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Root finding method for problems of elastodynamics

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This paper presents a simple and efficient method for finding complex roots of dispersion equations occurring in many problems of elastodynamics. The method is characterized by high accuracy in root finding and absence of restrictions on function representation. The essence of the method is explained geometrically; initial guesses are found as the solutions to the appropriate problems of elastostatics. Numerical solutions to dispersion equations are obtained for two elastic isotropic waveguides: a plate of infinite cross-section and a rod of rectangular cross-section. For an infinite plate, the calculated results are in full conformity with those obtained by Newton-Raphson and bisection methods. For a waveguide of rectangular cross-section, the earlier unsolved problem of finding complex roots of dispersion equations is solved by the proposed method.

1. Introduction

Study of steady-state waves in elastic waveguides and vibrations of solids of finite dimensions necessarily involves the solving of the transcendental dispersion equations. A dispersion equation governs a relationship between frequency, ω, and propagation constant, γ ($\gamma = 2\pi/\lambda$, λ is the wavelength), for guided waves in an elongated medium

$$F(\omega, \gamma) = 0. \tag{1}$$

For an elastic medium, ω is always real while γ takes real, imaginary and complex values. For every value of frequency Eq. (1) has an infinite set of pointlike solutions γ corresponding to propagating (real γ) and non-propagating (imaginary and complex γ) modes. Solutions to the dispersion equation are mapped on an ω versus γ plot in a multitude of dispersion curves (modes). Each curve originates from $\omega = 0$ and extends to infinity.

Root finding for dispersion equations is an intractable problem, even in spite of the advanced development of computational techniques and methods. This is caused by both double infinities (the number of modes and a theoretically infinite frequency range) and complexity of Eq. (1) that often cannot be written in a closed analytical form. In engineering applications, a frequency range is usually restricted to finite interval $[0, \hat{\omega}]$. However, a huge amount of computations should be performed to find all solutions γ_i ($i = 1, 2, \ldots, N$) to Eq. (1) for every value of ω in $[0, \hat{\omega}]$. To solve this problem, iterative techniques based on linear interpolation [10] or extrapolation algorithms [9] are usually used. They are very fast in finding a single root, but when two roots are in a close proximity, such schemes become unstable [7]. Another approach to determination of the roots of dispersion equations is based on slower but safer techniques, such as Newton-Raphson or bisection [13]. However, application of these methods is complicated when the function F in Eq. (1) is represented in a non-closed form [14], for example, as the determinant of an infinite system [8].

For complex values of γ Eq. (1) can be written as a system of two nonlinear equations

$$f(\xi, \eta) = 0\,, \qquad g(\xi, \eta) = 0\,, \tag{2}$$

where $\gamma = \xi + i\eta$, $f = \mathrm{Re}\,[F(\omega^*, \gamma)]$, $g = \mathrm{Im}\,[F(\omega^*, \gamma)]$ for a given value of ω^*. In general, determination of the roots of Eqs. (2) is a complicated task always requiring additional information specific to the problem under consideration. While solving a dispersion equation, as such information we can use initial guesses at $\omega = 0$ corresponding to solutions of an appropriate static problem. If, in addition to good initial guesses, we have analytical expressions for functions and their derivatives, the Newton-Raphson or bisection methods are rather efficient. More complicated algorithms have a reputation of robustness even with a bad initial guess [14], but are rather difficult and time-consuming.

This paper addresses an alternative method for solvin the system (2). The method successfully combines the simplicity of realization with the absence of restrictions on the representation of dispersion equations. Being a certain combination of the coordinate-wise step and bisection methods, the proposed method is characterized by a local convergence and requires a sufficiently good trial solution. As initial guesses the solutions to elastostatic problems are used. The method permits convenient graphical interpretation and is more stable and faster in finding a single root than the traditional techniques. The rate of convergence of the method has the lowest values at the dispersion curve bends. The proposed method is rather efficient for the solving of the dispersion equations in many elastodynamic problems.

