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Parameter sensitivity formulation for viscoelastic structures

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The problem of sensitivity of viscoelastic response with respect to material parameters is studied in the paper. The direct differentiation method is employed. The FEM-related implementation issues are discussed. A number of numerical examples illustrates the theory.

1. Introduction

Constitutive modeling of materials exhibiting viscoelastic behaviour, together with applications of the finite element method to solve corresponding initial/boundary-value problems, can now be considered a classical subject with some three decade history, see [1] for a extensive in-depth review, for instance.

The objective of the current work is to discuss a constitutive formulation similar in spirit to those presented in [1, 2] and elaborate on it to put explicitly forward a formulation suitable for the so-called direct differentiation method (DDM) of parameter sensitivity assessment.

We confine ourselves to isotropic materials characterized by only two distinct sets of relaxation behaviour — one associated with the shear modulus and the other with the bulk modulus. No difficulties with the sensitivity assessment are expected to emerge if more complex behaviour is considered.

2. Basic constitutive equation

The following well known hereditary integral relationship between stress and strain is taken as the constitutive equation defining the class of linear viscoelastic materials on hand:

$$\sigma_{ij} = \int_0^\tau C_{ijkl}(\tau - \tau') \, \frac{\mathrm{d}\varepsilon_{kl}(\tau')}{\mathrm{d}\tau'} \, \mathrm{d}\tau' \tag{1}$$

where τ stands for the time coordinate while the dependence of all the functions on the spatial coordinates x_k is suppressed for compactness. The summation convention over twice repeating indices is assumed to hold. The integrating functions $C_{ijkl}(\tau)$ represent relaxation moduli and are expected to be decreasing functions of their argument. In fact, eq. (1) is based on the superposition principle according to which the total stress at time τ is obtained by superposing the effect at time τ of all the strain increment at times $\tau' < \tau$.

Linear viscoelasticity has the same limitations as linear elasticity: it is necessarily an approximate theory applicable only when the strain and rotation component are sufficiently small. Also, linear elasticity can be regarded as the limiting case of linear viscoelasticity in which the relaxation functions are independent of τ. The isotropic form of the viscoelastic stress-strain relation is given by

$$C_{ijkl}(\tau) = \frac{1}{3}\left[C_1(\tau) - C_2(\tau)\right]\delta_{ij}\delta_{kl} + \frac{1}{2}\,C_1(\tau)\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right] \tag{2}$$

where $C_1(\tau)$, $C_2(\tau)$ are independent relaxation functions and δ_{ij} is the Kronecker symbol. In the interest of algebraic simplicity the relaxation behaviour is often represented as

$$C_{ijkl}(\tau) = C^{\infty}_{ijkl} + \widetilde{C}_{ijkl}\,\exp\left[-\frac{\tau}{\varrho}\right] \tag{3}$$

where C^{∞}_{ijkl} are the so-called equilibrium moduli, $\widetilde{C}_{ijkl}$ the magnitudes of transient decay Fig. 1 and ϱ a relaxation time – only one relaxation time is considered in this section and later from section 4 onwards for convenience of notation but two (or more) of them as implied by the decomposition of Eq. (2) can be handled with ease as shown in Section 3.

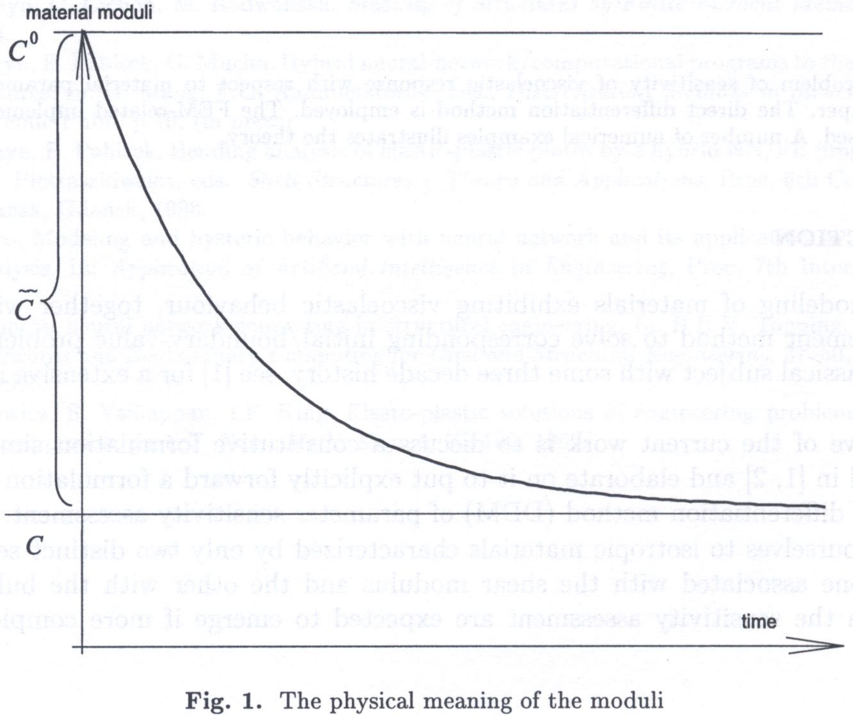

Fig. 1. The physical meaning of the moduli

At $\tau = 0$ the moduli C_{ijkl} become, cf. Fig. 1

$$C_{ijkl}(0) \overset{\mathrm{Df}}{=} C^{0}_{ijkl} = C^{\infty}_{ijkl} + \widetilde{C}_{ijkl} \tag{4}$$

known as the instantaneous (glassy) moduli. Using Eqs. (3), (4), the constitutive relation (1) is transformed to the form

$$\sigma_{ij}(\tau) = \int_0^{\tau} \left[C^{\infty}_{ijkl} + \widetilde{C}_{ijkl}\exp\left[-\frac{\tau-\tau'}{\varrho}\right]\right]\frac{\mathrm{d}\varepsilon_{kl}(\tau')}{\mathrm{d}\tau'}\,\mathrm{d}\tau'$$

$$= \int_0^{\tau} \left[C^{0}_{ijkl} - \widetilde{C}_{ijkl}\left[1 - \exp\left[-\frac{\tau-\tau'}{\varrho}\right]\right]\right]\frac{\mathrm{d}\varepsilon_{kl}(\tau')}{\mathrm{d}\tau'}\,\mathrm{d}\tau' = C^{0}_{ijkl}\epsilon_{kl}(\tau) - \sigma^{(c)}_{ij} \tag{5}$$

where the "creep" stress (difference between the purely elastic stress $C^{0}_{ijkl}\epsilon_{kl}$ based on the instantaneous moduli and the actual stress σ_{ij}) is defined as

$$\sigma^{(c)}_{ij}(\tau) = \int_0^{\tau} \widetilde{C}_{ijkl}\left[1 - \exp\left[-\frac{\tau-\tau'}{\varrho}\right]\right]\frac{\mathrm{d}\varepsilon_{kl}(\tau')}{\mathrm{d}\tau'}\,\mathrm{d}\tau' \tag{6}$$

