}, \dot{A}_{3\tau}, \dot{A}_{4\tau}, \dot{A}_{5\tau}, \dot{A}_{6\tau}).
\end{aligned} \tag{19}$$

By $(')$ we denote in (19) and in the sequel the right-hand value of the relevant quantity, e.g. for stress σ we have, $\sigma'_\tau = \sigma_\tau + \Delta\sigma_\tau$, where Δ is the symbol of a finite increment. The discontinuities which may appear in some state variables are induced by discontinuous changes in the external loading $\mathbf{p}(t)$ on the structure. At such time-discontinuity points t_τ we suppose an instantaneous elastoplastic material response, and consequently a finite increment of strain is defined as

$$\Delta\epsilon_\tau = \Delta\epsilon_\tau^l + \Delta\epsilon_\tau^{ve} + \Delta\epsilon_\tau^p. \tag{20}$$

for

$$\Delta\epsilon_\tau^{ve} \equiv \Delta\epsilon_\tau^{e} = \gamma_0 \Delta\sigma_\tau, \qquad \tau = 0, 1, 2, \ldots, \tag{21}$$

$$\Delta\dot{\epsilon}_\tau^{ve} = \gamma_0 \Delta\dot{\sigma}_\tau + \dot{\gamma}_0 \Delta\sigma_\tau, \qquad \tau = 0, 1, 2, \ldots. \tag{22}$$

The rate of stress depends on the degree of the polynomial approximation,
for the linear approximation:

$$\dot{\sigma}_\tau = \frac{1}{\vartheta_\tau}(\sigma_\tau - \sigma'_{\tau-1}), \tag{23}$$

for the quadratic approximation:

$$\dot{\sigma}_\tau = \frac{2}{\vartheta_\tau}(\sigma_\tau - \sigma'_{\tau-1}) - \dot{\sigma}'_{\tau-1}. \tag{24}$$

It should be remarked here that in the case when the linear approximation is used the increment of the velocity $\Delta\dot{\epsilon}_\tau^{ve}$ in (22) need not to be determined, only $\Delta\epsilon_\tau^{ve}$ in (21) is needed. Summing up, we want to point out that the increments of (20) can be determined by solving system (16), which now must be satisfied at the right-hand side of instant $t = t_\tau$.

Depending on the order of the time approximation applied and the character of time changes of loadings on the structure, we have to solve from one to three problems at the selected times $t = t_\tau$. These problems will be discussed in the sequel.

4.2. Time-step $t_{\tau-1} \to t_\tau^-$

Because the system (16) is nonlinear it is convenient to approach it in an incremental way. A typical time-step $t_{\tau-1} \to t_\tau$ consists in updating the known state of the system at previous time level $t_{\tau-1}$ to the state at current time level t_τ. Making use of the successive relation (17) we obtain the following updating formula for the viscoelastic strain vector

$$\boldsymbol{\epsilon}_\tau^{ve} = \mathbf{F}_\tau \boldsymbol{\sigma}_\tau + \tilde{\boldsymbol{\epsilon}}_{\tau,\tau-1} \tag{25}$$

in which $\mathbf{F}_\tau$ is a matrix that corresponds to the scalar γ_τ in (17) and plays a role similar to that of the flexibility matrix in elasticity, but here being updated from step to step. Associated with the scalar $\tilde{\epsilon}_{\tau,\tau-1}$ is the vector $\tilde{\boldsymbol{\epsilon}}_{\tau,\tau-1}$ by which the viscoelastic state of the system at the preceded instant is accounted for. At a time discontinuity point $t = t_\tau$, according to (22) the following relation between the finite increments holds

$$\Delta\dot{\boldsymbol{\epsilon}}_\tau^{ve} = \mathbf{F}_0 \Delta\dot{\boldsymbol{\sigma}}_\tau + \dot{\mathbf{F}}_0 \Delta\boldsymbol{\sigma}_\tau. \tag{26}$$

The updating of the plastic strain and plastic multiplier vectors is defined in a usual way

$$\begin{aligned} \boldsymbol{\lambda}_\tau &= \boldsymbol{\lambda}_{\tau-1} + \Delta\boldsymbol{\lambda}_\tau, \\ \boldsymbol{\epsilon}_\tau^{p} &= \boldsymbol{\epsilon}_{\tau-1}^{p} + \Delta\boldsymbol{\epsilon}_\tau^{p}. \end{aligned} \tag{27}$$

At any two successive time levels we may express the relations (16) in the following incremental form.

$$
\begin{array}{lrcllrcl}
 & \multicolumn{3}{l}{\text{time level } t = t_{\tau-1}} & & \multicolumn{3}{l}{\text{time level } t = t_{\tau}} \\
1) & \boldsymbol{\epsilon}_{\tau-1} & = & \mathbf{C}\,\mathbf{u}_{\tau-1}, & & \boldsymbol{\epsilon}_{\tau} & = & \mathbf{C}\,\mathbf{u}_{\tau}, \\
2) & \mathbf{p}_{\tau-1} & = & \mathbf{C}^{\mathrm{T}}\boldsymbol{\sigma}_{\tau-1}, & & \mathbf{p}_{\tau} & = & \mathbf{C}^{\mathrm{T}}\boldsymbol{\sigma}_{\tau}, \\
3) & \boldsymbol{\epsilon}_{\tau-1} & = & \boldsymbol{\epsilon}^{l}_{\tau-1} + \boldsymbol{\epsilon}^{ve}_{\tau-1} + \boldsymbol{\epsilon}^{p}_{\tau-1}, & & \boldsymbol{\epsilon}_{\tau} & = & \boldsymbol{\epsilon}^{l}_{\tau} + \boldsymbol{\epsilon}^{ve}_{\tau} + \boldsymbol{\epsilon}^{p}_{\tau}, \\
4) & \boldsymbol{\epsilon}^{ve}_{\tau-1} & = & \mathbf{F}_{\tau-1}\boldsymbol{\sigma}_{\tau-1} + \tilde{\boldsymbol{\epsilon}}_{\tau-1,\tau-2}, & & \boldsymbol{\epsilon}^{ve}_{\tau} & = & \mathbf{F}_{\tau}\boldsymbol{\sigma}_{\tau} + \tilde{\boldsymbol{\epsilon}}_{\tau,\tau-1}, \\
5) & \Delta\boldsymbol{\epsilon}^{p}_{\tau-1} & = & \mathbf{N}\,\Delta\boldsymbol{\lambda}_{\tau-1}, & & \Delta\boldsymbol{\epsilon}^{p}_{\tau} & = & \mathbf{N}\,\Delta\boldsymbol{\lambda}_{\tau}, \\
6) & \Delta\boldsymbol{\lambda}_{\tau-1} & \geq & \mathbf{0}, & & \Delta\boldsymbol{\lambda}_{\tau} & \geq & \mathbf{0}, \\
7) & \mathbf{f}_{\tau-1} & = & \mathbf{N}^{\mathrm{T}}\boldsymbol{\sigma}_{\tau-1} - \mathbf{H}\boldsymbol{\lambda}_{\tau-1} - \mathbf{k} \leq \mathbf{0}, & & \mathbf{f}_{\tau} & = & \mathbf{N}^{\mathrm{T}}\boldsymbol{\sigma}_{\tau} - \mathbf{H}(\boldsymbol{\lambda}_{\tau-1} + \Delta\boldsymbol{\lambda}_{\tau}) - \mathbf{k} \leq \mathbf{0}, \\
8) & 0 & = & (\Delta\boldsymbol{\lambda}_{\tau-1})^{\mathrm{T}} \cdot \mathbf{f}_{\tau-1}, & & 0 & = & (\Delta\boldsymbol{\lambda}_{\tau})^{\mathrm{T}} \cdot \mathbf{f}_{\tau}, \\
9) & \boldsymbol{\sigma}_{\tau-1} & = & \mathbf{M}\boldsymbol{\psi}_{\tau-1}, & & \boldsymbol{\sigma}_{\tau} & = & \mathbf{M}\boldsymbol{\psi}_{\tau}, \\
10) & \boldsymbol{\psi}_{\tau-1} & \geq & \mathbf{0} & & \boldsymbol{\psi}_{\tau} & \geq & \mathbf{0}, \\
11) & \mathbf{g}_{\tau-1} & = & \mathbf{M}^{\mathrm{T}}\boldsymbol{\epsilon}^{l}_{\tau-1} - \mathbf{l} \leq \mathbf{0}, & & \mathbf{g}_{\tau} & = & \mathbf{M}^{\mathrm{T}}\boldsymbol{\epsilon}^{l}_{\tau} - \mathbf{l} \leq \mathbf{0}, \\
12) & 0 & = & (\boldsymbol{\psi}_{\tau-1})^{\mathrm{T}} \cdot \mathbf{g}_{\tau-1}, & & 0 & = & (\boldsymbol{\psi}_{\tau})^{\mathrm{T}} \cdot \mathbf{g}_{\tau}.
\end{array}
\tag{28}
$$