2. Description of the method

The solutions to Eqs. (2) are common points to the zero contours of functions f and g, which divide the (ξ, η) plane into regions with positive and negative values of these functions. The idea of the proposed method consists in the following. Starting from an initial guess and analyzing the behavior of the zero contours, a solution is improved until a predetermined convergence criterion is satisfied. The root finding process can be most conveniently explained geometrically.

Let (ξ_0, η_0) be a solution to Eqs. (2) at $\omega^* = \omega - h_\omega$, where h_ω denotes a step chosen along $[0, \hat{\omega}]$. If h_ω is sufficiently small, then at $\omega^* = \omega$ zero contours of functions f and g have a common point in certain vicinity of (ξ_0, η_0) (see Fig. 1) according to the continuity property for dispersion curves [4]. Therefore, we can use (ξ_0, η_0) as an initial guess to solution (ξ_*, η_*) of Eqs. (2) at $\omega^* = \omega$. Thus, the method is locally convergent.

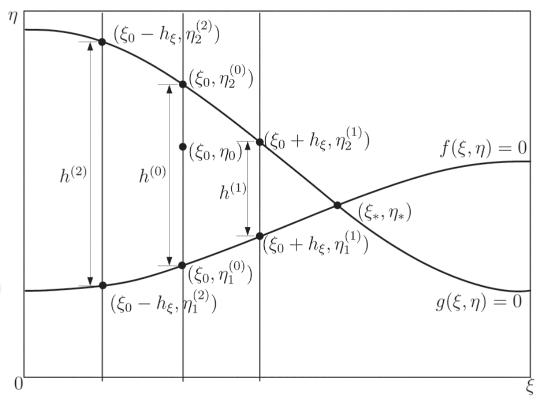

Fig. 1. Illustration of the method for finding the root (ξ_*, η_*) of Eqs. (2) with an initial guess (ξ_0, η_0)

Let h_ξ, h_η be the steps along the $O\xi$, $O\eta$ axes. Draw the line $\xi = \xi_0$ through (ξ_0, η_0). Then find the points of intersection $(\xi_0, \eta_1^{(0)})$ and $(\xi_0, \eta_2^{(0)})$ for this line with the zero contours by the change of sign for functions f and g while passing through the contours. Next, move from (ξ_0, η_0) both sides along the $O\xi$ axis and perform the described operations with $(\xi_0 + h_\xi, \eta_0)$ and $(\xi_0 - h_\xi, \eta_0)$. Finally, calculate the lengths of segments formed by three pairs of the points of intersection:

$$h^{(0)} = |\eta_2^{(0)} - \eta_1^{(0)}|\,, \qquad h^{(1)} = |\eta_2^{(1)} - \eta_1^{(1)}|\,, \qquad h^{(2)} = |\eta_2^{(2)} - \eta_1^{(2)}|\,. \tag{3}$$

If $h^{(1)} > h^{(0)}$, $h^{(2)} > h^{(0)}$ and $h^{(0)} \leq h_\eta$ in (3), then $(\xi_0, \frac{\eta_2^{(0)}+\eta_1^{(0)}}{2})$ is a solution to Eqs. (2) to within the accuracy $h = \max(h_\xi, h_\eta)$. To refine the solution, the values of h_ξ and h_η should be decreased.

If $h^{(2)} \geq h^{(0)} \geq h^{(1)}$ (or $h^{(1)} \geq h^{(0)} \geq h^{(2)}$) in (3), then the solution is located to the right (or left) of (ξ_0, η_0) along the $O\xi$-axis. Use $(\xi_0 + h_\xi, \frac{\eta_2^{(1)}+\eta_1^{(1)}}{2})$ (or $(\xi_0 - h_\xi, \frac{\eta_2^{(2)}+\eta_1^{(2)}}{2})$) as a next initial guess instead of (ξ_0, η_0). The described process is iterated to convergence.

The method must succeed that is guaranteed by the continuity property for dispersion curves [4]. If there are two or more roots in the vicinity of the initial guess, one of them will be found depending on the values of h_ξ and h_η. Changing h_ξ and/or h_η the remaining roots can be found.

