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# Parametric excitation of pipes through fluid flow

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In this paper the dynamic behaviour of a continuum inextensible pipe containing fluid flow is investigated. The fluid is considered to be incompressible, frictionless and its velocity relative to the pipe has the same but time-periodic magnitude along the pipe at a certain time instant.

The equations of motion are derived via Lagrangian equations and Hamilton's principle. The system is non-conservative, and the amount of energy carried in and out by the flow appears in the model. It is well-known, that intricate stability problems arise when the flow pulsates and the corresponding mathematical model, a system of ordinary or partial differential equations, becomes time-periodic.

The method which constructs the state transition matrix used in Floquet theory in terms of Chebyshev polynomials is especially effective for stability analysis of systems with multi-degree-of-freedom. The stability charts are created w.r.t. the forcing frequency $\omega$, the perturbation amplitude $\nu$ and the average flow velocity $U$.

**Keywords:** pulsatile flow, Floquet theory, Chebyshev polynomials

## 1. Introduction

The equation of motion of a simply supported continuum pipe derived from *Hamilton's principle* was already discussed by Housner [4] in connection with the vibrations of the Trans-Arabian pipeline. However, the correct equation for the general case was derived by Benjamin [1] using *Hamiltonian action-function.* In that paper the equation of motions of articulated pipes were also discussed. Analysis of different cases of elastic pipes carrying fluid yielded many paper in this field (see [2, 12, 7]). Semler and Païdoussis [10] have also overviewed the applicability of some numerical approaches in parametric resonances of cantilevered pipes.

## 2. Description of the model

In the following we consider a continuum elastic pipe shown in Fig. 1. One of its ends is attached to the wall while the other end can move freely. The motions are considered in the horizontal $xy$-plane. The masses per unit length of the pipe and the fluid are $M$ and $m$, respectively. The upstream

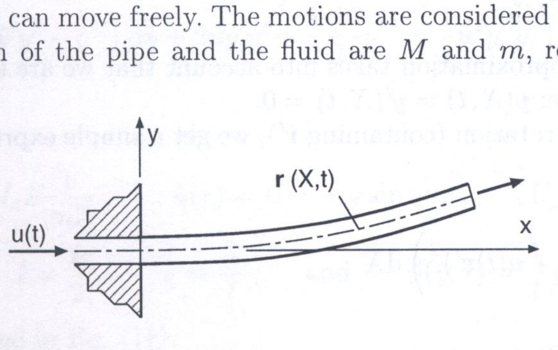


**Fig. 1.** Sketch of an elastic pipe with clamped-free ends

mass-flow $mu(t)$ is generally a periodic function of the time in the following manner:

$$u(t) = U(1 + \nu \sin \omega t)\,. \tag{1}$$

The length of the pipe is $L$ and its axis is inextensible (i.e. the cross-sectional area of the pipe remains constant):

$$x'^2 + y'^2 = 1 \tag{2}$$

when the position vector of the pipe axis is $\mathbf{r} = \mathrm{col}\,[x(X,t),\ y(X,t)]$ and $'$ denotes $\partial/\partial X$. Hence,

$$x(X,t) = \int_0^X \sqrt{1 - y'^2(\xi,t)}\,\mathrm{d}\xi \tag{3}$$

where $X$ is the identifier coordinate (i.e. the arc-length) along the pipe.

Let $\alpha$ be the angle between the pipe and the axis $x$. Then

$$\cos\alpha = x' = \frac{1}{\sqrt{1 + \tilde{y}'^2}} = \sqrt{1 - y'^2}$$

where $\tilde{y}(x,t) = y(X(x),t)$ is the graph of the axis in an orthogonal coordinate system, $\tilde{y}' = y'/x'$, and one can prove that $\tilde{y}'' = y''/(x')^4$ follows from the inextensibility of the pipe ($\tilde{y}'$ and $\tilde{y}''$ denote $\frac{\partial \tilde{y}}{\partial x}$ and $\frac{\partial^2 \tilde{y}}{\partial x^2}$, respectively). Thus, the curvature $\kappa$ of the pipe axis is

$$\kappa = \frac{-\tilde{y}''}{(1 + \tilde{y}'^2)^{\frac{3}{2}}} = \frac{-y''}{x'} \equiv \frac{-y''}{\sqrt{1 - y'^2}}\,. \tag{4}$$

### 2.1. Equations of motion

According to *Hamilton's principle*

$$\delta \int_{t_1}^{t_2} \{\mathcal{U} - \mathcal{T}\}\,\mathrm{d}t = \int_{t_1}^{t_2} \delta\mathcal{W}\,\mathrm{d}t \tag{5}$$

where $\mathcal{U}$, $\mathcal{T}$ and $\delta\mathcal{W}$ are the energy of strain, the whole kinetic energy and the virtual work, respectively.

