Computer Assisted Mechanics and Engineering Sciences, 6: 495–503, 1999.

Stability considerations in controller designs developed using the LF transformation

György Szász and George T. Flowers
Rotating Machines and Magnetic Bearing Laboratory
Department of Mechanical Engineering, Auburn University, Auburn, AL, USA

(Received March 8, 1999)

Controlling time periodic systems is a significant engineering challenge. One innovative approach that seems to be especially promising involves application of the Lyapunov Floquet (LF) transformation to eliminate time periodic terms from the system state matrices. Traditional control design techniques are then applied and the resulting gains transformed back to the original domain. Typically, the controller design process involves the use of an auxiliary control matrix and (assuming that the actual control matrix is time-varying and non-invertable) a pseudo inverse (which introduces approximations into the procedure). The degree to which the desired control results are achieved depends very strongly upon the impact of these approximations on the actual system dynamics. The research effort described below is concerned with investigating the performance of this LF control strategy and the existence of situations in which application of the procedure may produce undesirable behaviors.

Nomenclature

$M(t)$ – mass matrix
$C(t)$ – damping matrix
$K(t)$ – stiffness matrix
$F(t)$ – forcing function
$y(t)$ – second order state vector
$x(t)$ – first order state vector
$u(t)$ – input vector
$z(t)$ – auxiliary state vector
$e(t)$ – error
$A(t)$ – state matrix
$B_2(t)$ – input matrix
$\Phi(t)$ – state transition matrix
R – auxiliary state matrix
L – LF transformation matrix
T – system period
K_{aux} – auxiliary gain matrix
B_{aux} – auxiliary input matrix
$\dot{()}$ – time derivative
$()^\sharp$ – pseudo inverse

1. Introduction

Several authors have published work on methods for control synthesis of linear time-varying systems. These methods are largely based upon transformation of the original system to some canonical form where traditional control techniques can be applied. These transformations however tend to be not unique and tedious to implement, especially for higher dimensional systems [2]. A complicated fixed gain controllers had also been developed for time periodic systems, requiring two levels of iteration [3].

One of the most powerful results of Floquet theory is the Lyapunov Floquet theorem, which defines the Lyapunov Floquet (LF) transformation [9]. For control design purposes, the LF transformation has been used to eliminate time periodic terms from the system state matrices. This, generally, tends to make the controller design simpler but, unfortunately, there is a complication. That is, all time periodic terms cannot be eliminated in this way [4]. Specifically the control matrix does not necessarily become time invariant through the transformation and one is still left with a time periodic system. One approach is to employ an auxiliary system for the purpose of control design. The resulting gains are then transformed back to the original domain and implemented on the system, with the inherent involvement of a pseudo inverse. Therefore, there is an some intrinsic level of approximation involved in using this approach, which in some cases has a severe impact on ultimate system stability.

The occurrence and conditions for the degradation and possible loss of stability during the control design process is investigated here. It is intended to develop the governing equations in such a way that the effects of controller design on stability are clear. Note that the control design procedure using the LF transformation can be divided into three main steps:

- Applying the LF transformation
- Design of a controller for the constant coefficient system
- Obtaining the time varying gains

These steps are applied to a simple model and simulations are performed to illustrate the concepts that are identified. The following paragraphs detail the procedure and results of this investigation.

2. Transforming the Equations

Given a linear dynamical system of time periodic nature in the second order form our first step of the controller design process is to transform into a first order (state space) form. Let the second order system be

$$M(t)\,\ddot{\mathbf{y}}(t) + C(t)\,\dot{\mathbf{y}}(t) + K(t)\,\mathbf{y}(t) = B_2(t)\,\mathbf{v}(t),$$

then the first order system will be given by

$$\dot{\mathbf{x}}(t) = A(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t), \tag{1}$$

where

$$A(t) = \begin{pmatrix} 0 & I \\ -M^{-1}(t)K(t) & -M^{-1}(t)C(t) \end{pmatrix},$$

$$B(t) = \begin{pmatrix} 0 \\ -M^{-1}(t)B_2(t) \end{pmatrix},$$

and

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} \mathbf{y}(t) \\ \dot{\mathbf{y}}(t) \end{pmatrix}.$$