3. CONSTITUTIVE EQUATION WITH MULTIPLE EXPONENTIAL TERM

The constitutive equation with two exponential terms is considered in the form

$$
\begin{aligned}
C_{ijkl}(\tau) &= \lambda(\tau)\,\delta_{ij}\delta_{kl} + \mu(\tau)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) = \left[\frac{2}{3}\mu(\tau) - \kappa(\tau)\right]\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\
&= \frac{1}{3}\left[C_1(\tau) - C_2(\tau)\right]\delta_{ij}\delta_{kl} + \frac{1}{2}C_1(\tau)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \\
2\mu(\tau) &= C_1(\tau), \\
3\kappa(\tau) &= C_2(\tau), \\
\sigma_{ij}^{D} &= \int_0^{\tau} 2\mu(\tau - \tau')\frac{\mathrm{d}\varepsilon_{ij}^{D}(\tau')}{\mathrm{d}\tau'}\,\mathrm{d}\tau', \\
\sigma_{kk} &= \int_0^{\tau} 3\kappa(\tau - \tau')\frac{\mathrm{d}\varepsilon_{kk}}{\mathrm{d}\tau'}\,\mathrm{d}\tau',
\end{aligned}
$$

for which the relaxation moduli take the from

$$
C_{ijkl}(\tau) = C_{ijkl}^{\infty} + \widetilde{C}_{ijkl}^{\mu}\exp\left[-\frac{\tau}{\varrho_{\mu}}\right] + \widetilde{C}_{ijkl}^{\kappa}\exp\left[-\frac{\tau}{\varrho_{\kappa}}\right] \tag{7}
$$

where

$$
\begin{aligned}
\mu(\tau) &= \mu^{\infty} + \widetilde{\mu}\exp\left[-\frac{\tau}{\varrho_{\mu}}\right], \\
\kappa(\tau) &= \kappa^{\infty} + \widetilde{\kappa}\exp\left[-\frac{\tau}{\varrho_{\kappa}}\right], \\
C_{ijkl}^{\infty}(\tau) &= \left[\frac{2}{3}\mu^{\infty} - \kappa^{\infty}\right]\delta_{ij}\delta_{kl} + \mu^{\infty}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \\
C_{ijkl}^{\mu}(\tau) &= \left[\frac{2}{3}\widetilde{\mu}\exp\left[-\frac{\tau}{\varrho_{\mu}}\right]\right]\delta_{ij}\delta_{kl} + \widetilde{\mu}\exp\left[-\frac{\tau}{\varrho_{\mu}}\right]\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \\
&= \left[\frac{2}{3}\delta_{ij}\delta_{kl} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right]\widetilde{\mu}\exp\left[-\frac{\tau}{\varrho_{\mu}}\right], \\
C_{ijkl}^{\kappa}(\tau) &= -\widetilde{\kappa}\exp\left[-\frac{\tau}{\varrho_{\kappa}}\right]\delta_{ij}\delta_{kl}\,.
\end{aligned}
$$

The glassy moduli are defined as

$$
C_{ijkl}^{0}(\tau) = C_{ijkl}^{\infty} + \widetilde{C}_{ijkl}^{\mu}(\tau) + \widetilde{C}_{ijkl}^{\kappa}(\tau), \tag{8}
$$

while the constitutive equation may be written as follows:

$$
\sigma_{ij}(\tau) = C_{ijkl}^{0}\varepsilon_{kl} - {}^{\mu}\sigma_{ij}^{(c)} - {}^{\kappa}\sigma_{ij}^{(c)} \tag{9}
$$

where

$$
\begin{aligned}
{}^{\mu}\sigma_{ij}^{(c)} &= \int_0^{\tau}\widetilde{C}_{ijkl}^{\mu}\left[1 - \exp\left[-\frac{\tau - \tau'}{\varrho_{\mu}}\right]\right]\frac{\mathrm{d}\varepsilon_{kl}(\tau')}{\mathrm{d}\tau'}\,\mathrm{d}\tau' \\
{}^{\kappa}\sigma_{ij}^{(c)} &= \int_0^{\tau}\widetilde{C}_{ijkl}^{\kappa}\left[1 - \exp\left[-\frac{\tau - \tau'}{\varrho_{\kappa}}\right]\right]\frac{\mathrm{d}\varepsilon_{kl}(\tau')}{\mathrm{d}\tau'}\,\mathrm{d}\tau'
\end{aligned}
$$

All the derivations discussed belowed for the constitutive equation with one relaxation time can easily be extended to cover more general situations.

4. Incremental form of constitutive equations

It is clearly possible to develop a finite element methodology based on the constitutive equations (5), (6) — any such procedure will require solving a set of Volterra integrals of the type Eq. (6). However, guided by the widely demonstrated efficiency of finite element time incremental schemes a constitutive equation form suitable for such an approach is now looked for. It is emphasized that up to the numerical accuracy both the schemes are fully equivalent, the latter (i.e. incremental) is only believed to be less costly numerically. We begin with the observation that for any function

$$f(\tau) = \int_0^\tau g(\tau, \tau')\, \mathrm{d}\tau', \tag{10}$$

under appropriate smoothness assumptions, we quite generally have

$$\frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau} = g(\tau,\tau) + \int_0^\tau \frac{\mathrm{d}g(\tau,\tau')}{\mathrm{d}\tau}\, \mathrm{d}\tau'. \tag{11}$$

By applying this rule to Eq. (6) we arrive at

$$\frac{\mathrm{d}\sigma_{ij}^{(c)}(\tau)}{\mathrm{d}\tau} = \frac{1}{\varrho}\int_0^\tau \left[\widetilde{C}_{ijkl}\exp\left[-\frac{\tau-\tau'}{\varrho}\right]\right]\frac{\mathrm{d}\varepsilon_{kl}(\tau')}{\mathrm{d}\tau'}\,\mathrm{d}\tau' = \frac{1}{\varrho}\left[\widetilde{C}_{ijkl}\varepsilon_{kl}(\tau) - \sigma_{ij}^{(c)}(\tau)\right] \tag{12}$$

which is the time evolution equations for the creep stress $\sigma_{ij}^{(c)}$.

The constitutive description based on equations (5),(12) allows its easy incorporation in the standard time stepping finite element algorithms. In the simple form of Eq. (5) the stresses $\sigma_{ij}^{(c)}$ are treated as some internal parameters whose time evolution is governed by, again quite straightforward, Eq. (12).

5. Computational issues

An implicit time integration algorithm will be used to advance the solution in time τ, for spatially discretized problem described by means of the virtual work principle. Assuming that the virtual work principle is satisfied at the beginning of a typical time step $t \longrightarrow t + \Delta t$ (which amounts to saying that the equilibrium and stress boundary conditions are satisfied in a weak sense at t), the FEM solution generates the incremental nodal displacements which are then used to compute stresses and other field variables. Corresponding to the end of the step the so computed stresses will generally not satisfy principle of virtual work at $t \longrightarrow t + \Delta t$ which necessitates iteration improving the incremental displacement until the virtual work equation is satisfied within an acceptable tolerance.

The basic ingredients required by the above algorithm are the time integration scheme and the consistent (algorithmic) stiffness matrix assuring quadratic convergence of the iterative procedure. The implicit time integration of the constitutive equation to be performed at each spatial integration point within the FEM methodology is based on using the backward Euler scheme to compute the end-of-the-step value of the creep stress by Eq. (12) according to

$$\frac{{}^{t+\Delta t}\boldsymbol{\sigma}^{(c)} - {}^{t}\boldsymbol{\sigma}^{(c)}}{\Delta t} = \frac{1}{\varrho}\left[\widetilde{\mathbf{C}}\,{}^{t+\Delta t}\boldsymbol{\varepsilon} - {}^{t+\Delta t}\boldsymbol{\sigma}^{(c)}\right] \tag{13}$$

resolved for ${}^{t+\Delta t}\boldsymbol{\sigma}^{(c)}$ to yield

$${}^{t+\Delta t}\boldsymbol{\sigma}^{(c)} = \left[\frac{1}{\Delta t} + \frac{1}{\varrho}\right]^{-1}\left[\frac{{}^{t}\boldsymbol{\sigma}^{(c)}}{\Delta t} + \frac{\widetilde{\mathbf{C}}\,{}^{t+\Delta t}\boldsymbol{\varepsilon}}{\rho}\right]. \tag{14}$$

The constitutive equation (5) is then used as

$$ {}^{t+\Delta t}\boldsymbol{\sigma} = \mathbf{C}^0\,{}^{t+\Delta t}\boldsymbol{\varepsilon} - {}^{t+\Delta t}\boldsymbol{\sigma}^{(c)} \tag{15} $$