By substituting the stress from $(28)_9$ into $(28)_2$ and $(28)_7$, and making use of the split $(28)_3$, for $t = t_\tau$, we can reduce (28) to the conditions

$$
\begin{aligned}
\mathbf{C}^{\mathrm{T}}\mathbf{M}\boldsymbol{\psi}_{\tau} - \mathbf{p}_{\tau} &= \mathbf{0}, \\
\mathbf{H}\Delta\boldsymbol{\lambda}_{\tau} - \mathbf{N}^{\mathrm{T}}\mathbf{M}\boldsymbol{\psi}_{\tau} + \mathbf{k} + \mathbf{H}\boldsymbol{\lambda}_{\tau-1} &\geq \mathbf{0}, \\
\mathbf{M}^{\mathrm{T}}\mathbf{N}\Delta\boldsymbol{\lambda}_{\tau} - \mathbf{M}^{\mathrm{T}}\mathbf{C}\mathbf{u}_{\tau} + \mathbf{M}^{\mathrm{T}}\mathbf{F}_{\tau}\mathbf{M}\boldsymbol{\psi}_{\tau} + M^{\mathrm{T}}\tilde{\boldsymbol{\epsilon}}_{\tau,\tau-1} + \mathbf{l} + \mathbf{M}^{\mathrm{T}}\boldsymbol{\epsilon}^{p}_{\tau-1} &\geq \mathbf{0},
\end{aligned}
\tag{29}
$$

with

$$
\Delta\boldsymbol{\lambda}_{\tau} \geq \mathbf{0}, \quad \boldsymbol{\psi}_{\tau} \geq \mathbf{0} \quad \text{and} \quad (\Delta\boldsymbol{\lambda}_{\tau})^{\mathrm{T}} \cdot \mathbf{f}_{\tau} = 0, \quad (\boldsymbol{\psi}_{\tau})^{\mathrm{T}} \cdot \mathbf{g}_{\tau} = 0. \tag{30}
$$

The two unknown vectors of (29), $\Delta\boldsymbol{\lambda}_\tau$ and ψ_τ, should satisfy the non-negativity condition, cf. (30), and the orthogonality condition with respect to the conjugate variable, $(29)_2$ and $(29)_3$, respectively.

We can finally formulate the incremental problem under consideration (29) as the following nested linear complementarity problem (nLCP)— *Problem I*:

$$
\begin{gathered}
\mathbf{D}_{\tau}\mathbf{x}_{\tau} + \mathbf{y}_{\tau} = \mathbf{b}_{\tau,\tau-1}, \\
\mathbf{x}_{2,\tau} \geq \mathbf{0}, \quad \mathbf{x}_{3,\tau} \geq \mathbf{0}, \quad \mathbf{y}_{1,\tau} = \mathbf{0}, \quad \mathbf{y}_{2,\tau} \geq \mathbf{0}, \quad \mathbf{y}_{3,\tau} \geq \mathbf{0}, \quad \mathbf{x}_{\tau}^{\mathrm{T}}\mathbf{y}_{\tau} = 0
\end{gathered}
\tag{31}
$$

in which the following notations are used:

$$
\mathbf{D}_{\tau} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{C}^{\mathrm{T}}\mathbf{M} \\ \mathbf{0} & -\mathbf{H} & \mathbf{N}^{\mathrm{T}}\mathbf{M} \\ \mathbf{M}^{\mathrm{T}}\mathbf{C} & -\mathbf{M}^{\mathrm{T}}\mathbf{N} & -\mathbf{M}^{\mathrm{T}}\mathbf{F}_{\tau}\mathbf{M} \end{bmatrix}, \quad \mathbf{x}_{\tau} = \begin{Bmatrix} \mathbf{u}_{\tau} \\ \Delta\boldsymbol{\lambda}_{\tau} \\ \boldsymbol{\psi}_{\tau} \end{Bmatrix}, \quad \mathbf{b}_{\tau,\tau-1} = \begin{Bmatrix} -\mathbf{p}_{\tau} \\ \mathbf{k}_{\tau-1} \\ \tilde{\mathbf{l}}_{\tau,\tau-1} \end{Bmatrix}
$$

and given $\mathbf{p}_\tau$, $\mathbf{k}_{\tau-1} = \mathbf{k} + \mathbf{H}\boldsymbol{\lambda}_{\tau-1}$, $\tilde{\mathbf{l}}_{\tau,\tau-1} = \mathbf{l} + \mathbf{M}^{\mathrm{T}}(\tilde{\boldsymbol{\epsilon}}_{\tau,\tau-1} + \boldsymbol{\epsilon}^{p}_{\tau-1})$.

4.3. Step $t_\tau^- \to t_\tau^+$ at a time discontinuity point $t = t_\tau$

At these times t_τ at which the load vector suffers a jump discontinuity in time, i.e. $\Delta\mathbf{p}_\tau \neq \mathbf{0}$, we have to determine the state of the system at the right-hand side of t_τ, i.e. at $t = t_\tau^+ \equiv t_\tau + 0$. This leads to a *Problem II* defined below which must be solved in addition to *Problem I* in the case of both the linear and quadratic time approximations.

The state of the structural system at the time-discontinuity point $t = t_\tau$ is defined by the following relations.

$$
\begin{array}{rlcllcl}
 & \text{time level } t = t_\tau - 0 & & & \text{time level } t = t_\tau + 0 & & \\
1) & \boldsymbol{\epsilon}_\tau & = & \mathbf{C}\,\mathbf{u}_\tau, & \boldsymbol{\epsilon}'_\tau & = & \mathbf{C}\,\mathbf{u}'_\tau, \\
2) & \mathbf{p}_\tau & = & \mathbf{C}^{\mathrm{T}}\boldsymbol{\sigma}_\tau, & \mathbf{p}'_\tau & = & \mathbf{C}^{\mathrm{T}}\boldsymbol{\sigma}'_\tau, \\
3) & \boldsymbol{\epsilon}_\tau & = & \boldsymbol{\epsilon}^l_\tau + \boldsymbol{\epsilon}^{ve}_\tau + \boldsymbol{\epsilon}^p_\tau, & \boldsymbol{\epsilon}'_\tau & = & \boldsymbol{\epsilon}^{l\prime}_\tau + \boldsymbol{\epsilon}^{ve}_\tau + \Delta\boldsymbol{\epsilon}^{ve}_\tau) + \boldsymbol{\epsilon}^p_\tau + \Delta\boldsymbol{\epsilon}^{p\prime}_\tau), \\
4) & \boldsymbol{\epsilon}^{ve}_\tau & = & \mathbf{F}_\tau\boldsymbol{\sigma}_\tau + \tilde{\boldsymbol{\epsilon}}_{\tau,\tau-1}, & \Delta\boldsymbol{\epsilon}^{ve}_\tau & = & \mathbf{F}_0\,\Delta\boldsymbol{\sigma}_\tau, \\
5) & \Delta\boldsymbol{\epsilon}^p_\tau & = & N\,\Delta\boldsymbol{\lambda}_\tau, & \Delta\boldsymbol{\epsilon}^{p\prime}_\tau & = & \mathbf{N}\,\Delta\boldsymbol{\lambda}'_\tau, \\
6) & \Delta\boldsymbol{\lambda}_\tau & \geq & \mathbf{0}, & \Delta\boldsymbol{\lambda}'_\tau & \geq & \mathbf{0}, \\
7) & \mathbf{f}_\tau & = & \mathbf{N}^{\mathrm{T}}\boldsymbol{\sigma}_\tau - \mathbf{H}\boldsymbol{\lambda}_\tau - \mathbf{k} \leq \mathbf{0}, & \mathbf{f}'_\tau & = & \mathbf{N}^{\mathrm{T}}\boldsymbol{\sigma}'_\tau - \mathbf{H}\boldsymbol{\lambda}_\tau + \Delta\boldsymbol{\lambda}'_\tau) - \mathbf{k} \leq \mathbf{0}, \\
8) & 0 & = & (\Delta\boldsymbol{\lambda}_\tau)^{\mathrm{T}}\cdot\mathbf{f}_\tau, & 0 & = & (\Delta\boldsymbol{\lambda}'_\tau)^{\mathrm{T}}\cdot\mathbf{f}'_\tau, \\
9) & \boldsymbol{\sigma}_\tau & = & \mathbf{M}\boldsymbol{\psi}_\tau, & \boldsymbol{\sigma}'_\tau & = & \mathbf{M}\boldsymbol{\psi}'_\tau, \\
10) & \boldsymbol{\psi}_\tau & \geq & \mathbf{0} & \boldsymbol{\psi}'_\tau & \geq & \mathbf{0}, \\
11) & \mathbf{g}_\tau & = & \mathbf{M}^{\mathrm{T}}\boldsymbol{\epsilon}^l_\tau - \mathbf{l} \leq \mathbf{0}, & \mathbf{g}'_\tau & = & \mathbf{M}^{\mathrm{T}}\boldsymbol{\epsilon}^{l\prime}_\tau - \mathbf{l} \leq \mathbf{0}, \\
12) & 0 & = & (\boldsymbol{\psi}_\tau)^{\mathrm{T}}\cdot\mathbf{g}_\tau, & 0 & = & (\boldsymbol{\psi}'_\tau)^{\mathrm{T}}\cdot\mathbf{g}'_\tau.
\end{array}
\tag{32}
$$