Applying the LF transformation

To apply the LF transformation to (1) its derivative is required. The transformation has the form [5],

$$\mathbf{x}(t) = L(t)\,\mathbf{z}(t). \tag{2}$$

Differentiate (2) to get

$$\dot{\mathbf{x}}(t) = \dot{L}(t)\mathbf{z}(t) + L(t)\,\dot{\mathbf{z}}(t),$$

and substitute the result into (1) to obtain

$$\dot{L}(t)\mathbf{z}(t) + L(t)\,\dot{\mathbf{z}}(t) = A(t)L(t)\,\mathbf{z}(t) + B(t)\,\mathbf{u}(t),$$

which is then premultiplied by $L^{-1}(t)$

$$L^{-1}(t)\,\dot{L}(t)\,\mathbf{z}(t) + \dot{\mathbf{z}}(t) = L^{-1}(t)A(t)L(t)\,\mathbf{z}(t) + L^{-1}(t)B(t)\,\mathbf{u}(t).$$

Rearranging we get

$$\dot{\mathbf{z}}(t) = L^{-1}(t)[A(t)L(t) - \dot{L}(t)]\mathbf{z}(t) + L^{-1}(t)B(t)\,\mathbf{u}(t),$$

and since

$$R = L^{-1}(t)[A(t)L(t) - \dot{L}(t)],$$

the following is true

$$\dot{\mathbf{z}}(t) = R\mathbf{z}(t) + L^{-1}(t)B(t)\,\mathbf{u}(t). \tag{3}$$

3. Control design

It should be noted that the second term on the right hand side in (3) is still time periodic, which prevents us from applying traditional controller design techniques. One solution to this problem is to consider an auxiliary system of a similar form but where the term in question is indeed constant. We shall denote the matrix defining this new constant term by $\tilde{B}$ so

$$\dot{\mathbf{z}}(t) = R\mathbf{z}(t) + \tilde{B}\mathbf{v}(t).$$

It is of course required that form a controllable pair. Introducing the control law $\mathbf{v}(t) = -\tilde{K}(t)\,\mathbf{z}(t)$, the above takes on the following form

$$\dot{\mathbf{z}}(t) = (R - \tilde{B}\tilde{K})\,\mathbf{z}(t), \tag{4}$$

where $\tilde{K}$ is selected such that the above system is stable [1]. The time periodic gains were previously found from an error analysis between the transformed and the auxiliary systems [8]. For a better understanding of why and how stability is compromised an alternative development is presented here. The flowchart in Fig. 1 shows how both developments arrive at the same result.

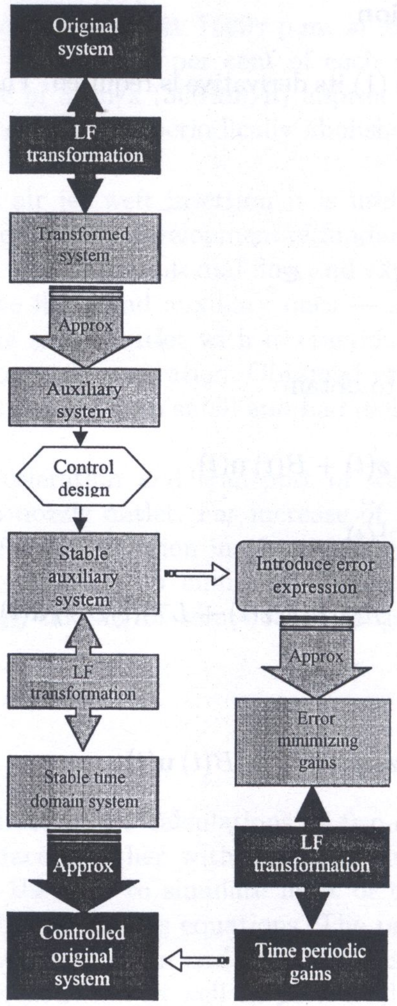

Fig. 1. Control design processes

4. Time periodic gains

In order to obtain the time periodic gains we transform (4) back into time periodic domain. Taking

$\mathbf{z}(t) = L^{-1}(t)\,\mathbf{x}(t)$

and differentiating,

$$\dot{\mathbf{z}}(t) = \dot{L}^{-1}(t)\,\mathbf{x}(t) + L^{-1}(t)\,\dot{\mathbf{x}}(t),$$

is obtained. Applying this to (4) now gives

$$\dot{L}^{-1}(t)\,\mathbf{x}(t) + L^{-1}(t)\,\dot{\mathbf{x}}(t) = (R - \tilde{B}\tilde{K})L^{-1}(t)\,\mathbf{x}(t).$$