It is seen by Eqs. (14), (15) that once the incremental strain ${}^{t+\Delta t}\boldsymbol{\varepsilon}$ is computed, no iteration at the integration point level is needed to calculate the value of ${}^{t+\Delta t}\boldsymbol{\sigma}$. Also the tangent operator consistent with the time integration scheme adopted has the form

$$ \mathbf{C}^* = \frac{\partial\,{}^{t+\Delta t}\boldsymbol{\sigma}}{\partial\,{}^{t+\Delta t}\boldsymbol{\varepsilon}} = \mathbf{C}^0 - \left[\frac{1}{\Delta t} + \frac{1}{\varrho}\right]^{-1} \frac{\widetilde{\mathbf{C}}}{\varrho} = \mathbf{C}^0 - \frac{\Delta t\,\widetilde{\mathbf{C}}}{\Delta t + \varrho} \tag{16} $$

The constitutive equation may be rewritten as

$$ {}^{t+\Delta t}\boldsymbol{\sigma} = \mathbf{C}^0\,{}^{t+\Delta t}\boldsymbol{\varepsilon} - \left[\frac{1}{\Delta t} + \frac{1}{\varrho}\right]^{-1} \left[\frac{{}^{t}\boldsymbol{\sigma}^{(c)}}{\Delta t} + \frac{\widetilde{\mathbf{C}}\,{}^{t+\Delta t}\boldsymbol{\varepsilon}}{\varrho}\right] = \mathbf{C}^*\,{}^{t+\Delta t}\boldsymbol{\varepsilon} - \frac{\varrho}{\Delta t + \varrho}\,{}^{t}\boldsymbol{\sigma}^{(c)} \tag{17} $$

The global equilibrium is enforced at $t + \Delta t$ by using the virtual work equations in the form

$$ \int_\Omega {}^{t+1}\boldsymbol{\sigma}^T\,\delta\boldsymbol{\varepsilon}\,\mathrm{d}\Omega = \int_\Omega \rho\,{}^{n+1}f^T\,\delta u\,\mathrm{d}\Omega + \int_{\partial\Omega_{\boldsymbol{\sigma}}} t^T\,\delta u\,\mathrm{d}\partial\Omega_{\boldsymbol{\sigma}} \tag{18} $$

in which ρf and t are the volumetric and boundary surface load vectors acting upon the system.

By introducing in Eq. (18) the standard finite element displacement approximation as

$$ \mathbf{u} = \mathbf{N}\mathbf{q}, \qquad \Delta\mathbf{u} = \mathbf{N}\Delta\mathbf{q}, \qquad \boldsymbol{\varepsilon} = \mathbf{B}\mathbf{q} $$

and substituting Eq. (17) we arrive at

$$ \begin{aligned} &({}^{t}\mathbf{q} + \Delta\mathbf{q})^T \left[\int_\Omega \mathbf{B}^T\mathbf{C}^0\mathbf{B}\,\mathrm{d}\Omega\right] \delta\mathbf{q} - \left[\frac{1}{\Delta t} + \frac{1}{\varrho}\right]^{-1} \frac{1}{\Delta t} \left[\int_\Omega {}^{t}\boldsymbol{\sigma}^{(c)T}\mathbf{B}\,\mathrm{d}\Omega\right] \delta\mathbf{q} \\ &- \left[\frac{1}{\Delta t} + \frac{1}{\varrho}\right]^{-1} \frac{1}{\varrho} ({}^{t}\mathbf{q} + \Delta\mathbf{q})^T \left[\int_\Omega \mathbf{B}^T\widetilde{\mathbf{C}}\mathbf{B}\,\mathrm{d}\Omega\right] \delta\mathbf{q} \\ &= \left[\int_\Omega \rho({}^{t}f + \Delta f)^T\mathbf{N}\,\mathrm{d}\Omega\right] \delta\mathbf{q} + \left[\int_{\partial\Omega_\sigma} ({}^{t}\mathbf{t} + \Delta t)^T\mathbf{N}\,\mathrm{d}\partial\Omega_\sigma\right] \delta\mathbf{q}. \end{aligned} \tag{19} $$

Introducing the notation

$$ {}^{t+\Delta t}\mathbf{Q}^T = \left[\int_\Omega \rho({}^{t}\mathbf{f} + \Delta\mathbf{f})^T\mathbf{N}\,\mathrm{d}\Omega\right] \delta\mathbf{q} + \left[\int_{\partial\Omega_\sigma} ({}^{t}\mathbf{t} + \Delta\mathbf{t})^T N\,\mathrm{d}\partial\Omega_\sigma\right] \delta\mathbf{q}, \tag{20} $$

$$ {}^{t}\mathbf{F}^0 = \left[\int_\Omega \mathbf{B}^T\mathbf{C}^0\mathbf{B}\,\mathrm{d}\Omega\right] {}^{t}\mathbf{q}, \tag{21} $$

$$ {}^{t}\mathbf{F}^{(c)} = \frac{\Delta t}{\Delta t + \varrho} \left[\int_\Omega \mathbf{B}^T\widetilde{\mathbf{C}}\mathbf{B}\,\mathrm{d}\Omega\right] {}^{t}\mathbf{q} + \frac{\varrho}{\Delta t + \varrho} \left[\int_\Omega \mathbf{B}^T\,{}^{t}\boldsymbol{\sigma}^{(c)}\,\mathrm{d}\Omega\right], \tag{22} $$

$$ {}^{t}\mathbf{F} = {}^{t}\mathbf{F}^0 - {}^{t}\mathbf{F}^{(c)}, \tag{23} $$

$$ \Delta\mathbf{F}^{(c)} = \frac{\Delta t}{\Delta t + \varrho} \left[\int_\Omega \mathbf{B}^T\widetilde{\mathbf{C}}\mathbf{B}\,\mathrm{d}\Omega\right] \Delta\mathbf{q} = \frac{\Delta t}{\Delta t + \varrho} \widetilde{\mathbf{K}}\,\Delta\mathbf{q}, \tag{24} $$

$$\mathbf{K}^0 = \int_\Omega \mathbf{B}^T\mathbf{C}^0\mathbf{B}\,\mathrm{d}\Omega, \tag{25}$$

$$\widetilde{\mathbf{K}} = \int_\Omega \mathbf{B}^T\widetilde{\mathbf{C}}\mathbf{B}\,\mathrm{d}\Omega, \tag{26}$$

$$\mathbf{K}^* = \int_\Omega \mathbf{B}^T\mathbf{C}^*\mathbf{B}\,\mathrm{d}\Omega = \mathbf{K}^0 - \frac{\Delta t}{\Delta t+\varrho}\widetilde{\mathbf{K}}, \tag{27}$$

observing the symmetry of the above three stiffness matrices and using the arbitrariness of the variation δu allows to rewrite Eq. (19) as

$$\left[\mathbf{K}^0 - \frac{\Delta t}{\Delta t+\varrho}\widetilde{\mathbf{K}}\right]\Delta\mathbf{q} = {}^{t+\Delta t}\mathbf{Q} - {}^t\mathbf{F}^0 + {}^t\mathbf{F}^{(c)}, \tag{28a}$$

$$\left[\mathbf{K}^0 - \frac{\Delta t}{\Delta t+\varrho}\widetilde{\mathbf{K}}\right]\Delta\mathbf{q} = {}^{t+\Delta t}\mathbf{Q} - {}^t\mathbf{F}, \tag{28b}$$

or, more compactly

$$\mathbf{K}^*\Delta\mathbf{q} = {}^{t+\Delta t}\mathbf{Q} - {}^t\mathbf{F}^0 + {}^t\mathbf{F}^{(c)}, \tag{29a}$$

$$\mathbf{K}^*\Delta\mathbf{q} = {}^{t+\Delta t}\mathbf{Q} - {}^t\mathbf{F}, \tag{29b}$$