The bending moment of the beam-like pipe is a linear function of the curvature: $M_z = \kappa I_z E$. The energy of strain of a beam is

$$\mathcal{U} = \frac{I_z E}{2} \int_0^L \kappa^2\,\mathrm{d}X \approx \frac{I_z E}{2} \int_0^L y''^2\left(1 + y'^2\right)\mathrm{d}X \tag{6}$$

where the fourth degree approximation takes into account that we are investigating the stability of the trivial equilibrial shape: $y(X,t) = y'(X,t) = 0$.

Neglecting the terms of rotation (containing $\dot{\mathbf{r}}'$), we get a simple expression for the kinetic energy of the pipe and the fluid:

$$\mathcal{T} = \int_0^L \left(\frac{1}{2}M\dot{\mathbf{r}}^2 + \frac{m}{2}\left(\dot{\mathbf{r}} + u(t)\mathbf{r}'\right)^2\right)\mathrm{d}X \tag{7}$$

where $\cdot$ ('dot') denotes $\partial/\partial t$.

The external forces changing the momentum of the flow between upstream and downstream at the ends of the pipe are

$$\mathbf{F} = -\int_0^L mu\left(\dot{\mathbf{r}} + u\mathbf{r}'\right)' \, \mathrm{d}X \equiv -mu\left[\dot{\mathbf{r}} + u\mathbf{r}'\right]_0^L \equiv \mathbf{F}_L + \mathbf{F}_0 \,. \tag{8}$$

Thus, the virtual work of these forces is $\delta\mathcal{W} = \mathbf{F}_L\delta\mathbf{r}_L + \mathbf{F}_0\delta\mathbf{r}_0$.

Putting the expressions of $\mathcal{U}$, $\mathcal{T}$ and $\delta\mathcal{W}$ in Eq. (5) we get

$$\int_{t_1}^{t_2}\int_0^L \left\{ I_zE\left(y''\delta y''\left(1+y'^2\right) + y''^2 y'\delta y'\right) - (M+m)\,\dot{\mathbf{r}}\delta\dot{\mathbf{r}} - mu\left(\mathbf{r}'\delta\dot{\mathbf{r}} + \dot{\mathbf{r}}\delta\mathbf{r}'\right)\right\} \mathrm{d}X\,\mathrm{d}t = -\int_{t_1}^{t_2} \left[mu\left(\dot{\mathbf{r}} + u\mathbf{r}'\right)\delta\mathbf{r}\right]_0^L \mathrm{d}t. \tag{9}$$

After integrating by parts (excluding the term of $I_zE$) and eliminating $\delta x$ one can obtain

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^L \Big\{ & I_zE\left(y''\delta y''\left(1+y'^2\right) + y''^2 y'\delta y'\right) \\ & + \delta y\left(\mathcal{G}\langle y(X,t)\rangle - \left(1+\frac{y'^2}{2}\right) y'\mathcal{G}\langle x(X,t)\rangle\right) \\ & + \delta y\left(1+\frac{3y'^2}{2}\right) y'' \int_X^L \mathcal{G}\langle x(\xi,t)\rangle \,\mathrm{d}\xi \Big\} \,\mathrm{d}X\,\mathrm{d}t = 0\end{aligned} \tag{10}$$

where

$$\mathcal{G}\langle z(X,t)\rangle = (M+m)\,\ddot{z} + 2mu\dot{z}' + m\dot{u}z' + mu^2 z''.$$

