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Rigidity of square grids with holes

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Bolker and Crapo gave a graph theoretical model of square grid frameworks with diagonal rods of certain squares. Using this model there are very fast methods for connected planar square grid frameworks to determine their (infinitesimal) rigidity when we can use diagonal rods, diagonal cables or struts, long rods, long cables or struts. But what about square grids containing some kind of holes? We will show that the model can be extended to the problem of holes too.

Keywords: grids, rigidity, frameworks, graphs

1. Preliminaries

In the case of hall structures and blocks of flats the rigid frame of the roof is often a grid of squares (or rectangles). To prevent the horizontal deformations we have to make this grid rigid. Moreover the ground-plan of the building is not a rectangle in general, and the building may have a court inside. But how can we check the rigidity of the construction? And which are the minimal rigid structures? In this paper we will show a fast method to solve these problems. (We have to mention that in practice the buildings have much more rigidifying elements than they would require in the minimal case.)

Throughout, every framework consists of rigid rods and rotatable joints (in the plane). The deformations of a square grid can be described with the rotations of the rods in rows and columns (see Fig. 1). A diagonal rod in a square of the grid results that the rotations of the corresponding row and column must be equal. If we use a northeast-southwest position cable or a northwest-southeast position strut then the rotation of the corresponding row must be greater than or equal to the rotation of the corresponding column. If we change the position of the cable or the strut the inequality will be reversed. We can observe that the effect of the cable and the strut is the same, so we will use only cables.

Long rods and cables can connect any two joints of the grid, but the horizontal and vertical ones cannot prevent the infinitesimal motions so we will ignore them. The range of a long rod consists of those rows and columns of the grid which are touched by the rod (see Fig. 2).

Let the rotations of rows and columns be denoted by α_i's and β_j's, respectively. Considering an infinitesimal rotation α_i there is a horizontal relative translation α_i (the edge length of the grid is supposed to be 1) between the bottom and the top of the i-th row. This translation can be projected to the diagonal: $\alpha_i \cos\gamma$ (where γ denotes the angle of the long rod to the horizontal line). Similarly, β_j produces a vertical relative translation β_j, and its projection is $-\beta_j \sin\gamma$. Together all the rotations in the range of the long rod must result no change of distance between the marked

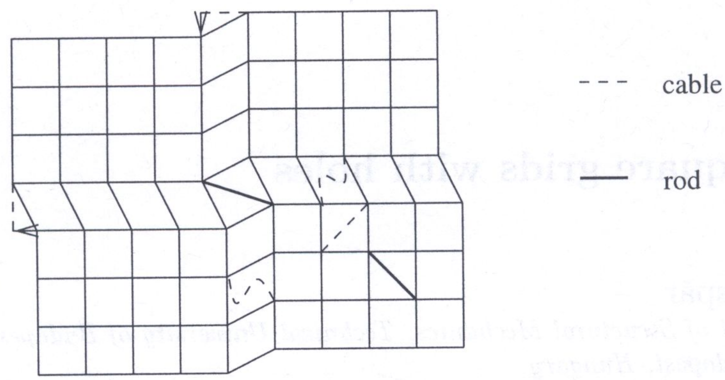

Fig. 1. Deformations of a square grid

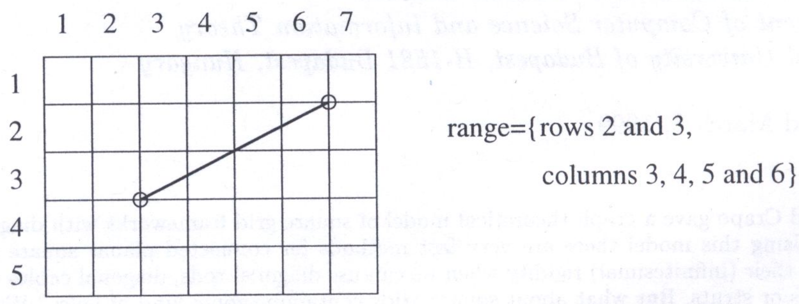

Fig. 2. The range of a long rod

points (the ends of the long diagonal):

$$(\alpha_3 + \alpha_2)\cos\gamma - (\beta_3 + \beta_4 + \beta_5 + \beta_6)\sin\gamma = 0.$$

Multiplying by the length of the rod:

$$4(\alpha_3 + \alpha_2) - 2(\beta_3 + \beta_4 + \beta_5 + \beta_6) = 0,$$

where the coefficients are the number of columns and rows in the range, respectively.