By moving over to the right-hand side the term involving $\widetilde{\mathbf{K}}$ we obtain

$$\mathbf{K}^0\Delta\mathbf{q} = {}^{t+\Delta t}\mathbf{Q} - {}^t\mathbf{F}^0 + {}^{t+\Delta t}\mathbf{F}^{(c)}(\Delta\mathbf{q}), \tag{30a}$$

$$\mathbf{K}^0\Delta\mathbf{q} = {}^{t+\Delta t}\mathbf{Q} - {}^t\mathbf{F} + \Delta\mathbf{F}^{(c)}(\Delta\mathbf{q}). \tag{30b}$$

We observe that using the formulation given in Eqs. (29) reduces the procedure to just one non-iterative solution of the linear algebraic equation system at each incremental step. In contrast, the procedure based on Eqs. (30) requires the direct iterative solution defined by the recursive formula

$$\mathbf{K}^0\Delta\mathbf{q}^{(i+1)} = {}^{t+\Delta t}\mathbf{Q} - {}^t\mathbf{F}^0 + {}^{t+\Delta t}\mathbf{F}^{(c)}(\Delta\mathbf{q}^{(i)}), \qquad i = 1, 2, \dots \,. \tag{31}$$

We also note that whereas $\mathbf{K}^0$ remains constant during the whole process, the tangent matrix $\mathbf{K}^*$ shares this property only if the time step length remains unchanged.

6. Sensitivity analysis

We assume now that our interest lies in finding a computationally effective technique of evaluating the gradient of any response functional with respect to a material parameter (say, h) entering the theory — such a gradient is referred to as the response first order sensitivity. It has been shown in the literature, see [3] for instance, that the gradient of any response functional can be expressed in terms of the displacement sensitivity $\mathrm{d}\mathbf{u}/\mathrm{d}h$. Thus, our objective now is to develop a system of equations to be used for the computation of $\mathrm{d}\mathbf{u}/\mathrm{d}h$ — in the context of the FEM methodology the goal is clearly fulfilled once a technique is developed to determine the value of $\mathrm{d}\mathbf{q}/\mathrm{d}h$.

The so-called direct differentiation (DDM) method is used in this paper. The method requires differentiation of the governing equation with respect to the parameter h — any material parameter entering the theory can be substituted for h in specific applications.

For the sake of derivation compactness we observe that

$$\frac{\mathrm{d}\,{}^t\mathbf{F}}{\mathrm{d}h} = \frac{\mathrm{d}\,{}^t\mathbf{F}^0}{\mathrm{d}h} - \frac{\mathrm{d}\,{}^t\mathbf{F}^{(c)}}{\mathrm{d}h},$$

$$\frac{\mathrm{d}\,{}^t\mathbf{F}^0}{dh} = \left[\int_\Omega \mathbf{B}^T \frac{\partial \mathbf{C}^0}{\partial h}\,\mathrm{d}\Omega\right] {}^t\mathbf{q} + \mathbf{K}^0 \frac{\mathrm{d}\,{}^t\mathbf{q}}{dh},$$

$$\frac{\mathrm{d}\,{}^t\mathbf{F}^{(c)}}{\mathrm{d}h} = \frac{\partial}{\partial h}\left[\frac{\Delta t}{\Delta t+\varrho}\right]\widetilde{\mathbf{K}}\,{}^t\mathbf{q} + \left[\frac{\Delta t}{\Delta t+\varrho}\right]\left[\int_\Omega \mathbf{B}^t \frac{\partial \widetilde{\mathbf{C}}}{\partial h}\mathbf{B}\,\mathrm{d}\Omega\right] {}^t\mathbf{q} + \left[\frac{\Delta t}{\Delta t+\varrho}\right]\widetilde{\mathbf{K}}\frac{\mathrm{d}\,{}^t\mathbf{q}}{\mathrm{d}h}$$

$$+ \frac{\partial}{\partial h}\left[\frac{\varrho}{\Delta t+\varrho}\right]\left[\int_\Omega \mathbf{B}^T\,{}^t\boldsymbol{\sigma}^{(c)}\,\mathrm{d}\Omega\right] + \left[\frac{\varrho}{\Delta t+\varrho}\right]\left[\int_\Omega \mathbf{B}^T \frac{\partial\,{}^t\boldsymbol{\sigma}^{(c)}}{\partial h}\,\mathrm{d}\Omega\right],$$

$$\frac{\mathrm{d}\,{}^{t+\Delta t}\mathbf{Q}}{\mathrm{d}h} = \mathbf{0},$$

$$\frac{\partial \mathbf{K}^0}{\partial h} = \int_\Omega \mathbf{B}^T \frac{\partial \mathbf{C}^0}{\partial h}\mathbf{B}\,\mathrm{d}\Omega,$$

$$\frac{\partial \widetilde{\mathbf{K}}}{\partial h} = \int_\Omega \mathbf{B}^T \frac{\partial \widetilde{\mathbf{C}}}{\partial h}\mathbf{B}\,\mathrm{d}\Omega.$$

Using the above results we now differentiate Eq. (29b) with respect to h to obtain

$$\mathbf{K}^*\frac{\mathrm{d}\,\Delta\mathbf{q}}{\mathrm{d}h} = -\left[\frac{\partial \mathbf{K}^0}{\partial h} - \left[\frac{\Delta t}{\Delta t+\varrho}\right]\frac{\partial \widetilde{\mathbf{K}}}{\partial h} - \frac{\partial}{\partial h}\left[\frac{\Delta t}{\Delta t+\varrho}\right]\widetilde{\mathbf{K}}\right]\Delta\mathbf{q} - \frac{\mathrm{d}\,{}^t\mathbf{F}}{\mathrm{d}h}\left(\frac{\mathrm{d}\,{}^t\mathbf{q}}{\mathrm{d}h}, \frac{\mathrm{d}\,{}^t\boldsymbol{\sigma}^{(c)}}{\mathrm{d}h}\right). \tag{32}$$

This equation can be solved for $\mathrm{d}\Delta\mathbf{q}/\mathrm{d}h$ provided the right-hand side is known — in fact the solution would be quite straightforward bearing in mind that the matrix $\mathbf{K}^*$ is available in decomposed form which has just been needed at the equilibrium solution stage. In order to see how the right-hand vector in Eq. (32) can be effectively evaluated we observe that:

- the stiffness matrices $\mathbf{K}^0$ and $\mathbf{K}^*$ are by the definition explicitly given functions of the parameter h,
- $\mathrm{d}\,{}^t\mathbf{q}/\mathrm{d}h$ and $\mathrm{d}\,{}^t\boldsymbol{\sigma}^{(c)}/\mathrm{d}h$ are known and have only to be recovered from the computer memory.

To elaborate on the last statement we first note that the displacement sensitivity accumulation along the solution path takes place in accordance with the update rule of the recursive form

$$\frac{\mathrm{d}\,{}^{t+\Delta t}\mathbf{q}}{\mathrm{d}h} = \frac{\mathrm{d}\,{}^t\mathbf{q}}{\mathrm{d}h} + \frac{\mathrm{d}\,\Delta\mathbf{q}}{\mathrm{d}h} \tag{33}$$

Thus, having the beginning-of-the-step sensitivity and adding to it the incremental sensitivity computed by Eq. (32) allows to obtain the end-of-the-step sensitivity. This rule can obviously employed at any time instant in the course of the process.