Premultiply by $L(t)$ to get

$$L(t)\,\dot{L}^{-1}(t)\,\mathbf{x}(t) + \dot{\mathbf{x}}(t) = L(t)(R - \tilde{B}\tilde{K})L^{-1}(t)\,\mathbf{x}(t),$$

and rearrange to get

$$\dot{\mathbf{x}}(t) = L(t)[(R - \tilde{B}\tilde{K})L^{-1}(t) - \dot{L}^{-1}(t)]\mathbf{x}(t),$$

or

$$\dot{\mathbf{x}}(t) = L(t)[RL^{-1}(t) - \dot{L}^{-1}(t)]\mathbf{x}(t) - L(t)\tilde{B}\tilde{K}L^{-1}(t)\,\mathbf{x}(t).$$

But

$$L(t)[RL^{-1}(t) - \dot{L}^{-1}(t)] = A(t),$$

so as a result

$$\dot{\mathbf{x}}(t) = A(t)\,\mathbf{x}(t) - L(t)\tilde{B}\tilde{K}L^{-1}(t)\,\mathbf{x}(t). \tag{5}$$

Linear transformations do not affect the stability of the system so the above expression still retains whatever stability characteristics were achieved for the constant coefficient auxiliary system. It is, however unsuitable for implementation because its form does not conform to that of (3).

Approximating the time periodic gains

To resolve this problem we assume that when controlled (1) will have the more conventional form

$$\dot{\mathbf{x}}(t) = A(t)\,\mathbf{x}(t) - B(t)K(t)\,\mathbf{x}(t), \tag{6}$$

$K(t)$ being the the time periodic gain. To find an expression for K(t), we subtract (5) from (6) which yields

$$B(t)K(t)\,\mathbf{x}(t) = L(t)\tilde{B}\tilde{K}L^{-1}(t)\,\mathbf{x}(t),$$

or

$$K(t) = B^{\sharp}L(t)\tilde{B}\tilde{K}L^{-1}(t). \tag{7}$$

where $B^{\sharp}$ is the Moore-Penrose Pseudo inverse [6]. In cases where the B matrix is actually invertable there will be no loss of stability. In the vast majority of practical applications however the B matrix will not be invertable and ultimate stability will be a direct function of the quality of the pseudo inverse. In the following example we will show how even in the case of a simple two mass system the pseudo inverse can cause a critical loss of the stability achieved for the auxiliary system.

5. Sample dynamical system

Consider the dynamical system depicted in Fig. 2.

Let $M_1 = 1$ kg, $M_2 = 1$ kg, $K_1 = 1$ N/m, $K_2 = 1 + \alpha\cos(1 * t)$ N/m. The mass and stiffness matrices of this system are given by