The linear complementarity problem associated with (32) has a form analogous to the prior nLCP and is stated as the following *Problem II*:

$$
\begin{gathered}
\mathbf{D}_0\mathbf{x}'_\tau + \mathbf{y}'_\tau = \mathbf{b}'_{\tau,\tau-1}, \\
\mathbf{x}'_{2,\tau} \geq \mathbf{0}, \quad \mathbf{x}'_{3,\tau} \geq \mathbf{0}, \quad \mathbf{y}'_{1,\tau} = \mathbf{0}, \quad \mathbf{y}'_{2,\tau} \geq \mathbf{0}, \quad \mathbf{y}'_{3,\tau} \geq \mathbf{0}, \quad (\mathbf{x}'_\tau)^{\mathrm{T}}\mathbf{y}'_\tau = 0.
\end{gathered}
\tag{33}
$$

where

$$
\mathbf{D}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{C}^{\mathrm{T}}\mathbf{M} \\ \mathbf{0} & -\mathbf{H} & \mathbf{N}^{\mathrm{T}}\mathbf{M} \\ \mathbf{M}^{\mathrm{T}}\mathbf{C} & -\mathbf{M}^{\mathrm{T}}\mathbf{N} & -\mathbf{M}^{\mathrm{T}}\mathbf{F}_0\mathbf{M} \end{bmatrix}, \quad \mathbf{x}'_\tau = \begin{Bmatrix} \mathbf{u}'_\tau \\ \Delta\boldsymbol{\lambda}'_\tau \\ \boldsymbol{\psi}'_\tau \end{Bmatrix}, \quad \mathbf{b}'_{\tau,\tau-1} = \begin{Bmatrix} -\mathbf{p}'_\tau \\ \mathbf{k}_\tau \\ \tilde{\mathbf{l}}'_{\tau,\tau-1} \end{Bmatrix}
$$

and given $\mathbf{p}'_\tau$, $\mathbf{k}_\tau = \mathbf{k} + \mathbf{H}\boldsymbol{\lambda}_\tau$, $\tilde{\mathbf{l}}'_{\tau,\tau-1} = \mathbf{l} + \mathbf{M}^{\mathrm{T}}(\tilde{\boldsymbol{\epsilon}}_{\tau,\tau-1} + \boldsymbol{\epsilon}^p_\tau) + \mathbf{M}^{\mathrm{T}}(\mathbf{F}_\tau - \mathbf{F}_0)\mathbf{M}\boldsymbol{\psi}_\tau$.

In this way, the original continuous-in-time problem (16) has been transformed into a series of linear complementarity problems (31) and (33), which describe completely the considered unilateral viscoelastic-plastic contact problem at time t_τ. In fact, the incremental procedure we have advanced allows us to analyze quite a complex, also non-proportional loading programme of the slackened system in which, within a typical time step, plastic strains are assumed holonomic whilst stresses are approximated with a linear or quadratic polynomial. At all times t_τ at which the external load on the system is time-continuous it is sufficient to solve the nLCP (31), but otherwise the nLCP (33) must be solved additionally afterwards using the known vector $\mathbf{x}_\tau$, the solution of *Problem I*.

For the solution of the nested linear complementarity problems we have devised a principle pivoting scheme described in Sec. 5. In reference to Lemma 2 of Sec. 2 we want to stress that in the derived LCP problems (31) and (33) the properties of the matrix $\mathbf{D}_\tau$, $\tau = 0, 1, 2, \ldots$ depend on both the kinematical and physical properties of the structural system. Firstly, the matrix $-\mathbf{D}_\tau$ is positive semi-definite if the matrices $\boldsymbol{H}$ and $\mathbf{M}^{\mathrm{T}}\mathbf{F}_\tau\mathbf{M}$ are positive semi-definite. This requirement is satisfied in the case of the used Prager's plastic hardening rule and the convexity of the yield and clearance regions, together with the non-negative hardening modulus $H'(\mathbf{x}) \geq 0$ and viscoelastic parameter $\gamma_\tau(\mathbf{x}) \geq 0$ for all points of the system. Secondly, the existence and uniqueness of a solution to our problem depends also on the data $\mathbf{b}_{\tau,\tau-1}$ which contain information on the applied loading and changes in the physical (plastic strain hardening) and kinematical properties of the structural system in the course of the deformation process. It is clear that the development of plastic strains (hinges) may convert the initially stable structure made of a viscoelastic- perfectly plastic material into a structural mechanism unable to sustain any additional loading.

When using the quadratic polynomial time approximation we need the rate of stress $\dot{\boldsymbol{\sigma}}'_\tau$ at the right-hand side of the time t_τ at which there is a jump in the external load, $\Delta\mathbf{p}_\tau \neq \mathbf{0}$ or/and in its speed $\Delta\dot{\mathbf{p}}_\tau \neq \mathbf{0}$. To this end, an additional *Problem III* must be considered. First, an increment $\Delta\dot{\mathbf{u}}_\tau$ is determined from the equation — *Problem IIIa*:

$$\tilde{\mathbf{K}}_0\,\Delta\dot{\mathbf{u}}_\tau = \Delta\dot{\mathbf{p}}_\tau + \tilde{\mathbf{C}}^{\mathrm{T}}\tilde{\mathbf{F}}_0^{-1}\dot{\tilde{\mathbf{F}}}_0\,\Delta\tilde{\boldsymbol{\sigma}}_\tau \tag{34}$$

where

$$\tilde{\mathbf{K}}_0 = \tilde{\mathbf{C}}^{\mathrm{T}}\tilde{\mathbf{F}}_0^{-1}\tilde{\mathbf{C}}$$

is the stiffness matrix of the structural subsystem, containing only the contributions of these elements of the foundation which are in contact with the structure. With the superscript $\tilde{}$ we denote here the restriction to this subsystem. The boundary conditions on $\Delta\dot{\mathbf{u}}_\tau$ can easily be derived from those imposed on the displacement vector $\mathbf{u}_\tau$. In deriving (34) we made use of (26) and the equation — *Problem IIIb*:

$$\tilde{\mathbf{C}}^{\mathrm{T}}\,\Delta\dot{\tilde{\boldsymbol{\sigma}}}_\tau = \Delta\dot{\mathbf{p}}_\tau. \tag{35}$$

Having determined $\Delta\dot{\mathbf{u}}_\tau$ we can calculate

$$\Delta\dot{\tilde{\boldsymbol{\epsilon}}}_\tau^{ve} = \tilde{\mathbf{C}}\,\Delta\dot{\mathbf{u}}_\tau,$$

$$\Delta\dot{\tilde{\boldsymbol{\sigma}}}_\tau = \tilde{\mathbf{F}}_0^{-1}\,\Delta\dot{\tilde{\boldsymbol{\epsilon}}}_\tau^{ve} - \tilde{\mathbf{F}}_0^{-1}\dot{\tilde{\mathbf{F}}}_0\Delta\,\tilde{\boldsymbol{\sigma}}_\tau,$$

and finally

$$\dot{\boldsymbol{\sigma}}'_\tau = \dot{\boldsymbol{\sigma}}_\tau + \Delta\dot{\boldsymbol{\sigma}}_\tau. \tag{36}$$

The equation (34) is valid provided that the matrix $\tilde{\mathbf{K}}_0$ is non-singular, which depends both on the matrix $\tilde{\mathbf{C}}$ and the matrix $\tilde{\mathbf{F}}_0$. Of course, when the matrix $\tilde{\mathbf{F}}_0$ is equal to zero, then $\tilde{\mathbf{K}}_0$ is singular. This is the case with any viscoelastic model for which the coefficient $\gamma_0 = 0$, in particular with the Kelvin-Voigt model. (In reference to the constitutive equation (12) we recall that γ_0 is defined as follows: $\gamma_0 = a_1/b_1$, if $b_1 \neq 0, b_2 = 0$, and $\gamma_0 = a_2/b_2$ if $b_2 \neq 0$.) In the situation $\tilde{\mathbf{F}}_0 = \mathbf{0}$ it is sufficient to satisfy equation (35). However, it should be noted that the system of equations (35), in which $\tilde{\mathbf{C}}^{\mathrm{T}}$ is a $m \times n$ rectangular matrix with $m > n$, may not possess a solution and if any, the solution may be non-unique. Following the idea presented in [12] we can show that there exists a unique solution to (35) if rank $\tilde{\mathbf{C}}^{\mathrm{T}} = n$ and the structural subsystem whose elements contribute to the matrix $\tilde{\mathbf{C}}$ is statically determinate. The requirement on the rank of $\tilde{\mathbf{C}}^{\mathrm{T}}$, i.e. rank $\tilde{\mathbf{C}}^{\mathrm{T}} = n$ is equivalent to the following condition det $\tilde{\mathbf{C}}\tilde{\mathbf{C}}^{\mathrm{T}} \neq 0$.

The numerical results for examples of practical importance in engineering are presented in Sec. 6, including the case of the Kelvin-Voigt material in Example 1.

5. Computational algorithm for the nLCP

There are basically two types of algorithms for solving the linear complementarity problem: direct methods and iterative methods, see [26, 42, 1, 29, 31, 6, 19], for example. The direct methods are based on the process of pivoting on the elements of the underlying matrix, which eventually amounts to removing some variables from the base and introducing the complementary variables into the base. The direct methods are finite procedures in the sense that they terminate after a finite

Algorithm for the nLCP

Phase I.