In order to see the way to compute the end-of-the-step value of $\mathrm{d}\boldsymbol{\sigma}^{(c)}/\mathrm{d}h$ on the basis of its value the time increment earlier we differentiate Eq. (14) to obtain

$$\frac{\mathrm{d}\,{}^{t+\Delta t}\boldsymbol{\sigma}^{(c)}}{\mathrm{d}h} = \frac{\partial}{\partial h}\left[\frac{1}{\Delta t}+\frac{1}{\varrho}\right]^{-1}\left[\frac{{}^t\boldsymbol{\sigma}^{(c)}}{\Delta t} + \frac{\widetilde{\mathbf{C}}\;{}^{t+\Delta t}\boldsymbol{\varepsilon}}{\varrho}\right]$$

$$+ \left[\frac{1}{\Delta t}+\frac{1}{\varrho}\right]^{-1}\left[\frac{\mathrm{d}\,{}^t\boldsymbol{\sigma}^{(c)}}{\mathrm{d}h}\frac{1}{\Delta t} + \frac{1}{\varrho}\left[\frac{\partial \widetilde{\mathbf{C}}}{\partial h}\,{}^{t+\Delta t}\boldsymbol{\varepsilon} + \widetilde{\mathbf{C}}\frac{\mathrm{d}\;{}^{t+\Delta t}\boldsymbol{\varepsilon}}{\mathrm{d}h}\right]\right]. \tag{34}$$

All the terms on the right-hand side are known: the beginning-of-the-step values are recovered from memory, while ${}^{t+\Delta t}\boldsymbol{\varepsilon}$ and $\mathrm{d}\,{}^{t+\Delta t}\boldsymbol{\varepsilon}/\mathrm{d}h$ are easily computable from

$${}^{t+\Delta t}\boldsymbol{\varepsilon} = \mathbf{B}\;{}^{t+\Delta t}\mathbf{q} \tag{35}$$

$$\frac{\mathrm{d}\;{}^{t+\Delta t}\boldsymbol{\varepsilon}}{\mathrm{d}h} = \mathbf{B}\frac{\mathrm{d}\;{}^{t+\Delta t}\mathbf{q}}{\mathrm{d}h} \tag{36}$$

Thus, using Eq. (34) we are able to establish the end-of-the-step value of $\mathrm{d}\,{}^{t+\Delta t}\boldsymbol{\sigma}^{(c)}/\mathrm{d}h$; since the above discussion is valid for any t (i.e. at any time step), the algorithms assures the effective way to advance the sensitivity solution along the whole solution path. It is worth observing that the need to carry along all the values of $\mathrm{d}\mathbf{q}/\mathrm{d}h$ makes it basically impossible to employ another technique of sensitivity assessment known as the adjoint system method — for details justifying this important statement the reader is referred to [3], though.

In case one is inclined to use the iterative equilibrium solution based on Eq. (31) the sensitivity analysis would require iterations as well. This is clearly seen from the differentiated equation (14) which may be presented as

$$\mathbf{K}^0 \frac{\mathrm{d}\,\Delta\mathbf{q}^{(i+1)}}{\mathrm{d}h} = \frac{\partial \mathbf{K}^0}{\partial h}\Delta\mathbf{q}^{(i+1)} - \frac{\mathrm{d}\,{}^t\mathbf{F}}{\mathrm{d}h} - \frac{\mathrm{d}\,\Delta\mathbf{F}^{(c)}}{\mathrm{d}h}\left(\frac{\mathrm{d}\,\Delta\mathbf{q}^{(i)}}{\mathrm{d}h}\right), \qquad 1,2,3\ldots \tag{37}$$

where

$$\frac{\mathrm{d}\,\Delta\mathbf{F}^{(c)}}{\mathrm{d}h} = \frac{\partial}{\partial h}\left[\frac{\Delta t}{\Delta t+\varrho}\right]\widetilde{\mathbf{K}}\Delta\mathbf{q} + \left[\frac{\Delta t}{\Delta t+\varrho}\right]\frac{\partial \widetilde{\mathbf{K}}}{\partial h}\Delta\mathbf{q} + \left[\frac{\Delta t}{\Delta t+\varrho}\right]\widetilde{\mathbf{K}}\frac{\mathrm{d}\,\Delta\mathbf{q}}{\mathrm{d}h}. \tag{38}$$

We summarize in Box 1 the (non-iterative) computational algorithm for sensitivity assessment at a typical time step $[t, t+\Delta t]$.

Box 1. The (non-iterative) computational algorithm for sensitivity assessment at a typical time step $[t, t+\Delta t]$

(a) On the basis of known values of all the necessary functions $({}^t\mathbf{q}, {}^t\boldsymbol{\sigma}^{(c)})$ at time t form the equation system (37) and solve it for $\Delta\mathbf{q}$.

(b) Using $\Delta\mathbf{q}$ and, additionally, $\mathrm{d}\,{}^t\mathbf{q}/\mathrm{d}h$ and $\mathrm{d}\,{}^t\boldsymbol{\sigma}^{(c)}/\mathrm{d}h$ compute the incremental displacement sensitivity by Eq. (32) with the already decomposed stiffness matrix $\mathbf{K}^*$.

(c) Update the function values

$$\begin{aligned}
{}^{t+\Delta t}\mathbf{q} &= {}^t\mathbf{q} + \Delta\mathbf{q},\\
{}^{t+\Delta t}\boldsymbol{\varepsilon} &= \mathbf{B}\,{}^{t+\Delta t}\mathbf{q},\\
\frac{\mathrm{d}\,{}^t\boldsymbol{\sigma}^{(c)}}{\mathrm{d}h} &= \left[\frac{1}{\Delta t}+\frac{1}{\varrho}\right]^{-1}\left[\frac{{}^t\boldsymbol{\sigma}^{(c)}}{\Delta t} + \frac{\widetilde{\mathbf{C}}\,{}^{t+\Delta t}\boldsymbol{\varepsilon}}{\varrho}\right],\\
{}^{t+\Delta t}\boldsymbol{\sigma} &= \mathbf{C}^0\,{}^{t+\Delta t}\boldsymbol{\varepsilon} - {}^{t+\Delta t}\boldsymbol{\sigma}^{(c)} \quad \text{(if needed)},
\end{aligned}$$

and sensitivity values

$$\begin{aligned}
\frac{\mathrm{d}\,{}^{t+\Delta t}\mathbf{q}}{\mathrm{d}h} &= \frac{\mathrm{d}\,{}^t\mathbf{q}}{\mathrm{d}h} + \frac{\mathrm{d}\,\Delta\mathbf{q}}{\mathrm{d}h},\\
\frac{\mathrm{d}\,{}^{t+\Delta t}\boldsymbol{\varepsilon}}{\mathrm{d}h} &= \mathbf{B}\frac{\mathrm{d}\,{}^{t+\Delta t}\mathbf{q}}{\mathrm{d}h},\\
\frac{\mathrm{d}\,{}^{t+\Delta t}\boldsymbol{\sigma}^{(c)}}{\mathrm{d}h} &= \text{r.h.s. of eq. (34)},\\
\frac{\mathrm{d}\,{}^{t+\Delta t}\boldsymbol{\sigma}}{\mathrm{d}h} &= \frac{\partial \mathbf{C}^0}{\partial h}\,{}^{t+\Delta t}\boldsymbol{\varepsilon} + \mathbf{C}^0\frac{\mathrm{d}\,{}^{t+\Delta t}\boldsymbol{\varepsilon}}{\mathrm{d}h} - \frac{\mathrm{d}\,{}^{t+\Delta t}\boldsymbol{\sigma}^{(c)}}{\mathrm{d}h} \quad \text{(if needed)}.
\end{aligned}$$

(d) Go to next step with ${}^{t+\Delta t}\mathbf{q}$, ${}^{t+\Delta t}\boldsymbol{\sigma}^{(c)}$ and $\mathrm{d}\,{}^{t+\Delta t}\mathbf{q}/\mathrm{d}h$, $\mathrm{d}\,{}^{t+\Delta t}\boldsymbol{\sigma}^{(c)}/\mathrm{d}h$ stored in memory.

7. Numerical illustrations

The constitutive model presented in the paper has been implemented in an in-house object-oriented finite element system WIMES32 written in C++ (BC 4.52) programming language. Computation have been performed on a Cyrix P166 machine under Windows95 with 64 MB RAM.