So, in general, the long rod results the following equation:

$$|\{\text{columns in range}\}| \sum_{\substack{\text{rows in}\\ \text{range}}} \alpha_i - |\{\text{rows in range}\}| \sum_{\substack{\text{columns}\\ \text{in range}}} \beta_j = 0.$$

The effect of a long cable – similarly to the case of diagonal cables – is an inequality between the sums of rotations which depends on the position of the cable, for example in the case of a northeast-southwest position cable:

$$|\{\text{columns in range}\}| \sum_{\substack{\text{rows in}\\ \text{range}}} \alpha_i - |\{\text{rows in range}\}| \sum_{\substack{\text{columns}\\ \text{in range}}} \beta_j \geq 0.$$

(See also [3].)

If we have a $k \times l$ square grid we can define a bipartite graph $G(A, B; E)$ where $|A| = k$ and $|B| = l$, the vertices of A correspond to the rows of the grid and the vertices of B correspond to the columns of the grid [4]. If there is a diagonal rod in a square of the grid then we put an (undirected) edge between the vertices of the corresponding row and column. If there is a northeast-southwest position cable in the square then we put a directed edge from the vertex of the corresponding row to the vertex of the corresponding column, and we put a directed edge in the opposite direction if the cable has the other position.

Now let us consider this graph. If we take the strongly connected components of it we can state that the rotations of the rows and columns whose vertices are in the same component must be equal (see [2] and [1]). For example, if we have a $k \times l$ square grid with diagonal rods and cables then this structure is infinitesimally rigid if and only if its graph is strongly connected. If we have long rods and cables too then we get a very simple system of equations and inequalities.

Using this model we can determine very quickly whether a $k \times l$ square grid with rods and cables is rigid or not. The engineers' usual method, checking the rank of the rigidity (geometrical) matrix would require $O(k^3 l^3)$ operations. But in this way, checking the graph condition requires $O(k + l)$ operations only, i.e. it works in linear time and – in the case of long rods and cables – the size of the coefficient matrix is much less than the size of the rigidity matrix.

We have to mention that these methods work without changes in case of square grids with any shape without holes, we only have to define the vertices of the graph for the connected sections of rows and columns. Moreover, if we have a grid of paralelograms the only changes will be about the coefficients in the equations and inequalities of the long rods and cables.

In Section 2 we will show how this model can be applied in case of square grids containing holes and with diagonal rods of certain squares, and finally in Section 3 we will study the most general case of planar square grid frameworks.

2. Holes in the grid

Let us consider at first a $k \times l$ square grid containing one hole and some diagonal rods of certain squares, like in Fig. 3. We want to construct a graph similarly to the former cases. The number of vertices will increase because the rotations of the disconnected segments of the rows and columns which are cut by the hole can be different, so we need different vertices for the segments in the graph. For example, the graph of the framework in Fig. 3 has 23 vertices, 11 vertices from rows and 12 from columns. They will be numbered from top to bottom and from left to right, so their rotations will be denoted by α_1, α_2, α'_3, α''_3, α'''_3, α'_4, α''_4, …, α_7 and β_1, β_2, β'_3, β''_3, β'_4, β''_4, …, β_8.

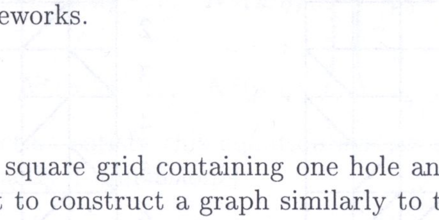

Fig. 3. Walking around the hole

"Walking" around the hole let us sum the rotations of the touched row-segments with positive or negative signs depending on the direction of our walk. We can observe that this sum must be equal to zero, because after walking around the hole we arrive at the initial point. Similar equation holds for the touched column-segments, too. In Fig. 3 it means that

$$\begin{aligned} &\alpha'_5 + \alpha'_4 + \alpha'_3 - \alpha''_3 + \alpha''_3 - \alpha'''_3 - \alpha''_4 - \alpha''_5 = 0, \\ &\beta'_3 + \beta'_4 + \beta'_5 + \beta'_6 - \beta''_6 - \beta''_5 - \beta''_4 - \beta''_3 \;\; = 0. \end{aligned} \tag{1}$$

The framework is rigid if and only if the equation system (1) has a one dimensional set of solutions, because the constant vectors, which form a one dimensional subspace, realize the congruent motions of the framework.