From Eq. (3) one can express the derivatives of $x$ as the function of the derivatives of $y$. Thus, we can eliminate all the derivatives of $x$ from Eq. (10). After neglecting the fifth and higher order terms we obtain the equation of motion in dimensionless form which corresponds to the results in [9] presented by Semler *et al.*:

$$\begin{aligned}\int_{\tau_1}^{\tau_2}\int_0^2 \Big\{ & \left(y''\delta y''\left(1+y'^2\right) + y''^2 y'\delta y'\right) + \delta y\left(3\ddot{y} + 2\tilde{u}(\tau)\dot{y}'\left(1+y'^2\right)\right) \\ & + \delta y\left(\frac{1}{\mu}\tilde{u}^2(\tau)y''\left(1+y'^2\right) + 3y'\int_0^\xi \left(\ddot{y}'y' + \dot{y}'^2\right)\mathrm{d}\eta + \frac{\mathrm{d}\tilde{u}}{\mathrm{d}\tau}(2-\xi)\,y''\left(1+\frac{3}{2}y'^2\right)\right) \\ & - \delta y\, y'' \int_\xi^2 \left(3\int_0^\eta \left(\ddot{y}'y' + \dot{y}'^2\right)\mathrm{d}\tilde{\eta} + 2\tilde{u}\dot{y}'y' + \frac{1}{2}\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\tau}y'^2 + \frac{1}{\mu}\tilde{u}^2 y''y'\right)\mathrm{d}\eta \Big\}\,\mathrm{d}\xi\,\mathrm{d}\tau = 0\end{aligned} \tag{11}$$

where

$$\mu = \frac{3m}{M+m}\,, \qquad \alpha^2 = I_zE\frac{\mu}{ml^4}\,, \qquad \tilde{u}(\tau) = \tilde{U}(1+\nu\sin w\tau)\,, \qquad \tilde{U} = \frac{\mu}{\alpha l}U\,,$$

$$w = \frac{\omega}{\alpha}\,, \qquad \tau = \alpha t\,, \qquad l = \frac{L}{2}\,, \qquad \xi = \frac{X}{l} \qquad \text{and} \qquad \tilde{y}(\xi,\tau) = \frac{1}{l}\,y\Big(\xi l, \frac{\tau}{\alpha}\Big)$$

but the 'tildes' were dropped in Eq. (11).

The boundary conditions are as follows:

clamped end at $\xi = 0$: $\quad y(0) = y'(0) = 0$,

free end at $\xi = 2$: $\quad y''(2) = y'''(2) = 0$.

### 2.2. Discretizing the equation of motion

We use *Galerkin's method* for discretizing Eq. (11). Assuming

$$y(\xi,\tau) = \sum_{i=1}^{n} y_i(\tau)\varphi_i(\xi) \tag{12}$$

where $\varphi_i(\xi)$ is the appropriate base function that satisfies the boundary conditions and $n$ is the number of modes approximated by base functions. Substituting the form (12) of $y(\xi,t)$ in Eq. (11) the integral form will be

$$\begin{aligned}\int_{\tau_1}^{\tau_2} \delta y_i \Big\{ & K_{0ij} y_j + 3M_{ij}\ddot{y}_j + 2\tilde{u}(\tau)C_{ij}\dot{y}_j + \frac{1}{\mu}\tilde{u}^2(\tau)K_{1ij}y_j + \frac{\mathrm{d}\tilde{u}(\tau)}{\mathrm{d}\tau}\left(2K_{1ij} - K_{20ij}\right)y_j \\ & + K_{01ijkl}y_j y_k y_l + 3\left(M_{1ijkl} - M_{10ijkl}\right)\ddot{y}_j y_k y_l + 3\left(M_{1ijkl} - M_{10ijkl}\right)\dot{y}_j \dot{y}_k y_l \\ & + 2\tilde{u}\left(C_{1ijkl} - C_{10ijkl}\right)\dot{y}_j y_k y_l + \frac{1}{\mu}\tilde{u}^2\left(K_{11ijkl} - K_{12ijkl}\right)y_j y_k y_l \\ & + \frac{1}{2}\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\tau}\left(6K_{11ijkl} - 3K_{21ijkl} - C_{10ijkl}\right)y_j y_k y_l \Big\}\,\mathrm{d}\tau = 0\end{aligned} \tag{13}$$

where the $\sum$-s were dropped according to Einstein's convention. Furthermore,

$$M_{ij} = \int_0^2 \varphi_i\varphi_j\,\mathrm{d}\xi\,, \qquad C_{ij} = \int_0^2 \varphi_i\varphi_j'\,\mathrm{d}\xi\,, \qquad K_{0ij} = \int_0^2 \varphi_i''\varphi_j''\,\mathrm{d}\xi\,, \qquad K_{1ij} = \int_0^2 \varphi_i\varphi_j''\,\mathrm{d}\xi\,,$$