7.1. Uniaxial test example

The plane stress configuration is shown in Fig. 2. The material constants are taken as $E^0 = 500\,\text{kPa}$, $E^\infty = 100\,\text{kPa}$, $\varrho = 1\,\text{s}$, $\nu = 0.3$. Load history is shown in Fig. 3. 200 time steps of 0.2 sec are considered. Numerical results are displayed in Figs. 4–9.

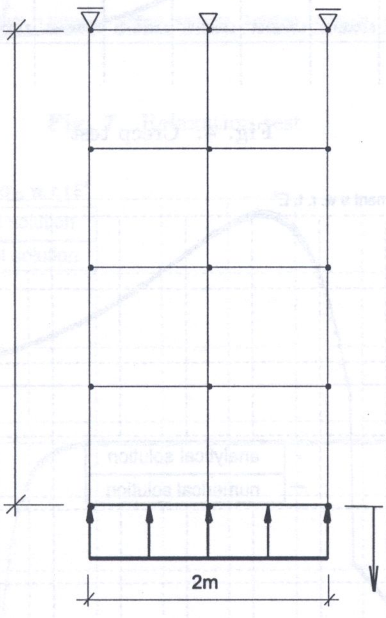

Fig. 2. FEM mesh with linear (Lagrange-type) 4 node elements — imposed displacement for relaxation test and for creep test