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix},$$

$$K = \begin{pmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{pmatrix}.$$

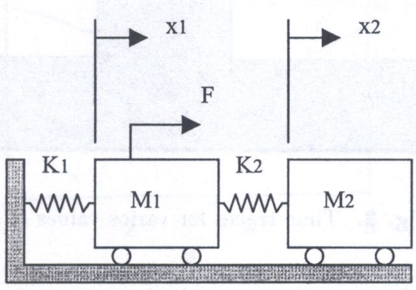

Fig. 2. Sample dynamical system

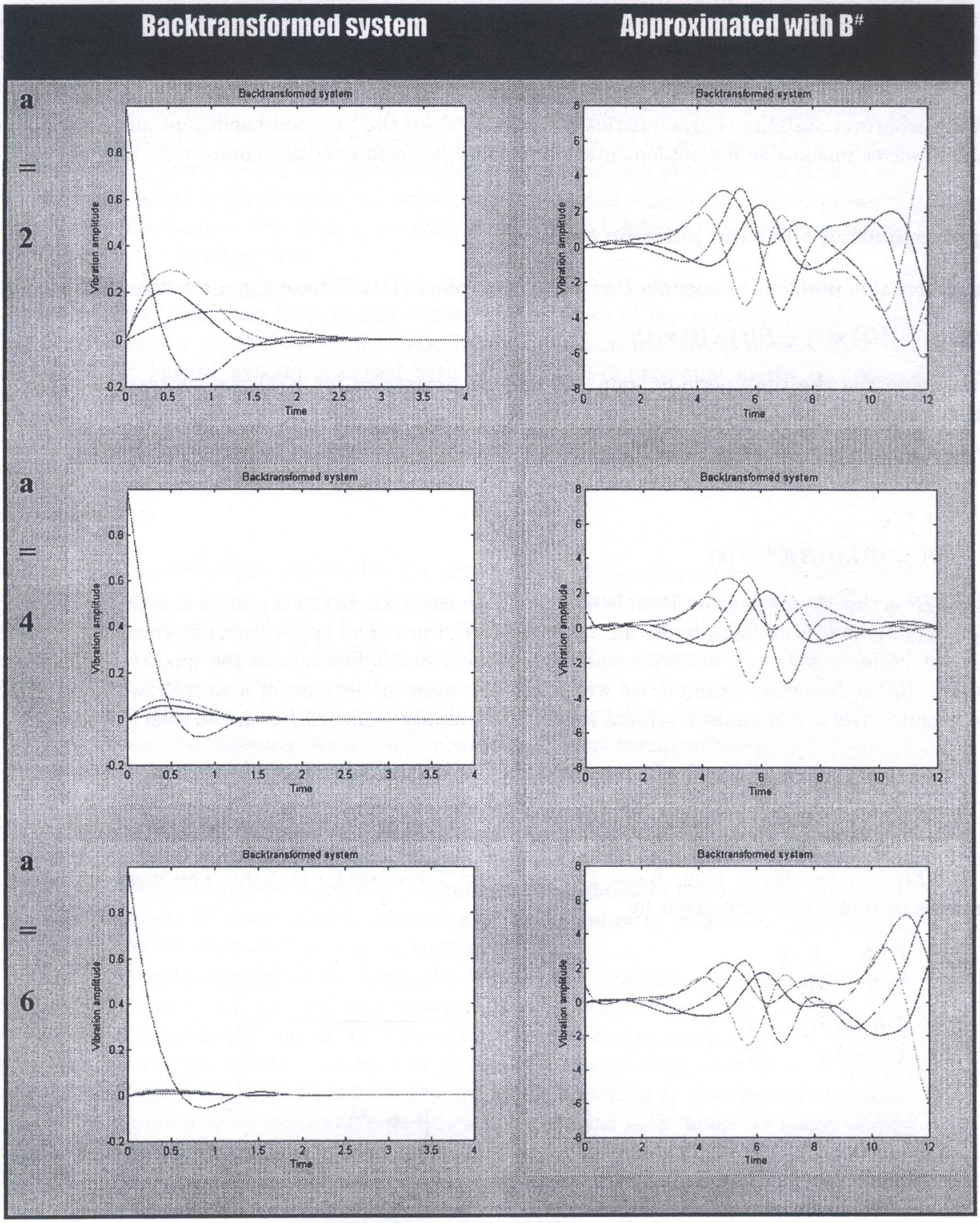

Fig. 3. Time traces for varios values of a

Following the procedure described previously, we form the first order system to be controlled and apply the Lyapunov Floquet transformation. For α=2 the uncontrolled system is unstable so a controller is designed in the time invariant domain. The usual choice for the auxiliary matrix, $\tilde{B}$, is the identity matrix, as it is simple and always results in a controllable pair (as required). Pole placement is used to determine the gains that will drive the constant coefficient system stable. The poles were selected to according to the general expression.

$$P = a(-1 \quad -1 \quad -0.1 \quad -0.1)\,.$$

Various values of a were used to investigate the effectiveness of the controller. Figure 3 shows a number of time traces that correspond to various values of a.

It can be observed that all time traces show stability when the backtransformed gains, in the form of (5), are used. If on the other hand we evaluate (6) with (7) as the expression for the gains, only $a = 4$ produces a stable system.

6. Stability Charts

To further investigate the range of parameters that result in stable control designs, a number of numerical simulations were performed for the system of Fig. 2. In order to vary the strength of the periodic term in the system, one of the variable parameters was selected to be α. The variable other parameter, a, was the value premultiplying the desired poles. This quantity, in essence, determined gain magnitude.