- **Step 0.** Introduce variables x_i, $1 \le i \le n_1$ into the base.

Phase II.

- **Step 1.** *IF*

$$b_i^k \ge 0 \quad \forall i \in I \equiv \{n_1+1, n_1+2, \ldots, n\},$$

THEN pair $(\boldsymbol{x}^\star, \boldsymbol{y}^\star) = (\mathbf{0}, \boldsymbol{b}^k)$ is solution of (7), EXIT 1

ELSE GO TO **Step 2.**

- **Step 2.** Determine

$$\omega_m = \max_{i \in I} \left\{ \frac{-b_i^k}{a_i^k} \,\middle|\, a_i^k > 0 \right\} = \frac{-b_m^k}{a_m^k}.$$

IF $d_{mm}^k < 0$, *then* GO TO **Step 3**,

ELSE GO TO **Step 3′**.

- **Step 3.** Perform principal pivoting with element d_{mm}^k,

Set $k = k+1$,

GO TO **Step 1.**

- **Step 3′.** *IF*

$$d_{im}^k \le 0 \quad \forall i \in I,$$

THEN there is no solution to the system (7), EXIT 2

ELSE Determine

$$\min_{i \in I} \left\{ \frac{b_i^k + \omega_k a_i^k}{d_{im}^k} \,\middle|\, d_{im}^k > 0 \right\} = \frac{b_r^k + \omega_k a_r^k}{d_{rm}^k},$$

Perform principal pivoting with elements d_{rm}^k and d_{mr}^k,

Set $k = k+1$,

GO TO **Step 1.**

number of computations finding the solution if it exists, or giving the answer that no solution to the underlying LCP exists. Due to the particular structure of the matrix $\mathbf{D}_\tau$ in (31) and (33), we decided to solve our nested LCPs by a direct method. We remark that the matrix $\mathbf{D}_\tau$ is bisymmetric with the submatrix $\mathbf{D}_{11} = \mathbf{0}$ and in the case of viscoelastic–ideal plastic behaviour of the structural system also the submatrix $\mathbf{D}_{22} \equiv \mathbf{H}$ is zero. Moreover, there are circumstances discussed above in which $\mathbf{D}_{33} = \mathbf{0}$. An iterative scheme for the slackened skeletal elastic-plastic structures is proposed by Gawęcki and Janińska [10] who make use of another reasoning.

The algorithm we develop for solving the nLCP (31) is based on a parametric method and operates in two phases. Within *Phase I* the components x_i, $1 \le i \le n_1$, of the sign-unrestricted vector $\boldsymbol{x}_1$ are made base variables, and within *Phase II* the so-called *extended* LCP is solved:

$$\begin{aligned}&\mathbf{D}\mathbf{x} + \mathbf{y} = \mathbf{b} + \omega\mathbf{a},\\ &\mathbf{x}_2 \ge \mathbf{0}, \qquad \mathbf{x}_3 \ge \mathbf{0}, \qquad \mathbf{y}_1 = \mathbf{0}, \qquad \mathbf{y}_2 \ge \mathbf{0}, \qquad \mathbf{y}_3 \ge \mathbf{0}, \qquad \mathbf{x}^{\mathrm{T}}\mathbf{y} = 0,\end{aligned} \tag{37}$$

where $\omega \ge 0$, $\mathbf{a} \in \mathcal{R}^n$, and with $1 \le i \le n$,

$$a_i = \begin{cases} 1, & \text{if } b_i < 0,\\ 0, & \text{otherwise.}\end{cases}$$

We recall that the parametric method consists in such consecutive changes of the parameter ω, i.e. $\omega \in \{\omega_0, \omega_1, \ldots, \omega_k, \ldots\}$, so that for each iteration k, with $k = 1, 2, \ldots$, the extended system

$$\begin{aligned}&\mathbf{D}^k\mathbf{x}^k + \mathbf{y}^k = \mathbf{b}^k + \omega_k\mathbf{a}^k,\\ &\mathbf{x}_2^k \ge \mathbf{0}, \qquad \mathbf{x}_3^k \ge \mathbf{0}, \qquad \mathbf{y}_1^k = \mathbf{0}, \qquad \mathbf{y}_2^k \ge \mathbf{0}, \qquad \mathbf{y}_3^k \ge \mathbf{0}, \qquad (\mathbf{x}^k)^{\mathrm{T}}\mathbf{y} = 0,\end{aligned} \tag{38}$$

possesses a solution [42]. Finally, for $\omega = \omega_\star = 0$ we obtain a pair $(\mathbf{x}^\star, \mathbf{y}^\star)$, if exists, which is the solution of the original problem (7). By the notation $\mathbf{D}^k, \mathbf{x}^k$, etc., we denote the quantities which are obtained after k pivot transformations on the system. We remark that in the used method there is no need to pre-set a starting solution $\mathbf{x}^0$.

Remarks

It must be emphasized that the introduction to the base of the sign-unrestricted variables within the step 0 of the algorithm cannot be done directly because the corresponding matrix $\mathbf{D}_{11}$ is equal to zero. So we carry out this process by performing pivotal transformations with elements of matrices $\mathbf{D}_{13}$ and $\mathbf{D}_{31}$ as pivots. Due to this particular property of our LCP (31) we call it the *nested* LCP. Of course, the nested LCP is a special case of the *mixed* LCP [6].

6. Results of numerical experiments

This section contain the numerical results which we have obtained with our material model and the proposed numerical algorithm for some typical engineering test problems. We start with the bending problem of a simply supported beam made of different viscoelastic materials. The main purpose of Example 1 is to test the convergence rate of the piecewise linear approximation for the prescribed piecewise quadratic history of loading. Notice that in Example 1 it suffices to divide the beam into only two finite elements in order to get numerical results which are exact as concerns both the statics and kinematics of the problem. More specifically, the description of the kinematics with only a few elements, the number of which is determined by the number of forces applied and supports of the beam, is exact under the assumption that the lumped plastic hinges may appear in the cross-sections where the forces are applied. The latter is true for a beam with ideal I–cross-section and if the material of the beam is perfectly plastic. In other cases, where the plastic hardening modulus H' is positive and/or the beam may come in contact with a foundation, the finite element discretization of the beam must be finer. The question of spatial discretization is, *inter alia*, analyzed in the next examples of the interaction problem between a beam and its foundation. We have included the results for three examples, with a number of cases concerning various boundary conditions (clearances) and material models. In the calculations of Examples 2 through 4 the piecewise linear time approximation method was applied, with a constant time step during each caluculation process. In Examples 2 and 4 the foundation is located beneath the beam in distance $l^+ = 0.005$ m, whilst in Example 3 the chain of beam elements rests freely on the foundation. Example 3 is a "small" challenge from the numerical viewpoint, cf. Demkowicz and Świerczek [7]. In Examples 2 through 4 the material response of the beam is assumed to be elastic- perfectly plastic with plastic hinges lumped at nodes. The foundation is treated as the viscoelastic-plastic medium of Winkler-type. The beam is discretized with the two-node four-parameter cubic elements and the foundation is modeled with the truss elements located at the beam nodes. In all the examples included the material of the beam is characterized by the Young modulus $E = 2.0 \cdot 10^5$ MPa, $H' = 0.0$, inertia moment of the cross-section $I = 10^{-4}$ m^4, yield moment $M_Y = 0.16$ MNm. The viscoelastic response of the foundation is described by the standard three-parameter solid, Fig. 1. We have assumed the following material parameters: $E_1 = 60.0$ MPa/m, $E_2 = 30.0$ MPa/m, $\eta_2 = 10^3$ MPa·hr (hour), $H' = 10.0$ MPa/m, $\sigma_Y = 0.15$ MPa/m. For the purposes of the numerical examples presented here the values of the material parameters of the viscoelastic-plastic models are chosen to represent properties of a generic material (different for the beam and for the foundation) rather than a specific material. The gap between the beam and the foundation is modelled as clearance strains with the modulus l^+ equally distributed under the beam (l^- is assumed to be a sufficiently great number).

Example 1.