$$K_{20ij} = \int_0^2 \xi\varphi_i\varphi_j''\,\mathrm{d}\xi\,, \qquad K_{01ijkl} = \int_0^2 \left(\varphi_i''\varphi_j' + \varphi_i'\varphi_j''\right)\varphi_k''\varphi_l'\,\mathrm{d}\xi\,, \qquad C_{1ijkl} = \int_0^2 \varphi_i\varphi_j'\varphi_k'\varphi_l'\,\mathrm{d}\xi\,,$$

$$K_{11ijkl} = \int_0^2 \varphi_i\varphi_j''\varphi_k'\varphi_l'\,\mathrm{d}\xi\,, \qquad K_{21ijkl} = \int_0^2 \xi\varphi_i\varphi_j''\varphi_k'\varphi_l'\,\mathrm{d}\xi\,, \qquad M_{1ijkl} = \int_0^2 \varphi_i\varphi_l'\int_0^{\xi}\varphi_j'\varphi_k'\,\mathrm{d}\eta\,\mathrm{d}\xi\,,$$

$$M_{10ijkl} = \int_0^2 \varphi_i\varphi_l''\int_{\xi}^{2}\int_0^{\eta}\varphi_j'\varphi_k'\,\mathrm{d}\tilde{\eta}\,\mathrm{d}\eta\,\mathrm{d}\xi\,, \qquad C_{10ijkl} = \int_0^2 \varphi_i\varphi_l''\int_{\xi}^{2}\varphi_j'\varphi_k'\,\mathrm{d}\eta\,\mathrm{d}\xi\,,$$

$$K_{12ijkl} = \int_0^2 \varphi_i\varphi_l''\int_{\xi}^{2}\varphi_j''\varphi_k'\,\mathrm{d}\eta\,\mathrm{d}\xi\,.$$

The integral is zero for arbitrary $\delta y_i$. Hence, its coefficient (i.e. the expression in the braces in Eq. (13)) must be zero.

However, $\ddot{y}_j$ is also in the nonlinear terms:

$$3\left(\mathbf{M} + \left(\mathbf{M}_{1kl} - \mathbf{M}_{10kl}\right)y_k y_l\right)\ddot{\mathbf{y}}$$

where for sake of brevity we write the coefficients partially in matrix representation (instead of the indices $i, j$) and keeping Einstein's convention in the third and fourth indices $(k, l)$.

If we multiply the term of $\ddot{\mathbf{y}}$ with $\mathbf{I} - (\mathbf{M}_{1mn} - \mathbf{M}_{10mn})\,\mathbf{M}^{-1} y_m y_n$ we get $3\mathbf{M}\ddot{\mathbf{y}}$ where the terms of order higher than four were neglected.

Applying this matrix-multiplication on the other terms it yields

$$\begin{aligned}
&3\mathbf{M}\ddot{\mathbf{y}} + 2\tilde{u}(\tau)\mathbf{C}\dot{y} + \left(\mathbf{K}_0 + \frac{1}{\mu}\tilde{u}^2(\tau)\mathbf{K}_1 + \frac{\mathrm{d}\tilde{u}(\tau)}{\mathrm{d}\tau}\mathbf{K}_2\right)\mathbf{y} \\
&\quad + 3\left(\mathbf{M}_{1kl} - \mathbf{M}_{10kl}\right)\ddot{\mathbf{y}}y_k y_l + 2\tilde{u}\left(\mathbf{C}_{1kl} - \mathbf{C}_{10kl} - \tilde{\mathbf{I}}_{kl}\mathbf{C}\right)\dot{\mathbf{y}}y_k y_l \\
&\quad + \left(\mathbf{K}_{01kl} - \tilde{\mathbf{I}}_{kl}\mathbf{K}_0 + \frac{1}{\mu}\tilde{u}^2\left(\mathbf{K}_{11kl} - \mathbf{K}_{12kl} - \tilde{\mathbf{I}}_{kl}\mathbf{K}_1\right)\right)\mathbf{y}y_k y_l \\
&\quad + \frac{1}{2}\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\tau}\left(6\mathbf{K}_{11kl} - 3\mathbf{K}_{21kl} - \mathbf{C}_{10kl} - 2\tilde{\mathbf{I}}_{kl}\mathbf{K}_2\right)\mathbf{y}y_k y_l = 0
\end{aligned} \tag{14}$$

where

$$\mathbf{y} = [y_j]\ , \qquad \tilde{\mathbf{I}}_{kl} = \left(\mathbf{M}_{1kl} - \mathbf{M}_{10kl}\right)\mathbf{M}^{-1}\ , \qquad \mathbf{K}_2 = 2\mathbf{K}_1 - \mathbf{K}_{20}\ .$$