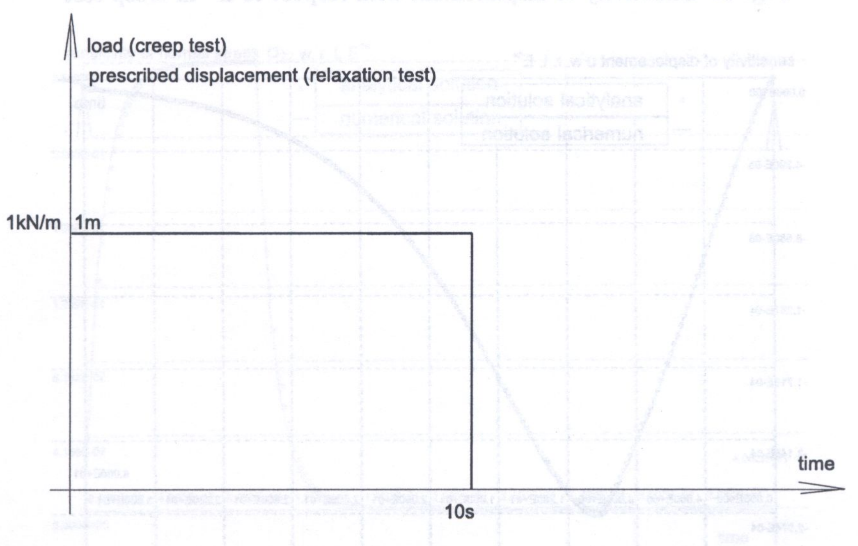

Fig. 3. Load history

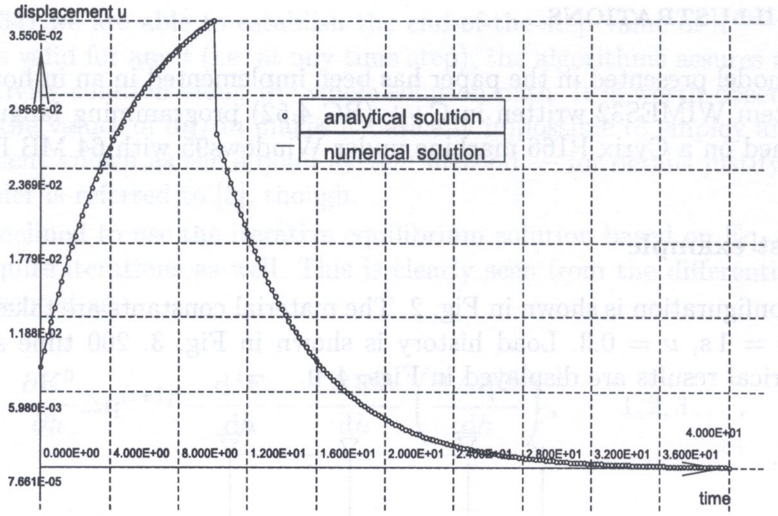

Fig. 4. Creep test

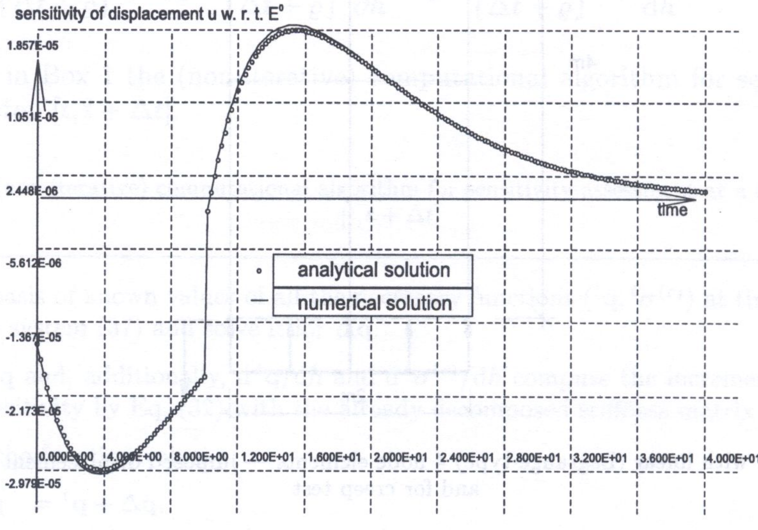

Fig. 5. Sensitivity of displacement with respect to E^0 in creep test

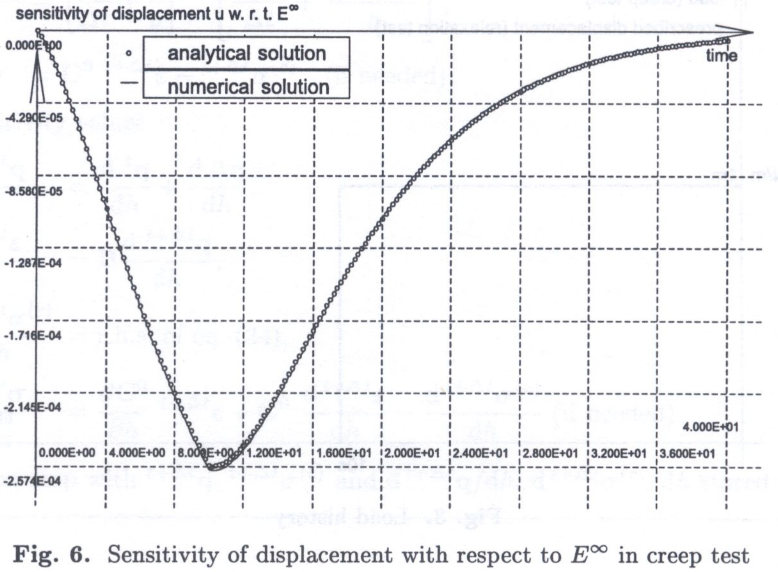

Fig. 6. Sensitivity of displacement with respect to E^{∞} in creep test

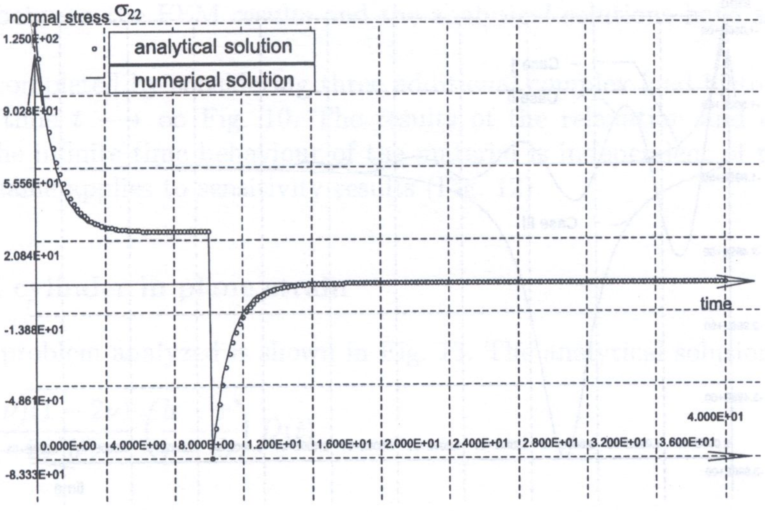

Fig. 7. Relaxation test

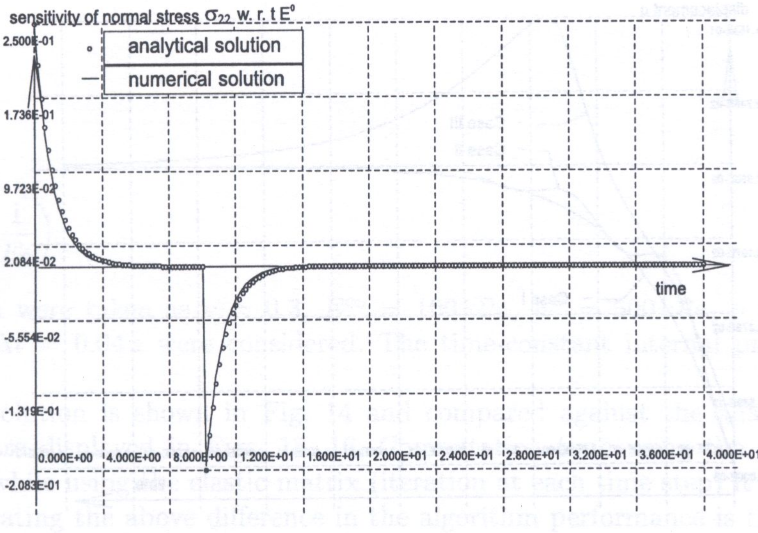

Fig. 8. Sensitivity of stress with respect to E^0 in relaxation test

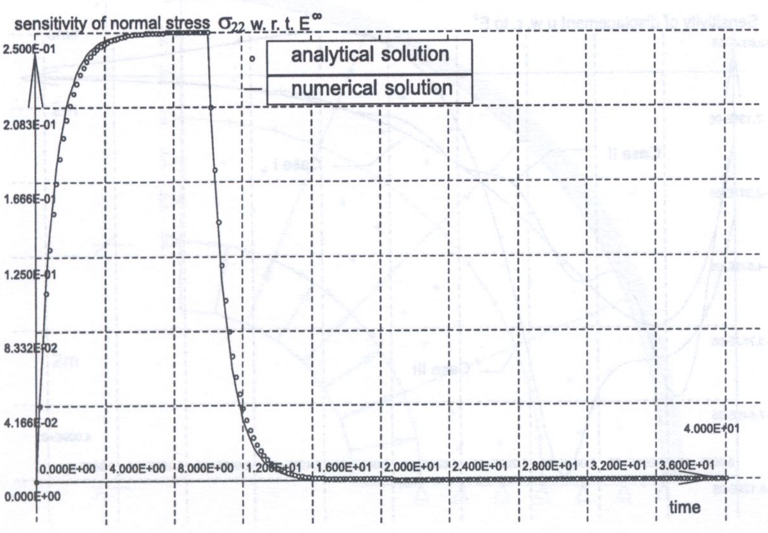

Fig. 9. Sensitivity of stress with respect to E^∞ in relaxation test

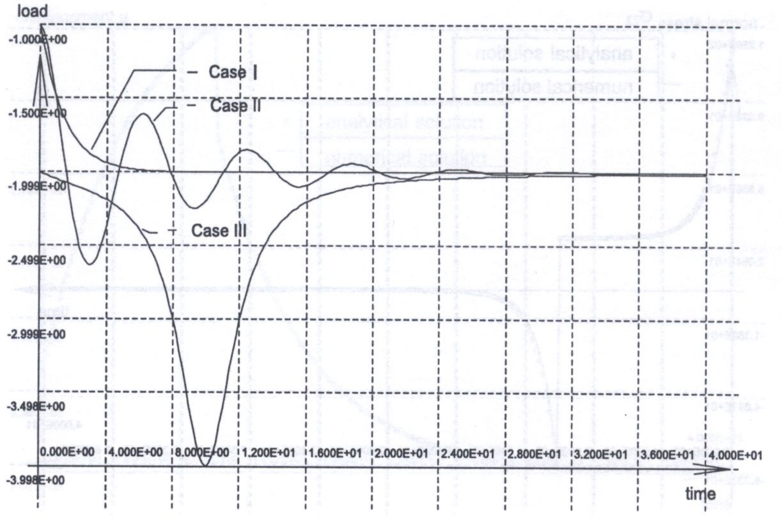

Fig. 10. Complex load histories

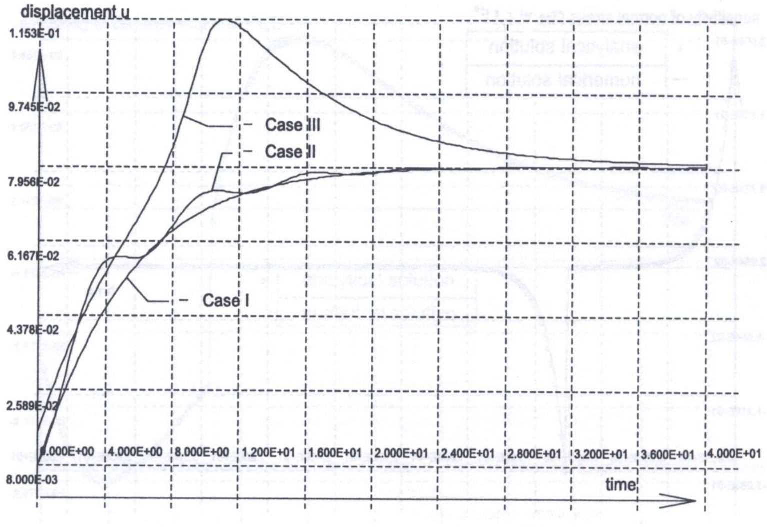

Fig. 11. Creep test for three load cases

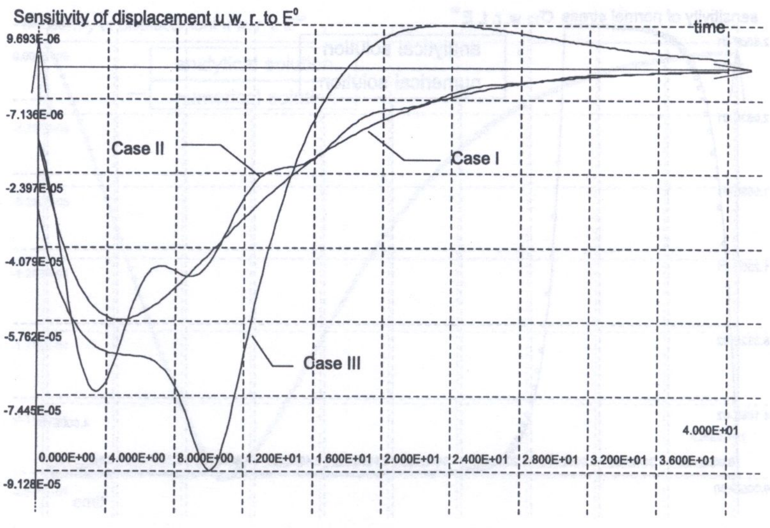

Fig. 12. Sensitivity of displacement with respect to E^0

The differences between the FEM results and the analytical solutions have always been within 1%.