In spite of the fact that in general the method is rather fast, there exists a 'pathological' input of data that causes extremely slow convergence. Such cases can be easily recognized by fulfilment of one of the conditions $|\eta_1^{(0)}| >> |\eta_0|$ or $|\eta_2^{(0)}| >> |\eta_0|$ and demand special consideration. Graphically, this means that the zero contours are parallel or almost parallel to the coordinate axes. In this case lines $\xi = \eta$ instead of $\xi = const$ should be used.

It only remains to discuss practical criteria for determination of initial guesses at the first iteration when $\omega = 0$. These are the solutions to appropriate static problems. Since the zero contours of functions f and g may be located arbitrarily, initial guesses should be sufficiently good to provide a local convergence of the method. If the governing equation of the static problem can be written analytically. Then the Newton-Raphson, bisection or other methods can be used for root finding. In more complicated cases the most straightforward way is to map out the zero contours of both functions and to find the roots as points of intersection of the contours.

3. Application of the method

To illustrate the capabilities of the proposed method, we apply it to the problem about determination of complex roots of dispersion equations. Let us consider harmonic longitudinal waves in two elastic isotropic waveguides: an infinite plate and a waveguide of rectangular cross-section (a rectangular waveguide).

Dispersion equation for an infinite plate is attributed originally to Lord Rayleigh (1888). This is the fundamental equation, the roots of which are also eigenvalues associated also with vibrations of circular disks [1] or rectangular rods [12]. Solutions to this equation were obtained and well studied in the sixties of the past century [11, 13]. Here complex roots of the equation are calculated to illustrate the accuracy of the proposed method.

In a rectangular waveguide the presence of two pairs of boundary faces leads to considerable complication of the wave propagation process compared to that in an infinite plate. It is known [4, 5, 12] that for a rectangular waveguide with arbitrary side ratio the exact solution in a closed form cannot be constructed. As a result, various approximate and numerical approaches were developed to derive the dispersion equation in this case. The most powerful tool for performing the guided wave analysis in a rectangular waveguide is an analytical method of superposition [4, 8]. The use of this method permits to find the dispersion relation as the determinant of an infinite system of equations. Study of asymptotic behavior of unknowns in the system enables developing effective reduction methods for such systems [4] providing reliable results in a wide frequency range. In this paper complex roots of the reduced systems for longitudinal waves are calculated by means of the proposed method. To the author's knowledge, similar results are known only for a static problem for a rectangular rod [15], whereas for a dynamic problem there are no reliable solutions in the literature.

3.1. Infinite plate

Let us consider an elastic plate $-b \le x \le b$, $-\infty < y, z < \infty$ made of homogeneous isotropic material that is characterized by density ρ, shear modulus G, and Poisson's ratio ν, or by velocities of compressional $c_1 = \sqrt{2(1-\nu)/(1-2\nu)}c_2$ and shear $c_2 = \sqrt{G/\rho}$ waves. Plane waves propagating in the positive z direction are assumed to be harmonic, that is $\mathbf{U}(x,z) = \mathbf{u}(x)\exp(i(\gamma z - \omega t))$. A solution to the Lamé equations of motion describing small motions of the elastic medium can be obtained in terms of the scalar and vector potentials by the method of separation of variables for waves symmetric and antisymmetric with respect to the middle plane of the plate $x = 0$ [2, 11]. Further we will discuss only symmetric (longitudinal) waves. Satisfaction of the zero stress boundary conditions at the traction-free faces $x = \pm b$ leads to the following dispersion equation (further details on the solution process can be found elsewhere [2, 4]):

$$F(\gamma,\Omega) = (2\gamma^2 - \Omega^2)\cos\frac{\pi\alpha}{2}\sin\frac{\pi\beta}{2} + 4\alpha\beta\gamma^2\sin\frac{\pi\alpha}{2}\cos\frac{\pi\beta}{2} = 0\,. \tag{4}$$

Here $\alpha^2 = \Omega^2/k^2 - \gamma^2$, $\beta^2 = \Omega^2 - \gamma^2$, $k^2 = 2(1-\nu)/(1-2\nu)$, $\Omega = 2\omega b/\pi c_2$ is the nondimensional frequency.