As the first example we consider a simply supported beam, shown in Fig. 2, made of different viscoelastic materials and loaded with the point force $P = P(t)$ whose time evolution is depicted in Fig. 2a. We assumed the following material parameters $E_1 = E_2 = E = 2.0 \cdot 10^5$ MPa, $\eta_1 = \eta_2 = \eta = 4.0 \cdot 10^5$ MPa·hr, which are used in the governing differential equations of the following viscoelastic models [37], for the standard model see (13):

- the Maxwell model

$$\frac{1}{\eta}\sigma + \frac{1}{E}\dot{\sigma} = \dot{\epsilon}^{ve}, \tag{39}$$

- the Kelvin-Voigt model

$$\sigma = E\,\epsilon^{ve} + \eta\,\dot{\epsilon}^{ve}, \tag{40}$$

- the Burgers model

$$\sigma + \left(\frac{\eta_1}{E_1} + \frac{\eta_1}{E_2} + \frac{\eta_2}{E_2}\right)\dot{\sigma} + \frac{\eta_1\eta_2}{E_1E_2}\,\ddot{\sigma} = \eta_1\dot{\epsilon}^{ve} + \frac{\eta_1\eta_2}{E_2}\,\ddot{\epsilon}^{ve}. \tag{41}$$

Figure 2b shows the time evolution of the deflection of the midspan point for the different viscoelastic material models when the load P varies piecewise linearly with a jump at $t = 4.00$ hr. Note that in this case the piecewise linear time approximation furnishes the exact solution for any time step $\vartheta_\tau \leq 4.00$ hr, in particular in order to determine the states at $t = 4.00$ and $t = 8.00$ it suffices to take $\vartheta_1 = \vartheta_2 = 4.00$, and then we just solve *Problem I* at times 4.00, 8.00 and *Problem II* at $t = 4.00$. The same results we can obtain by employing the piecewise quadratic time approximation, but now at times $t = 0.00$ and $t = 4.00$ we have to additionally solve *Problem III*. However, if the evolution of P is piecewise quadratic, then the other time approximation furnishes the exact solution to the whole deformation process which is illustrated in Fig. 2c for the characteristic midspan deflection. In other words, with the piecewise quadratic approximation we can determine exactly the state of the beam at $t = 4.00$ and $t = 8.00$ by taking $\vartheta_1 = \vartheta_2 = 4.00$ and solving *Problem I* at these times and *Problems II* and *III* at $t = 4.00$. Moreover, it is interesting to remark that the deflection $u(t)$ is a non-monotone function for times $t > 4.00$ while the evolution of the load is monotone. This effect is even more pronounced in case of the quadratic load history.

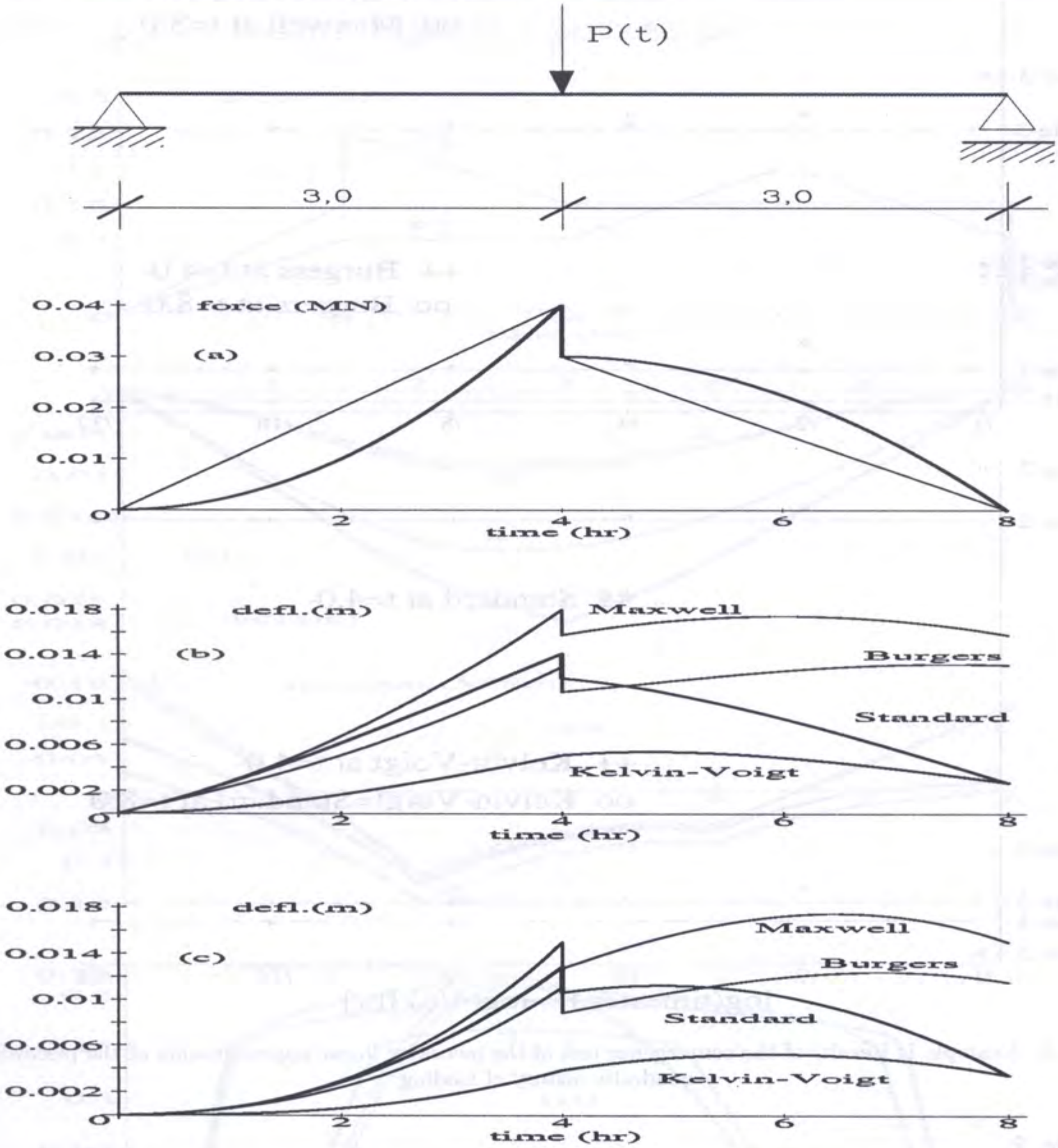

Fig. 2. Example 1. Simply supported beam subjected to force $P(t)$. (a) Piecewise linear (thin line) and piecewise quadratic (thick line) history of P; Deflections of the midspan of the beam for different viscoelastic models under the piecewise linear (b), and piecewise quadratic (c) evolution of P.

Figure 3 shows the convergence results of the midspan deflection of the beam for the piecewise linear approximation of the quadratic evolution of loading $P(t)$. The time step ϑ_i is varying (but being constant along the time interval [0.00,8.00]) from $\vartheta_0 = \bar{\vartheta} = 4.00$ hr to $\vartheta_5 = 0.125$ hr by the formula $\vartheta_i = \bar{\vartheta}/2^i$ with $i = 0, 1, \ldots, 5$. (On the abscissa of Fig. 3 the locations of $\log \vartheta_i$ are marked merely with the coefficients $1/2^i$.)

As can be seen the piecewise linear variant of the time integration method exhibits a very good convergence rate, since the error is from about 0.6 % for the standard model to 2 % for the Kelvin-Voigt model at $t = 4.00^-$, when the basic time step is divided into only four steps, i.e. $\vartheta_i = \bar{\vartheta}/4 = 1.00$ hr. Due to the particular changes of load $P(t)$ the errors at $t = 8.00$ are even smaller.

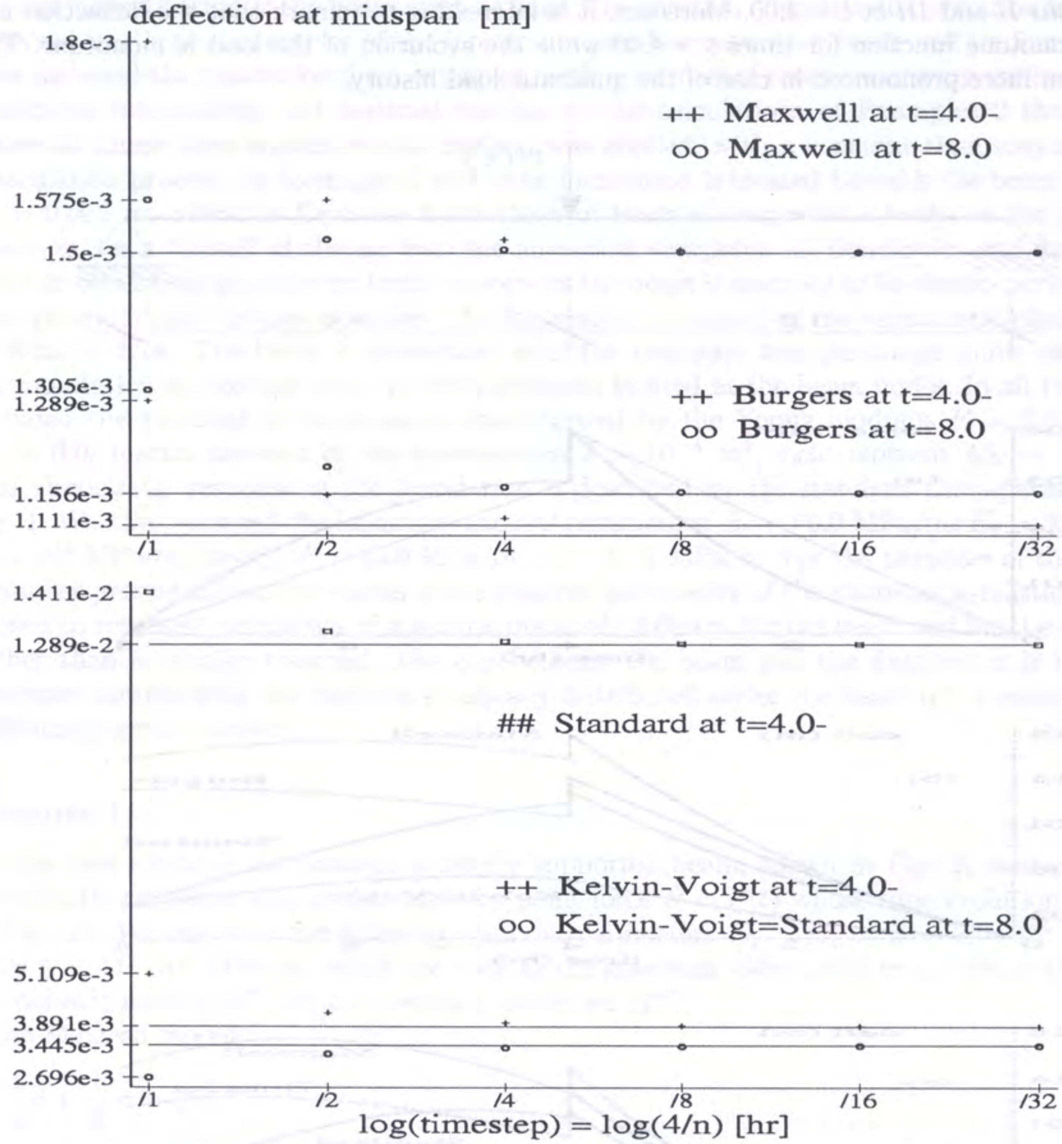

Fig. 3. Example 1. Results of the convergence test of the piecewise linear approximation of the piecewise quadratic history of loading

Example 2.