The base functions were searched as linear combination of *Krylov functions*. The following set of functions satisfies the boundary conditions of the analyzed model:

$$\varphi_i(\xi) = U(\beta_i\xi) - \frac{S(2\beta_i)}{T(2\beta_i)}V(\beta_i\xi)$$

where $S(x)$, $T(x)$, $U(x)$ and $V(x)$ are the *Krylov functions* and $\beta_i$ is the root of $\cosh 2x \cos 2x = -1$ (see Table 1).

**Table 1.** Critical values of the flow-velocity $\tilde{U}$ at $\mu = 1$

| $\beta_n$ | Number of base functions, $n$ | $\tilde{U}_{cr}$ | Poincaré–Liapunov constant |
|---|---|---|---|
| 0.9375 | 1 | $\infty$ | — |
| 2.3470 | 2 | 3.3419 | −0.0805 |
| 3.9274 | 3 | 4.2904 | −0.2752 |
| 5.4978 | 4 | 4.2716 | −0.5367 |
| 7.0686 | 5 | 4.2408 | |
| 8.6394 | 6 | 4.2411 | |

## 3. Stability analysis

### 3.1. Autonomous case

If the perturbation amplitude of the flow velocity is zero ($\nu = 0$) Eq. (14) will be a system of autonomous differential equations ($\tilde{u}(\tau) \equiv \tilde{U}$ and $\tilde{u}'(\tau) \equiv 0$).

Using only the first base function ($n = 1$) the system appears to be always stable:

$$1.5\ddot{y}_1 + \tilde{U}\dot{y}_1 + \left(0.3863 + 0.1073\frac{\tilde{U}^2}{\mu}\right)y_1 = 0$$

because $\tilde{U}$ and $\mu$ are always greater than zero.

In case of $n = 2$ the coefficients of the characteristic polynomial $\lambda^4 + \sum_{k=0}^{3} a_k \lambda^k$ are as follows

$$a_0 = 0.0736\frac{\tilde{U}^4}{\mu^2} + 0.4381\frac{\tilde{U}^2}{\mu} + 2.6051,$$

$$a_1 = \left(-0.1954\frac{\tilde{U}^2}{\mu} + 6.9150\right)\tilde{U},$$

$$a_2 = \left(0.8461 - \frac{1.0363}{\mu}\right)\tilde{U}^2 + 10.372,$$

$$a_3 = 1.3333\tilde{U}.$$

Applying *Routh-Hurwitz criterion* for stability analysis of the roots of the characteristic polynomial one can obtain at $\mu = 1$ that the system is asymptotically stable if and only if

$$0 < \tilde{U} < 3.3419\ .$$

At the upper bound a pair of pure imaginary roots cross the imaginary axis, i.e. *Hopf bifurcation* can occur.

In Table 1 are listed the critical values of $\tilde{U}$ w.r.t. the number of base function. In each case the system is stable below these values ($0 < \tilde{U} < \tilde{U}_{cr}$) and we have to take in account a *Hopf bifurcation.*

Because of the symmetric nonlinearities (there isn't any second order term) the plane of the critical eigenvectors approximates the centre manifold in second order. Hence, the centre manifold reduction can be done easily. The bifurcation analysis results negative Poincaré-Liapunov constants ($n = 2$, 3, 4), i.e. *super-critical Hopf bifurcation* takes place in these cases.