Let $\varphi_1, \varphi_2, \ldots, \varphi_s$ be the common values of the rotations in the connected components of the graph of the framework. We have only two equations as the effect of the hole, so the number of connected components of the graph can be at most three if we want the framework to be rigid. Now the equation system (1) can be reduced to the following system:

$$N_1^r\varphi_1 + N_2^r\varphi_2 + N_3^r\varphi_3 = 0,$$
$$N_1^c\varphi_1 + N_2^c\varphi_2 + N_3^c\varphi_3 = 0,$$

where N_i^r equals the number of row-segments in the i-th component with positive sign minus the number of row-segments in the i-th component with negative sign, and similarly N_j^c equals the number of column-segments in the j-th component with positive sign minus the number of column-segments in the j-th component with negative sign.

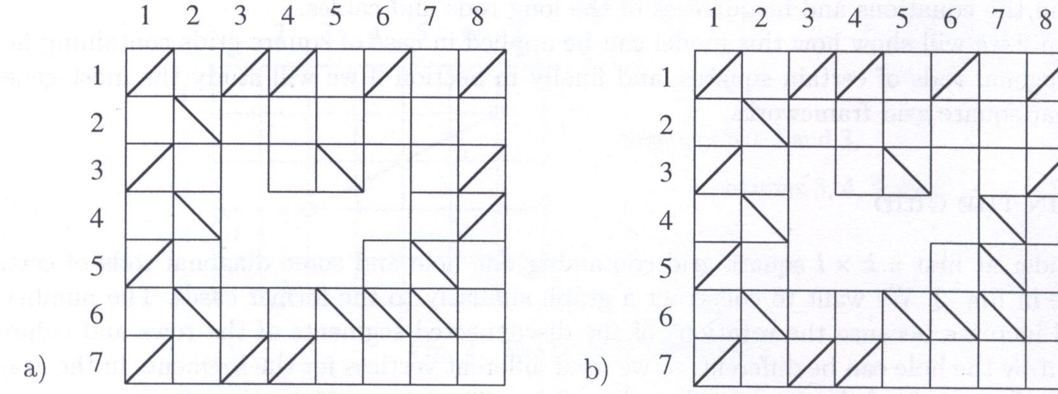

Fig. 4. Nonrigid and rigid systems of diagonal rods

For example the three component of the graph of the framework in Fig. 4a are the following: $\{\alpha_1, \alpha_3', \alpha_3''', \alpha_4'', \alpha_5', \alpha_7, \beta_1, \beta_3', \beta_3'', \beta_4', \beta_4'', \beta_6', \beta_8\}$; $\{\alpha_2, \alpha_4', \alpha_5'', \alpha_6, \beta_2, \beta_5'', \beta_6'', \beta_7\}$; $\{\alpha_3'', \beta_5'\}$, so the reduced equation system is:

$$0\varphi_1 + 0\varphi_2 + 0\varphi_3 = 0,$$
$$1\varphi_1 - 2\varphi_2 + 1\varphi_3 = 0.$$

The rank of the coefficient matrix of this system is 1, so the solutions form a 2-dimensional space, which is too large, the framework is nonrigid. But changing the place of one rod like in Fig. 4b the framework become rigid, its matrix is:

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

which has rank 2.

Now we obtained the following

Theorem 1 *A framework composed of a square grid with a hole in it and some diagonal rods of certain squares is rigid if and only if*

a) the graph of the framework is connected, or

b) the graph has two components and at least one of N_i^r's or N_j^c's is nonzero, or

c) the graph has three components and the rank of the matrix

$$\begin{pmatrix} N_1^r & N_2^r & N_3^r \\ N_1^c & N_2^c & N_3^c \end{pmatrix}$$

equals 2.

Since the sum of the three columns of the matrix above equals the zero vector it is sufficient to check whether $\det\begin{pmatrix} N_1^r & N_2^r \\ N_1^c & N_2^c \end{pmatrix} = 0$ or not.