In this example we consider the beam and the foundation schematically shown in Fig. 4 for the following combinations of material properties and changes in boundary conditions of the beam.

Case 1: elastic beam / viscoelastic foundation;

Case 2: elastic-plastic beam / viscoelastic foundation;

Case 3: elastic-plastic beam / viscoelastic-plastic foundation;

Case 4: Case 3 plus clearances at supports, $l^- = l^+ = 0.01$ rad;

Case 5: Case 4 plus clearance at midspan, $l^- = l^+ = 0.01$ rad.

The beam is subjected to two point loads P_1 and P_2 whose variation in time is shown in Fig. 4a. Figures 4b, c and d illustrate the impact of the various material models and clearances on the distributions of beam's displacements, bending moments and contact pressure at time $t = 8.00^+$ hr. Remarkable differences can be observed in Cases 3, 4 and 5; these are caused by plastic deformation of the foundation and the clearances in the beam. Notice the smaller contact pressures in Case 5 than in Case 4, although the corresponding displacements are larger. Figure 4e shows the distributions of foundation pressure after partial unloading by decreasing force P_1.

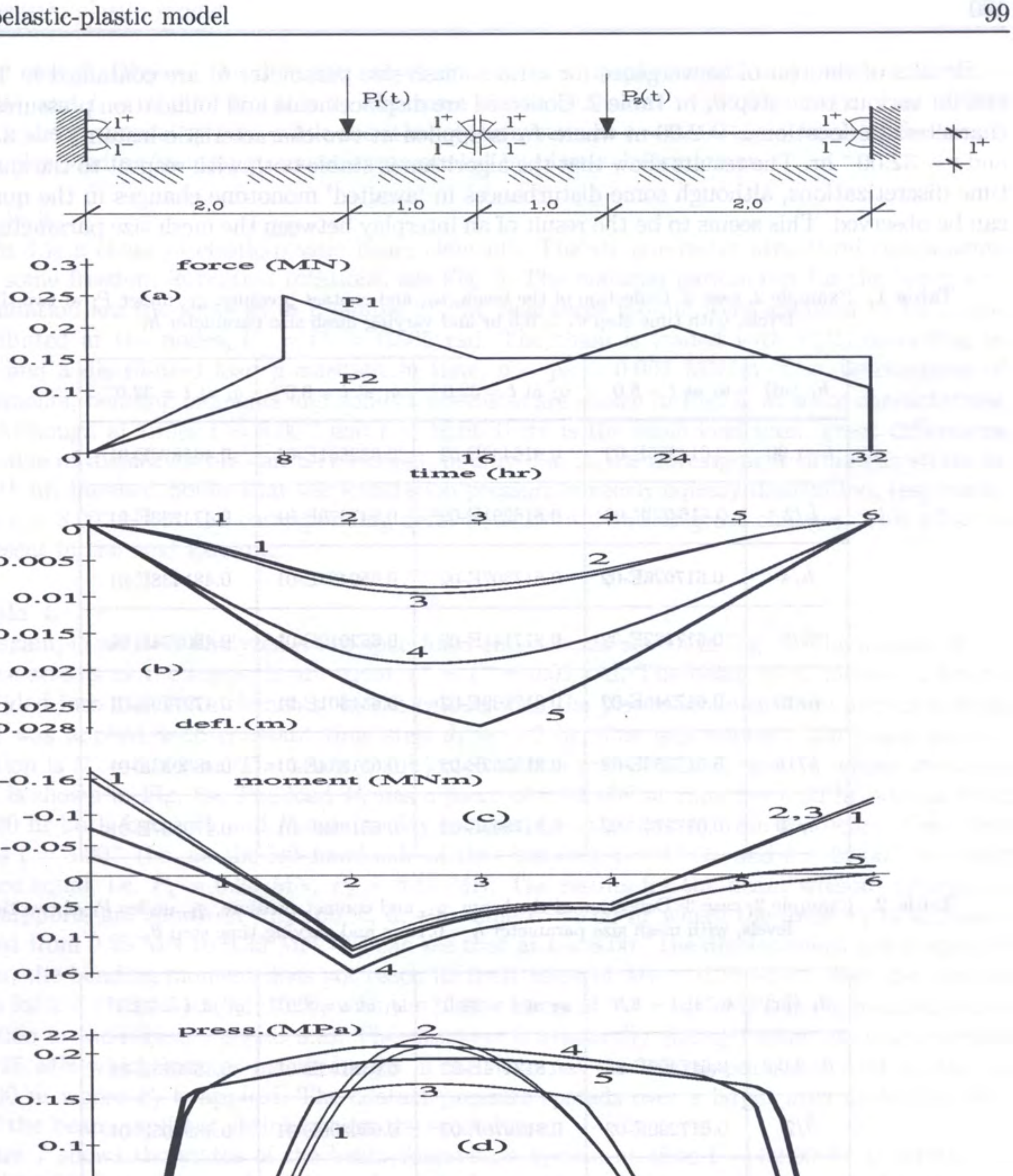

Fig. 4. Example 2. Fixed-ended beam with rotation clearances and its foundation, for five cases of material models and clearances (see text). (a) History of forces P_1 and P_2; (b) Deflections of the beam at $t = 8.00^+$ hr, (c) Bending moments at $t = 8.00^+$ hr, (d) Foundation pressure at $t = 8.00^+$ hr, (e) Foundation pressure at $t = 16.00$ hr

Results of the test of convergence for various mesh size parameter h_i are contained in Table 1, and for various time step ϑ_i in Table 2. Collected are displacements and foundation pressures at the characteristic location $x = 2.00$ m where P_1 is applied at two characteristic instants $t = 8.00^-$ hr and $t = 32.00^-$ hr. The results show that the algorithm is stable both with respect to the space and time discretizations, although some disturbances in 'awaited' monotone changes in the quantities can be observed. This seems to be the result of an interplay between the mesh size parameter h and

Table 1. Example 2, case 3. Deflection of the beam, u_1, and contact pressure, q_1, under P_1 at two time levels, with time step $\vartheta_\tau = 0.5$ hr and varying mesh size parameter h_i.

h_i (m)	u_1 at $t = 8.0^-$	u_1 at $t = 32.0^-$	q_1 at $t = 8.0^-$	q_1 at $t = 32.0^-$
$\bar{h}$=1.00	0.614295E-02	0.816433E-02	0.632661E-01	0.461876E-01
$\bar{h}/2$	0.616460E-02	0.816291E-02	0.647379E-01	0.474238E-01
$\bar{h}/4$	0.617026E-02	0.817307E-02	0.650467E-01	0.481438E-01
$\bar{h}/8$	0.617282E-02	0.817141E-02	0.652012E-01	0.480274E-01
$\bar{h}/10$	0.617246E-02	0.817169E-02	0.651801E-01	0.479779E-01
$\bar{h}/16$	0.617256E-02	0.816962E-02	0.651854E-01	0.480033E-01
$\bar{h}/20$	0.617272E-02	0.817092E-02	0.651958E-01	0.479907E-01

Table 2. Example 2, case 3. Deflection of the beam, u_1, and contact pressure, q_1, under P_1 at two time levels, with mesh size parameter $h = 0.10$ m and varying time step ϑ_i.