### 3.2. Non-autonomous case

It is well-known from the *Floquet theory* (see e.g. [3]) that the stability of a linear system with periodic coefficients (e.g. $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$, $\mathbf{A}(t+T) = \mathbf{A}(t)$) is determined by the characteristic multipliers (i.e. the eigenvalues of the *Floquet Transition Matrix* $\mathbf{C} = \mathbf{\Phi}(T)$, where $\mathbf{\Phi}(t)$ is the *State Transition Matrix*: $\dot{\mathbf{\Phi}} = \mathbf{A}\mathbf{\Phi}$ and $\mathbf{\Phi}(0) = \mathbf{I}$ is the identity matrix). Sinha and Wu [11] has developed a numerical method for stability analysis in any general case based on Chebyshev polynomials. If we present the solution vector as an expansion of these polynomials we obtain a set of linear algebraic equations from the differential equations. The solution of this system gives the coefficients of the Chebyshev expansion. In this way we obtain an approximation of the *Floquet Transition Matrix* $\mathbf{C}$ and we

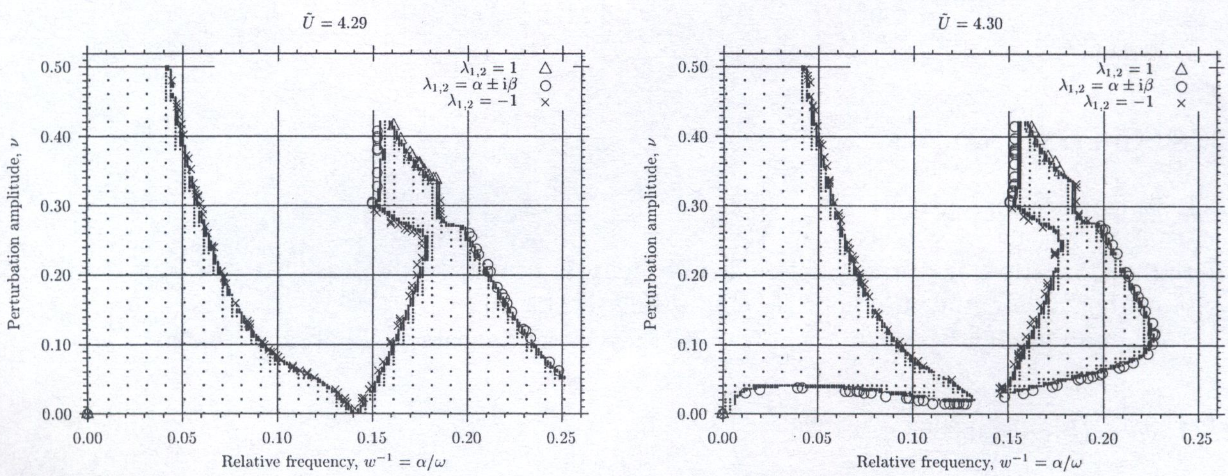


**Fig. 2.** Stability charts of a cantilevered pipe, $\tilde{U}_{cr} = 4.2904$ (3 modes)

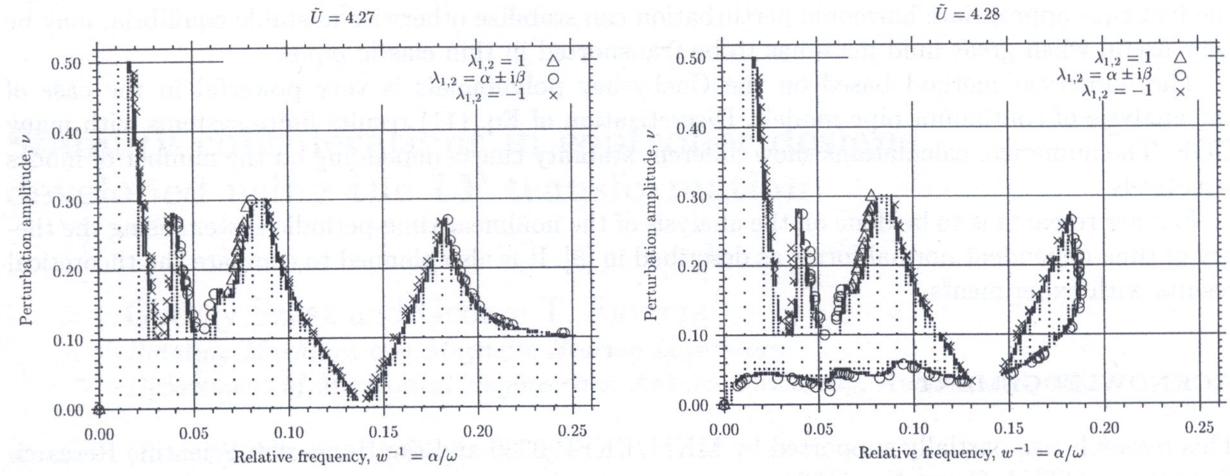


**Fig. 3.** Stability charts of a cantilevered pipe, $\tilde{U}_{cr} = 4.2716$ (4 modes)

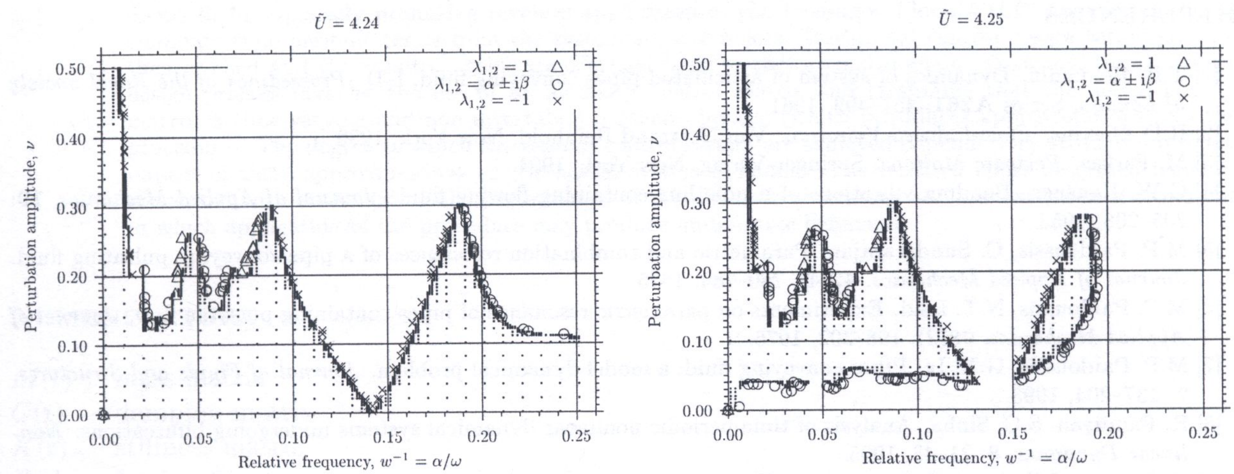


**Fig. 4.** Stability charts of a cantilevered pipe, $\tilde{U}_{cr} = 4.2408$ (5 modes)

can analyze the characteristic multipliers $\lambda_i(\mathbf{C})$ at particular system parameters. The linear system is asymptotically stable iff all the multipliers are located in the open unit disk of the complex plane. Fixing the parameters of our mechanical structures and choosing $\tilde{U}$, $\nu$ and $w$ for bifurcation parameters, Sinha's numerical method provides the stability domains of the system.