An efficient method for the analysis of the roots of Eq. (4) and their roots was proposed almost simultaneously by Mindlin [11] and Holden [6]. A large amount of complex roots of Eq. (4) was calculated by the secant method [13]; asymptotic formulae describing behavior of complex roots with large magnitudes were derived by Zlatin [17]. It was also shown [11] that in a plate at any frequency ω, there exists a finite number of waves with real and imaginary values of γ and an infinite number of waves with complex propagation constants.

Transcendental equations like Eq. (4) are solved quite easily with the assistance of modern computing devices, even for complex roots. We consider the dispersion equation for an infinite plate in order to both verify the proposed method and analyze properties of dispersion curves that are rather similar for all elastic waveguides.

Solutions to the appropriate static problem for an infinite plate are used as initial guesses for complex roots of Eq. (4). If in Eq. (4) Ω is allowed to approach zero in such a way that γ remains finite, the following equation is easily obtained [4]:

$$F(\gamma,0) = \sinh\pi\gamma + \pi\gamma = 0\,. \tag{5}$$

A characteristic feature of this equation is the independence of its roots from Poisson's ratio ν. The values of the complex roots of Eq. (5) can be found elsewhere [4, 13, 17]. The similar equation for

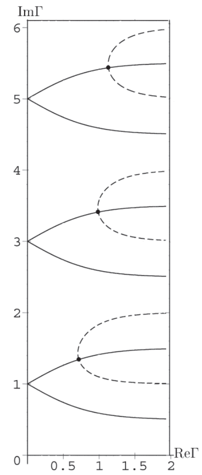

Fig. 2. Zero contours for $F(\gamma, 0)$ in Eq. (5)

the static problem for an anisotropic elastic plate was derived and studied in detail by Wojnar [16].

Zero contours for the functions in the right-hand side of Eq. (5) are presented in Fig. 2. Contours $\mathrm{Re}\,[F(\Gamma,0)] = 0$ are indicated by solid lines and contours $\mathrm{Im}\,[F(\Gamma,0)] = 0$ by dashed lines. Here and in the following nondimensional propagation constant $\Gamma = 2\gamma b/\pi$ is used for convenience. The relatively simple layout of the zero contours explains the simplicity of the root finding process.

Table 1 presents the first three complex roots of Eq. (4) for various values of frequency Ω at Poisson's ratio $\nu = 0.31$. As the frequency increases from zero to a certain finite value, these

Table 1. Complex branches of dispersion curves for longitudinal waves in an infinite plate, $\nu = 0.31$

Ω	The first branch	The second branch	The third branch
0.20	$0.7174{+}1.3315i$	$0.9881{+}3.4061i$	$1.1305{+}5.4323i$
0.60	$0.7248{+}1.2438i$	$0.9909{+}3.3757i$	$1.1321{+}5.4134i$
1.00	$0.7325{+}1.0499i$	$0.9958{+}3.3140i$	$1.1348{+}5.3755i$
1.40	$0.6951{+}0.6834i$	$1.0015{+}3.2191i$	$1.1382{+}5.3180i$
1.73	$0.5414{+}0.1855i$	$1.0050{+}3.1135i$	$1.1407{+}5.2552i$
3.50		$0.7678{+}1.9236i$	$1.1022{+}4.6462i$
3.86		$0.2402{+}1.4069i$	$1.0620{+}4.4527i$
4.60			$0.8585{+}3.9460i$
5.15			$0.2061{+}3.4272i$

roots form complex branches of dispersion curves for longitudinal waves in the infinite plate. The comprehensive analysis of dispersion equation (4) implemented in [4] shows that complex branches join at a minimum of the real branch and at a maximum of the complex one; their intersections are orthogonal. Analogous statement is held for imaginary and complex branches of the curves. Complex branches are also orthogonal to the $\Omega = 0$ plane. The performed calculations completely confirm these statements, and moreover, the continuity and smoothness of the complex branches. The same properties inhere in antisymmetrical waves in plates and, in general, in all types of harmonic waves in elastic waveguides of various cross-sections. The knowledge of these properties enables considerable simplification of the problem of finding complex roots and construction of dispersion curves for waveguides of finite cross-section, for example, for a rectangular waveguide.