ϑ_i (hr)	u_1 at $t = 8.0^-$	u_1 at $t = 32.0^-$	q_1 at $t = 8.0^-$	q_1 at $t = 32.0^-$
$\bar{\vartheta}$=2.00	0.617407E-02	0.812474E-02	0.648193E-01	0.487913E-01
$\bar{\vartheta}/2$	0.617290E-02	0.812970E-02	0.650890E-01	0.489005E-01
$\bar{\vartheta}/4$	0.617246E-02	0.817169E-02	0.651801E-01	0.479779E-01
$\bar{\vartheta}/8$	0.617230E-02	0.824082E-02	0.651860E-01	0.468053E-01
$\bar{\vartheta}/16$	0.617225E-02	0.829712E-02	0.651906E-01	0.460202E-01
$\bar{\vartheta}/32$	0.617224E-02	0.834231E-02	0.651906E-01	0.453424E-01
$\bar{\vartheta}/64$	0.617223E-02	0.837234E-02	0.651907E-01	0.448610E-01
$\bar{\vartheta}/128$	0.617223E-02	0.839269E-02	0.651908E-01	0.445972E-01
$\bar{\vartheta}/256$	0.617223E-02	0.840495E-02	0.651908E-01	0.443976E-01

the time step ϑ. Observe the different convergence rates of displacements and pressure at times $t = 8.00^-$ and $t = 32.00^-$ in Table 2. The main reason is that at $t = 8.00^-$ there is no plastic strains in the structural system, while the state of the system at $t = 32.00^-$ is a result of the longer and complex loading/unloding process accompanied by plastic deformations.

Example 3.
Example 3 is a chain of elastic-plastic beam elements. The six one-meter structural components possess some freedom in mutual rotations, see Fig. 5. The material parameters for the beam and the foundation are the same as in Example 2. The clearances moduli were assumed to be equally distributed at the nodes, $l^- = l^+ = 0.005$ rad. The chain is loaded with $P_1(t)$ according to Fig. 4a and a distributed load p constant in time, $p = p_0 = 0.002$ MN/m. The distributions of displacements, bending moments and contact pressures are shown in Fig. 5, at some characteristic times. Although at times $t = 8.00^-$ and $t = 16.00$ there is the same load level, great differences in the value of displacements can be observed, mainly due to the development of plastic strain at $t = 8.00^+$ hr. Further, notice that the foundation pressure is nearly equally distributed, (especially at time $t = 8.00^-$), but the corresponding deflections exhibit rather great changes. This effect is also present in the next example.

Example 4.
In this example we have analyzed a two-span fixed-ended beam shown in Fig. 6. The moduli of the clearance strains at the supports are equal, $l^+ = l^- = 0.01$ rad. The beam of 12 meters in length was divided into 120 finite elements. In the calculations the piecewise linear time approximation method was applied with constant time step $\vartheta_\tau = 1.0$ hr. The gap between the beam and its foundation is $l^+ = 0.005$ m. The beam is subjected to two point loads P_1 and P_2 whose variation in time is shown in Fig. 6a. The load P_1 has a jump of 0.10 MN at time $t = 8.00$ hr, and at time $t = 32.00$ hr both forces P_1 and P_2 are quickly removed (another jump) from the beam. Note that at times $t = 8.00^-$ (i.e. on the left-hand side of that instant), $t = 16.00$, and $t = 32.00^-$ the load levels are equal, i.e. $P_1 = 0.25$ MN, $P_2 = 0.15$ MN. The results for the beam without clearances at the supports are shown in Figs. 6b, c, d, e at time $t = 8.00$ hr where the force P_1 is suddenly increased from 0.25 MN to 0.35 MN. We can see that at $t = 8.00^-$ the displacement u is a smooth function, the bending moment does not reach its limit value of $M_Y = 0.16$ MNm, and the contact appears for $x \in (1.6, 4.5) \cup (8.1, 10.0)$, with the plastic stress of $\sigma_Y = 0.15$ MPa being reached in the foundation in the region $x \in (2.3, 3.8)$. The situation is drastically changed after the load increase from 0.25 MN which induces plastic hinges in the beam, at the left support, $x = 0.00$ m, and at $x = 3.00$ m where P_1 is applied. The contact pressure spreads over a larger area under the first span of the beam, whilst it shrinks under the second one.

Figure 7 shows the states of the beam/foundation system at time $t = 32.00$ hr at which the external load is instantaneously removed from the beam. Owing to the previous plastic deformation of the beam at its supports and beneath forces P_1 and P_2, there are some residual deformations in the beam, see Fig. 7a, which are accompanied by the residual bending moments, Fig. 7d. At $t = 32.00^+$ the foundation exhibits both permanent deformations, thick line in Fig. 7b and some viscoelastic ones which will eventually disappear in the case of the tree-parameter solid model we used. That the history of the deformation process plays a great role is again demonstrated in Fig. 7c, where the distributions of the foundation pressure at the same load level, $P_1 = 0.25$ MN, $P_2 = 0.15$ MN, but at different times can be compared. Note that under the first span of the beam the pressure is nearly equally distributed at time $t = 32.00^-$, as before in Example 3. The residual moment in the beam with clearances is different from that without clearances, compare case (d) and (c) in Fig. 7.

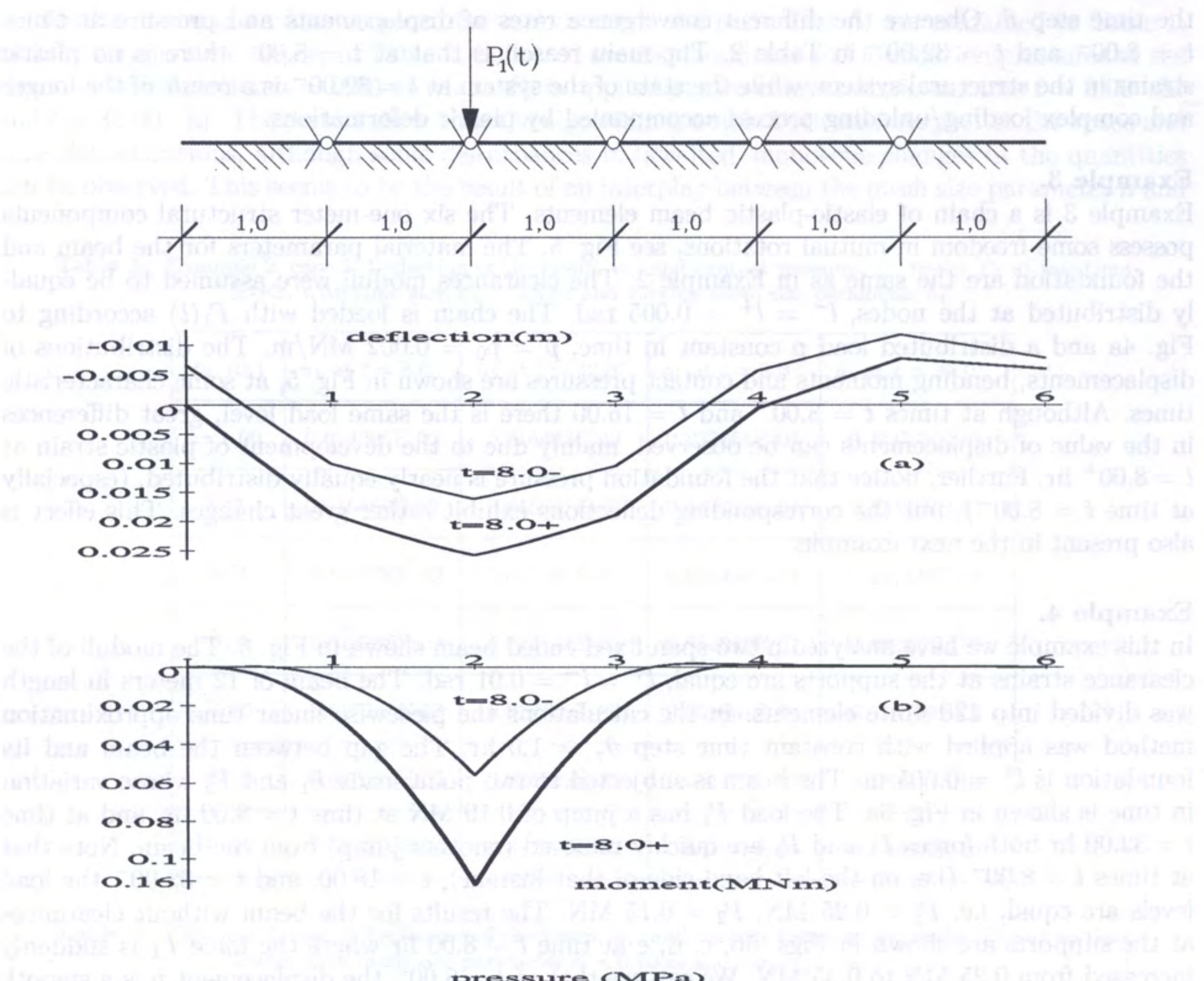

Fig. 5. Example 3. Chain of beam elements with rotation clearances resting on foundation, loaded with its dead load $p = p_0 = 0.002$ MN/m and the point load $P_1(t)$ with history shown in Fig. 4a; (a) Deflections, (b) Bending moments, and (c) Foundation pressures at the left and the right hand-side of instant $t = 8.00$ hr. (e) Deflections at the same load level but at different times: $t = 8.00^-, t = 16.00, t = 32.00^-$ hr