Figures 2–4 show slices of the three dimensional $(\tilde{U},\, w,\, \nu)$ theoretic stability domain around the critical value $\tilde{U}_{cr}$ obtained in the autonomous case. The dotted region represents the stable domain of the analyzed space. The symbols on the boundary show the type of stability loss, i.e. the way where the characteristic multipliers leave the unit circle while crossing the stability boundary.

It is worthwhile to notice that the stability charts in Fig. 2 differ significantly from the ones in Figs. 3–4. It means that the first three base functions are not sufficient to describe the exact behaviour of the model.

## 4. Conclusions

Equations of motion of a cantilevered pipe were derived via *Lagrangian equations* and *Hamilton's principle*. Nonlinear stability analysis were performed in autonomous case, while in non-autonomous (time-periodic) case the linearized system was analyzed. The stability charts show that the harmonic perturbation of the fluid velocity with certain values of $\nu$ and $w$ can destabilize the equilibrium of the pipe even for $\tilde{U} < \tilde{U}_{cr}$ as also pointed out by Païdoussis in [5] and [6]. On the other hand, the fact that appropriate harmonic perturbation can stabilize otherwise unstable equilibria, may be very useful when great fluid mass has to be transported in thin elastic pipes.

The numerical method based on the Chebyshev polynomials is very powerful in the case of the analysis of continuum pipe models. Discretization of Eq. (11) results finite systems with many DOF. The numerical calculations show different stability charts depending on the number of modes sensitively.

Further research is to be done on the analysis of the nonlinear time-periodic system using the theory of time-dependent normal forms as described in [8]. It is also planned to compare the theoretical results with experiments.

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