Figure 4 is a plot of the resulting stability measures, i.e. the largest Floquet multipliers. A stability chart can also be obtained by plotting only the stable and unstable regions in the space spanned by a and α (shown in Fig. 5). Inspection of these figures shows that, for low values of α (Case A, 0 to 1.2), increasing the control parameter tends to increase system stability. A sufficiently large control gain will always yield a stable system in such cases. For somewhat higher values of α (Case B, 1.2 to 2.7), there is a range of control gains for which the system is stable. For gain values outside of that range (both higher and lower), the system is unstable. In addition, there is a definitely an optimal value of control gain for which the system is most stable. Simply increasing the strength of the controller does not necessarily result in a more stable system and may actually drive the system unstable. For even higher values of α (Case C, 2.7 to 3.5), no level of control gain can be found that will result in system stability. As a result, it is clear that the control design procedure does not produce an acceptable result (namely stability) for very high values of periodicity, which is a disturbing result.

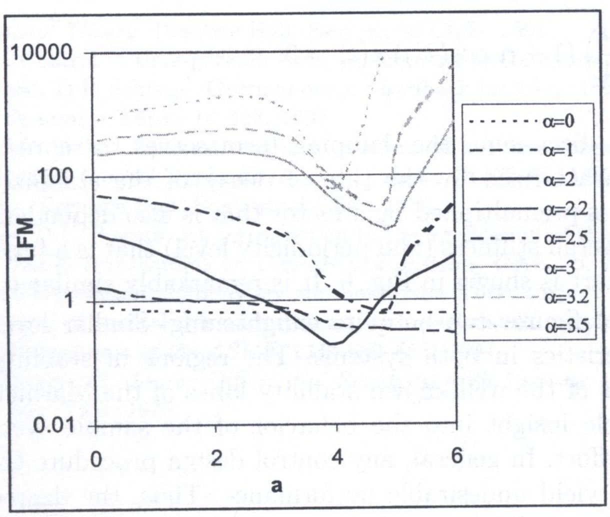

Fig. 4. Largest Floquet multiplier (LFM) levels as a function of a and α

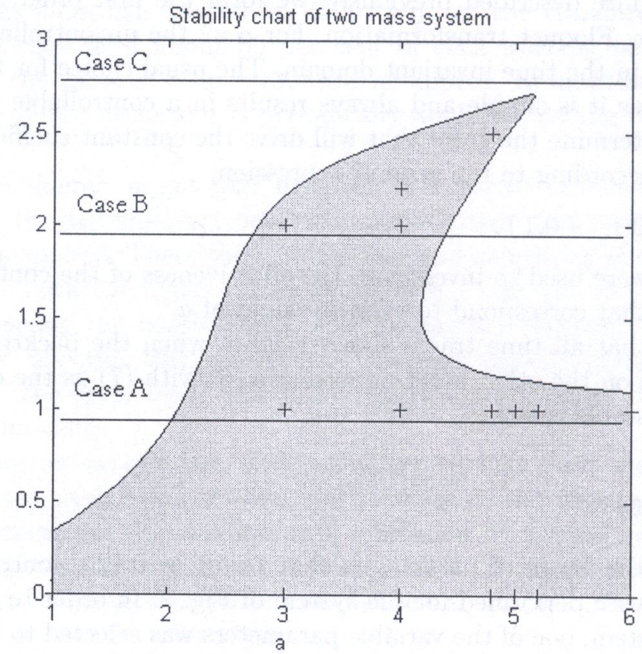

Fig. 5. Stability chart for the two mass system

While the above system is rather simplified, it is still sufficiently complex that the overall mechanisms that drive the unstable behavior are difficult to identify. However, the dependence of system stability on the level of periodicity is reminiscent of the characteristics of the damped Mathieu equation [7],

$$\ddot{\mathbf{x}}(t) + c\dot{\mathbf{x}}(t) + (\delta + 2\epsilon\cos(2t))\,\mathbf{x}(t) = 0. \qquad (8)$$

Please note that the sample dynamical system with the controller implemented using the auxiliary system approach has a periodic term that is inherently dependent upon the level of control effort. In order to obtain qualitative similarity between the sample dynamical system and the damped Mathieu equation, (8) is recast in the following form:

$$\ddot{\mathbf{x}}(t) + \left(\frac{a}{10}\right)\dot{\mathbf{x}}(t) + \left(\frac{a}{4}\right)(1 + \alpha\cos(2t))\,\mathbf{x}(t) = 0.$$

For the purposes of this discussion, the damping term serves to represent the control effort (or feedback gain level), characterized by the rate of decay of the stabilized auxiliary system. The position dependent term is premultiplied by a factor that is also dependent upon the feedback gain level. This results in an overall stiffness (and periodicity level) that is a function of the control effort. The resulting stability chart is shown in Fig. 6. It is remarkably similar to Fig. 5.