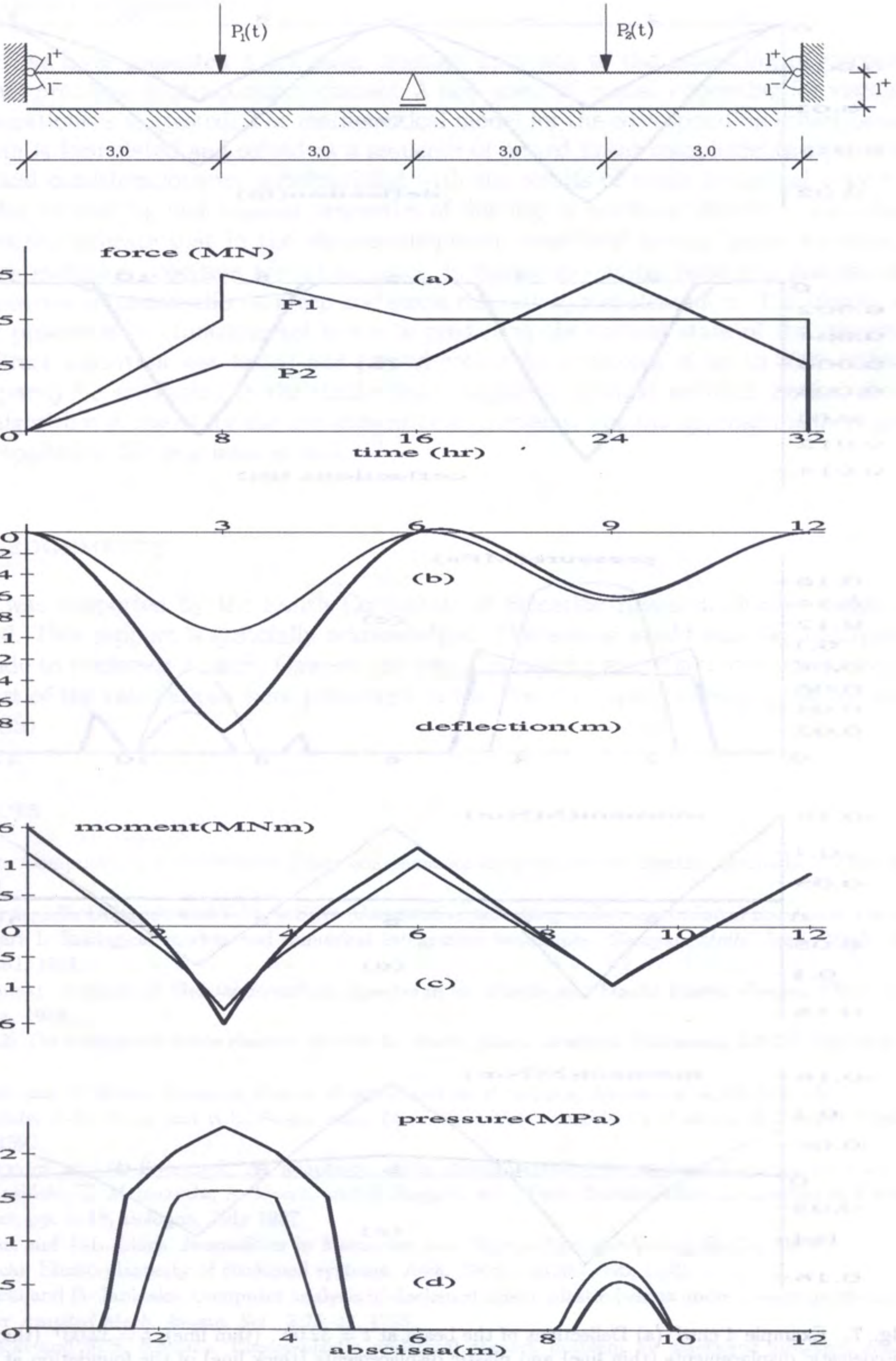

Fig. 6. Example 4. Two-span fixed-ended beam with rotation clearances and its foundation. (a) History of forces P_1 and P_2, (b) Deflections of the beam, (d) Bending moments and (e) Foundation pressures at the left (thin line) and the right (thick line) hand-side of instant $t = 8.00$ hr, for the beam without clearances at the supports, $l^- = l^+ = 0$

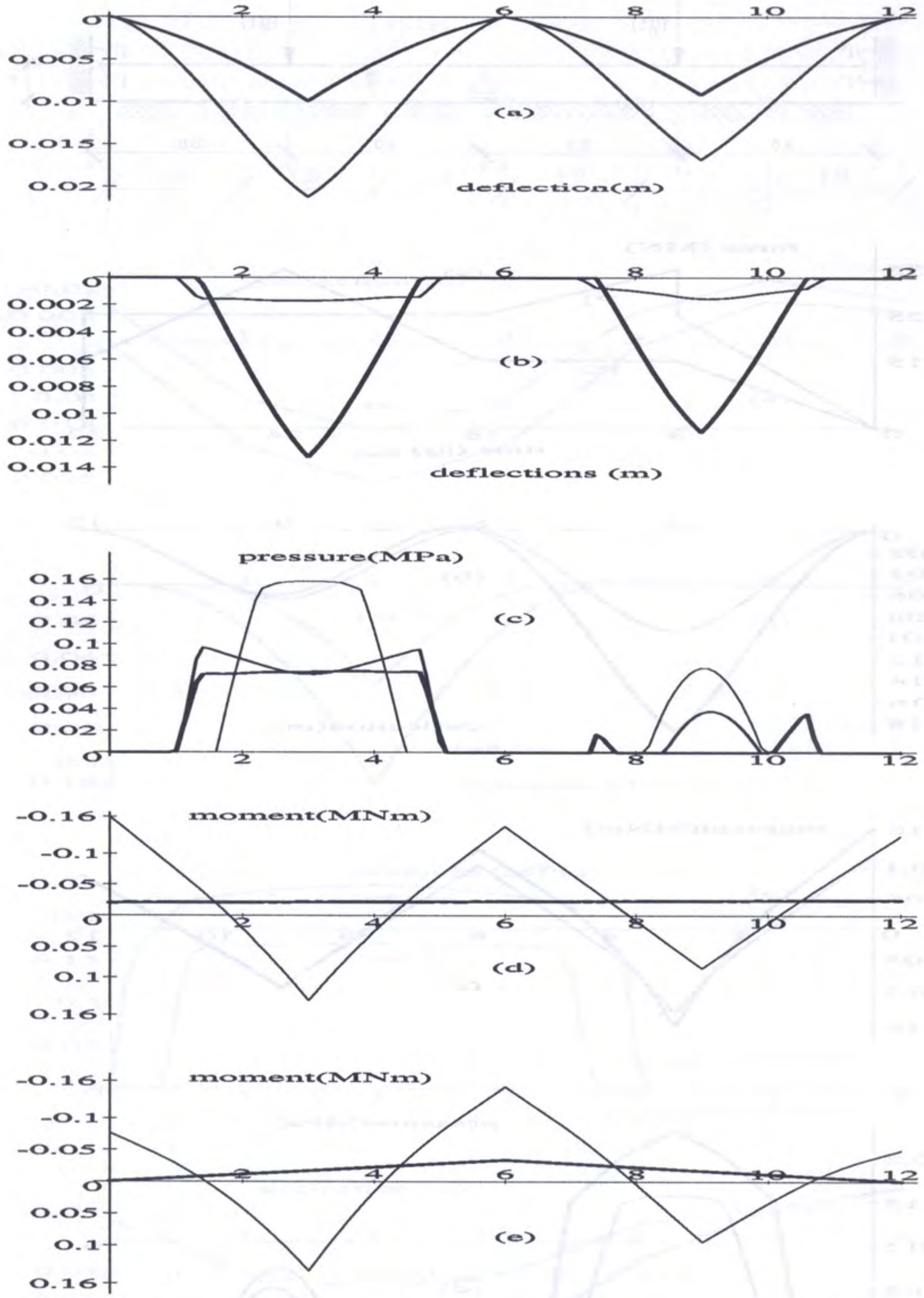

Fig. 7. Example 4 cntd. (a) Deflections of the beam at $t = 32.00^-$ (thin line), $t = 32.00^+$ (thick line). (b) Viscoelastic displacements (thin line) and plastic displacements (thick line) of the foundation at $t = 32.00^+$ hr. (c) Distribution of foundation pressure for the same level of loads at different times: $t = 8.00^-$ hr (thin line), $t = 16.00$ hr (medium line), $t = 32.00^-$ (thick line). Distribution of bending moments at $t = 32.00^-$ (thin line), $t = 32.00^+$ (thick line), for the beam without (d) and with (e) clearances at the supports

7. Concluding remarks

In the paper we have presented a computer-oriented approach to the modelling of viscoelastic-plastic contact problems in structural mechanics. A new material model accounting for viscous and plastic deformations is suggested. The mathematical model for the corresponding initial boundary value problem is formulated and solved as a sequence of nested linear complementarity problems. The theoretical considerations are substantiated with the results of many numerical experiments which revealed interesting and unusual properties of this highly nonlinear problem. The obtained numerical results indicate that in the viscoelastic-plastic structural system under variable loads some complex evolution processes are taking place, including unloading, reloading and redistribution of stresses due to plastic effects, creep and stress relaxation, and clearances. The history of the deformation process is the fundamental factor in predicting the current state of the system. The developed direct algorithm was tested and proved robust for problems of up to 2000 unknowns, which may partly be attributed to the stable time integration method we used. So far, the computational algorithm is tested for the one-dimensional problems, but the approach is more general and can be applied to 2D problems as well.

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