Numerical results obtained by the proposed method at $\nu = 0.30$ for symmetric and antisymmetric waves in the infinite plate agree within the accuracy of 10^{-4} with the data given in [13].

3.2. Rectangular waveguide

Let us consider harmonic waves with displacement vector $\mathbf{U}(x,y,z) = \mathbf{u}(x,y)\exp(i(\gamma z - \omega t))$ in a rectangular elastic waveguide $-a \le x \le a$, $-b \le y \le b$, $-\infty < z < \infty$ with the same material properties as for the considered infinite plate. Dispersion equations for waves in the rectangular waveguide can be derived analytically by means of the superposition method [4, 8]. The principal idea of the method consists in using two ordinary Fourier series in terms of complete systems of trigonometric functions in the coordinates x and y. Both series satisfy identically the Lamé equations of motion within the rectangular cross-section and have six sets of Fourier coefficients sufficient to implement the zero stress boundary conditions at the free waveguide faces. Because of interdependency each coefficient of one series depends on all coefficients of another series and vice versa. Therefore to get the solution to the problem one requires to solve an infinite system of linear algebraic equations [3]. For longitudinal waves symmetric with respect to the middle planes of the waveguide, an infinite system is written as follows:

$$
\begin{aligned}
&Y_k^L a\Delta_k^{(1)}(q)+\varepsilon_k\sum_{n=0}^{\infty}X_n^L b_n\left[\frac{2\alpha_n^2\beta_k^2}{\alpha_n^2+q_1^2}-\frac{2\alpha_n^2\beta_k^2}{\alpha_n^2+q_2^2}-\frac{\Omega_0^2\left(2\gamma^2-\Omega_2^2\right)}{\alpha_n^2+q_1^2}\right]=0,\\
&X_n^L b\Delta_n^{(1)}(p)+\varepsilon_n\sum_{k=0}^{\infty}Y_k^L c_k\left[\frac{2\alpha_n^2\beta_k^2}{\beta_k^2+p_1^2}-\frac{2\alpha_n^2\beta_k^2}{\beta_k^2+p_2^2}-\frac{\Omega_0^2\left(2\gamma^2-\Omega_2^2\right)}{\beta_k^2+p_1^2}\right]=0\,, \qquad k,n=0,1,2,\ldots,
\end{aligned}
\tag{6}
$$

where the notation

$$
\Delta_k^{(1)}(q)=c_k\left\{q_2\left(\gamma^2+\beta_k^2\right)\coth q_2a-\frac{\left(\gamma^2+\beta_k^2+q_2^2\right)^2}{4q_1}\coth q_1a\right\},
$$

$$
\Delta_n^{(1)}(p)=b_n\left\{p_2\left(\gamma^2+\alpha_n^2\right)\coth p_2b-\frac{\left(\gamma^2+\alpha_n^2+p_2^2\right)^2}{4p_1}\coth p_1b\right\},
$$

$$
\varepsilon_i=\begin{cases}\frac{1}{2}, & i=0;\\ 1, & i>0;\end{cases}\qquad c_k=\begin{cases}1, & k=0;\\ \frac{1}{\beta_k^2}, & k>0;\end{cases}\qquad b_n=\begin{cases}1, & n=0;\\ \frac{1}{\alpha_n^2}; & n>0;\end{cases}
$$

$$
\alpha_n=\frac{n\pi}{a},\qquad \beta_k=\frac{k\pi}{b},\qquad p_i^2=\alpha_n^2+\gamma^2-\Omega_i^2,\qquad q_i^2=\beta_k^2+\gamma^2-\Omega_i^2,\qquad i=1,2,
$$

$$
\Omega_1=\frac{\omega}{c_1},\qquad \Omega_2=\frac{\omega}{c_2},\qquad \Omega_0^2=\frac{\nu\Omega_1^2}{1-2\nu}.
$$

is introduced, and X_n, Y_k $(n,k=1,2,\ldots)$ are unknown coefficients. The only non-trivial solutions for X_n, Y_k are those, for which the determinant of system (6) is equal to zero. The equation formed by expanding the determinant is the required dispersion equation, which for a given value of a/b relates ω to γ with Poisson's ratio ν as a parameter.