Comparison of the two figures can be quite enlightening. Similar levels of periodicity produce similar stability characteristics in both systems. The regions of stability and instability can be associated with variations of the well-known stability lobes of the Mathieu equation. This analogy serves to give considerable insight into the behavior of the sample system and its responses to various levels of control effort. In general, any control design procedure that magnify the effects of time periodic terms may yield undesirable performance. Thus, the degree of approximation (and resulting amplification of periodic terms in the control matrix) has a considerable impact on the effectiveness of the control design that is selected.

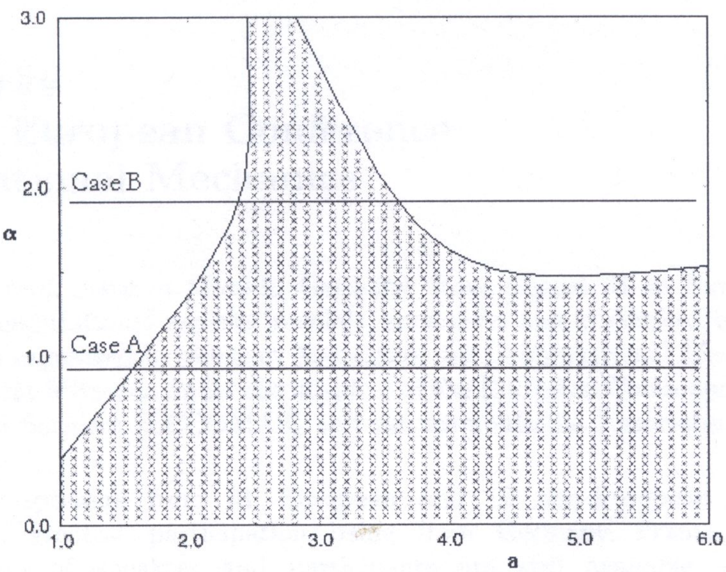

Fig. 6. Stability chart for modified Mathieu equation

7. Conclusion

An investigation of the stability characteristics of controlled time periodic dynamical systems has been presented. Specifically, a controller design procedure is considered in which the LF transform, coupled with an auxiliary system, is used. The fundamentals of the controller implementation are described and a sample dynamical system is investigated. The resulting dynamic characteristics are presented and discussed. Some interesting observations and conclusions are obtained. In particular, it is noted that, for systems with low controllability properties and high periodicity, the range of parameters that produce stable control designs can be somewhat limited. Therefore care must be taken when using this method to account for the above phenomenon. The stability region has to be established and should be used as a guide for stable control design. For an optimum design, the gains need to be hand picked and carefully analyzed.

References

[1] W.L. Brogan. *Modern Control Theory.* Prentice Hall, Englewood Cliffs, 1991.

[2] R.A. Calico, W.E. Wiesel. Control of time-periodic systems. *Journal of Guidance*, **7**(6): 671–676, 1984.

[3] A.J. Calise, M.E. Wasikowski, D.P. Schrage. Optimal output feedback for linear time periodic systems. *Journal of Guidance Control and Dynamics*, **15**(2): 16–423, 1992.

[4] P. Joseph, R. Pandiyan, S.C. Sinha. Optimal control of mechanical systems subjected to periodic loading via Chebysev polynomials. *Optimal Control Applications and Methods*, **14**: 75–90, 1993.

[5] T. Kailath. *Linear Systems.* Prentice Hall, Englewood Cliffs, 1980.

[6] V. Lovass-Nagy, R.J. Miller, D.L. Powers. An introduction to the application of the simplest matrix generalized inverse in system science. *IEEE Transactions on Circuits and Systems*, **CAS-25**: 766–771, 1978.

[7] A.H. Nayfeh. *Problems in Perturbation.* John Wiley and Sons, New York, 1985.

[8] S.C. Sinha, P. Joseph. Control of general dynamic systems with periodically varying parameters via Lyapunov Floquet transformation. *Transactions of the ASME*, **116**: 650–658, 1994.

[9] V.A. Yakubovich, V.M. Starzhinski. *Linear Differential Equations with Periodic Coefficients, Part I and Part II.* John Wiley and Sons, New York, 1975.