The example is completed by considering three additional complex load histories cases reaching the same value at time $t \longrightarrow \infty$ Fig. 10. The results of the relaxation and displacement tests demonstrate that the infinite time behaviour of the material is independent of the time history of load (Fig. 11), the same applies to sensitivity results (Fig. 12).

7.2. Thick walled cylinder in plane strain

One quarter of the problem analyzed is shown in Fig. 13. The analytical solution has the form

$$u_r = \frac{p_0\, a^2\, b(1-\nu)(1-2\nu)}{a^2 + (1-2\nu)b^2}\left(\frac{b}{r} - \frac{r}{b}\right) D(t) \tag{39}$$

where

$$D(t) = D_0 + D_1\left(1 - e^{-\frac{t}{\lambda}}\right)$$

$$\lambda = \frac{E_0 \varrho}{E_\infty}$$

$$D_0 = \frac{1}{E_0}$$

$$E_0 = E_\infty + E_1$$

$$D_1 = \left(\frac{1}{E_\infty} - \frac{1}{E_0}\right)$$

The numerical data were taken as $\nu = 0.3$, $E^\infty = 100\,\text{kPa}$, $E^0 = 500\,\text{kPa}$, $\varrho = 1\,\text{s}$, $t = [0, 40\text{s}]$. 100 time steps of $\Delta t = 0.04\,s$ were considered. The time-constant internal pressure is taken as $p = 0.1\,\text{kPa}$.

The numerical solution is shown in Fig. 14 and compared against the analytical curve. The sensitivity results are displayed in Figs. 15, 16. Computation time using the consistent tangent matrix was 5 min, while using the elastic matrix (iteration at each time step) it was 14.30 min. An implict factor generating the above difference in the algorithm performance is the degree of shape

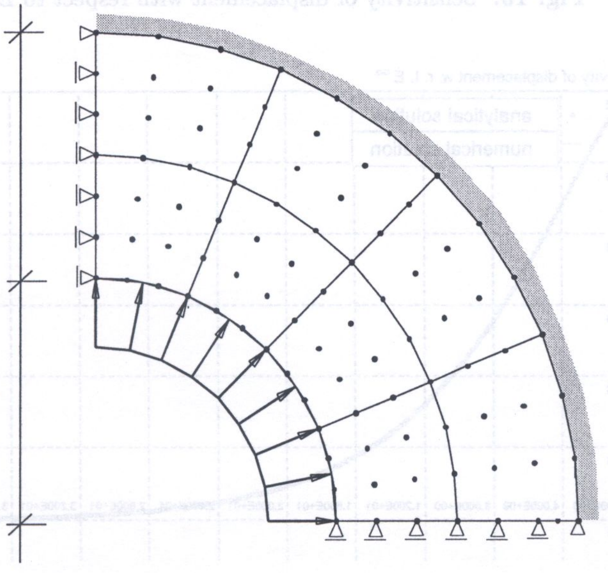

Fig. 13. One quarter of the thick walled cylinder

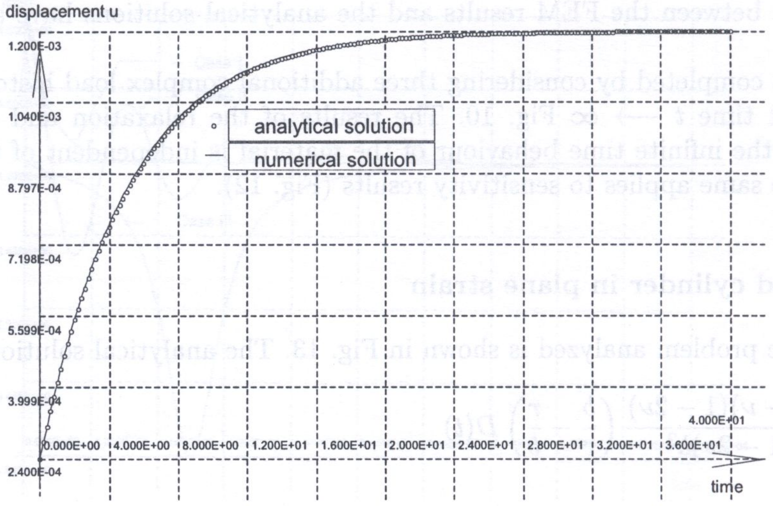

Fig. 14. Creep test

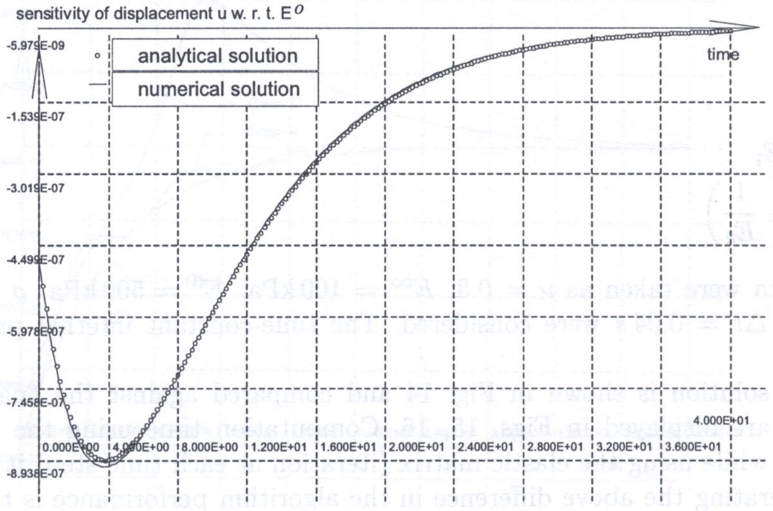

Fig. 15. Sensitivity of displacement with respect to E^0

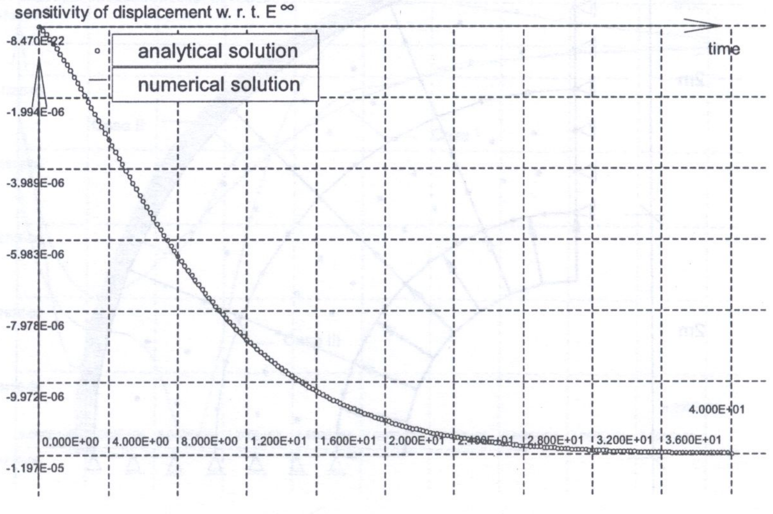

Fig. 16. Sensitivity of displacement with respect to E^{∞}

functions used the higher the degree, the greater the difference in performance due to computations needed to form the right-hand side of Eqs. (29), (31).

8. Conclusions

A linear viscoelastic material model including its parameter sensitivity and a finite element formulation has been discussed.

The backward Euler integration algorithm for the constitutive equation appears to assure a stable behaviour and good accuracy of the solution in both the creep and relaxation tests.

Two versions of the FEM equilibrium and sensitivity formulation have been developed and tested. The first is based on the elastic stiffness matrix and needs global iteration at each time step while the second based on the consistent tangent matrix is iteration-free. The first version is typically more than two times slower than the second one.

Object oriented approach has assured an easy and fast implementation of the model in an object oriented finite element system written in C++ programming language. Details of the implementation will be described elsewhere.

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