In a square waveguide owing to the diagonal symmetry of the cross-section, longitudinal waves are divided into pure longitudinal L modes with displacements symmetric relative to the diagonals, $u_x(x,y)=u_y(y,x)$, and the first screw S_1 modes that are antisymmetric relative to the diagonals, $u_x(x,y)=-u_y(y,x)$ [5]. In this case the number of unknowns as well as the number of equations in system (6) is halved, since $X_i=-Y_i$ for L-modes and $X_i=Y_i$ for S_1-modes, where $i=0,1,2,\ldots$.

For proper reduction of infinite system (6) to a finite one an important role plays the law of asymptotic behavior of the unknowns with large values of indices [4]:

$$
\lim_{n\to\infty}X_n=\lim_{k\to\infty}Y_k=A, \tag{7}
$$

where A is as yet unknown, in general, non-zero constant depending on frequency ω. A detailed description of the reduction procedure and determination of A are given in [4]. Here it is only worth to note that the solution of the finite system obtained by taking into account relation (7) provides the knowledge of all coefficients X_n and Y_k. This enables considerable increasing of the accuracy in finding the roots of the dispersion equation.

Numerical evaluation of the roots of Eq. (6) is carried out for a square waveguide. Real and imaginary roots were calculated and analyzed in [3, 8]. Here a special attention is paid to complex roots that, to our best knowledge, were not studied earlier.

Solution to the appropriate static problem for a rectangular rod was obtained recently by means of a finite element-transfer matrix procedure [15]. It was established that complex roots at $\Omega=0$ depend on Poisson's ratio ν as well as on sides ratio a/b. The complicated layout (see Fig. 3) of zero contours makes the root finding process very difficult. To determine complex roots at a non-zero frequency we use the solutions given in [15] at $\nu=0.25$. For other values of ν the initial guesses were determined as points of intersection of corresponding zero contours.

Complex roots of Eq. (6) for various frequencies Ω are presented by complex branches of dispersion curves in Fig. 4 for pure longitudinal and screw modes in a square waveguide at different

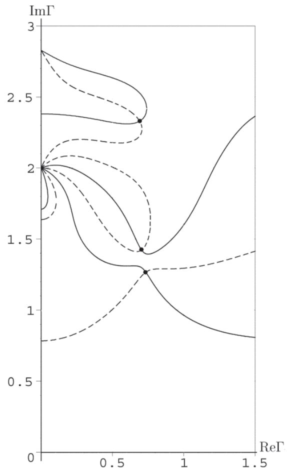

Fig. 3. Zero contours of the function entering the dispersion equation (6) for a square waveguide at $\Omega = 0$, $\nu = 0.25$

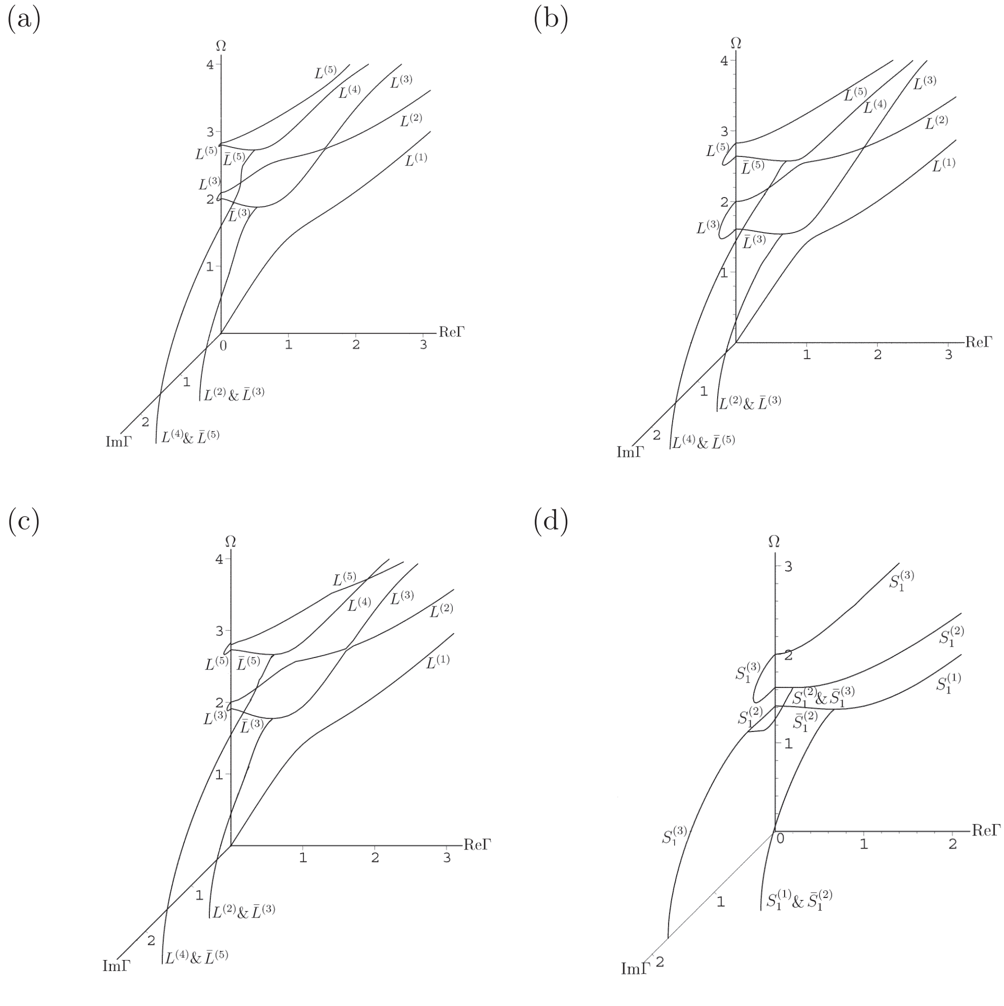

Fig. 4. Dispersion spectra for pure longitudinal L and screw S_1 modes in a square waveguide: (a) L-modes at $\nu = 0.3$, (b) L-modes at $\nu = 0.125$, (c) L-modes at $\nu = 0.25$,(d) S_1-modes at $\nu = 0.25$

values of ν. Each dispersion curve is designated by the symbol according to the symmetry type with the superscript in parenthesis representing the order of the mode. A bar over the symbol indicates that the reflection of a particular branch in the $\mathrm{Re}\,[\Gamma] = 0$ plane is shown. It is easy to see that behavior of complex curves in spectra for pure longitudinal waves is almost independent from Poisson's ratio. In the spectrum for screw modes in Fig. 4(d) there exists a complex branch joining the real branch with the imaginary one instead of the $\Omega = 0$ plane. The proposed method allows to determine this branch. In this case accuracy of the obtained results is confirmed by the fact that the calculated branch joins the extrema of real and imaginary branches, as it was assumed by the properties mentioned above.

4. Conclusion

This paper presents a rather efficient method for finding complex roots of dispersion equations. The method is characterized by the simplicity of realization and the absence of restrictions on the representation of equations. It is shown that the results obtained by this method are in good agreement with the data calculated by other methods for an elastic isotropic infinite plate. Calculations performed for complex roots of the dispersion equation for a rectangular waveguide, to our best knowledge, are completely new results. It has been established that these roots form smooth branches of dispersion curves with the properties typical for all elastic waveguides. The proposed method can be applied to problems dealing with waveguides of other cross-section as well as to vibration problems of plates and shells.

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