If we have p holes in the grid then we get a larger system of equations, the number of equations is equal to twice the number of holes. We can construct the bipartite graph of the framework: vertices correspond to the row-segments and the column-segments, the edges correspond to the diagonal rods just as above. The connected components of the graph will be numbered from 1 to q and the common rotation of the segments corresponding to the i-th component will be denoted by φ_i. $N^r(s,i)$ will equal the difference of the number of bordering row-segments of the s-th hole corresponding to the i-th component of the graph with positive and with negative sign which came from an arbitrary walk around the hole, and $N^c(s,j)$'s will be the same for the column-segments.

Now reducing the equation system on the basis of the graph of the framework we get the following system:

$$\begin{pmatrix} N^r(1,1) & N^r(1,2) & \dots & N^r(1,q) \\ N^c(1,1) & N^c(1,2) & \dots & N^c(1,q) \\ N^r(2,1) & N^r(2,2) & \dots & N^r(2,q) \\ \vdots & \vdots & \ddots & \vdots \\ N^c(p,1) & N^c(p,2) & \dots & N^c(p,q) \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

All the $(c, c, \dots, c)$ constant vectors satisfy this equation system so the framework is rigid if and only if the set of solutions is exactly 1-dimensional.

Theorem 2 *A framework composed of a square grid with holes in it and some diagonal rods of certain squares is rigid if and only if the rank of the matrix*

$$\begin{pmatrix} N^r(1,1) & N^r(1,2) & \dots & N^r(1,q) \\ N^c(1,1) & N^c(1,2) & \dots & N^c(1,q) \\ N^r(2,1) & N^r(2,2) & \dots & N^r(2,q) \\ \vdots & \vdots & \ddots & \vdots \\ N^c(p,1) & N^c(p,2) & \dots & N^c(p,q) \end{pmatrix}$$

is $q-1$, where q is the number of connected components of the graph of the framework.

The engineers' usual method about this problem is checking the rank of the rigidity matrix. In our new method we have to do the same with the coefficient matrix. So, why is this model better? Recall that the size of the rigidity matrix is proportional to the number of joints in the framework while the size of the coefficient matrix is proportional to the number of the row- and column-segments in the framework, which is much smaller. For example, if $k = l$ then checking the rank of the rigidity matrix requires ck^6 operations. Our method requires $O(k^3)$ operations if $p = O(k)$. Even in the extreme case of $p = O(k^2)$ the above constant c decreases.

3. The general case

Let us consider a square grid of any shape with some holes in it and some rods and cables between any pairs of joints. We can suppose that the rods and cables are neither horizontal nor vertical because these kinds of rods and cables cannot prevent the infinitesimal motions of the framework.

We can construct a directed bipartite graph in which the nodes correspond to the row- and column-segments and the edges correspond to the "short" diagonal rods and cables.

The (really) long rods and cables determine linear equations and inequalities, respectively, where the rotations of the row- and column-segments are the unknowns. Applying the graph of the framework (its strongly connected components exactly, to be precise) we can decrease the number of unknowns. This number will be denoted by u. The remaining diagonal cables which correspond to the directed edges leading from a strongly connected component of the graph to another determine linear inequalities, too. The total number of equations and inequalities will be denoted by L. If we fix the value of the rotation of a segment as zero we only have to decide whether the polyhedron in the u-dimensional real space determined by the linear equations and inequalities of long rods and cables contains only the origin of the space. But this can be done in at most $O([\max(L,u)]^3)$ time which is much better than the time of the original method with the rigidity matrix.

For example, let us consider the framework in Fig. 5a. The sets of variables correspond to the strongly connected components are the following: $\{\alpha_1, \alpha_2, \alpha_3', \alpha_3'', \alpha_4'', \alpha_5', \alpha_5'', \beta_1', \beta_2, \beta_3, \beta_4', \beta_6, \beta_7\}$; $\{\alpha_6, \alpha_7, \beta_1'', \beta_4''\}$; $\{\alpha_4'\}$; $\{\beta_5'\}$; $\{\beta_5''\}$. Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ and φ_5 denote the common values of the rotations within the respective strongly connected components. Now we have five unknowns, two equations from the hole, one equation from the long rod, one inequality from the long cable, one inequality from the diagonal cable which is in the intersection of a row and column from different components of the graph and one more equation fixing the value of one of the variables as zero:

$$\begin{aligned}
&2\varphi_1 + \varphi_3 - 3\varphi_1 &&= 0,\\
&\varphi_1 + \varphi_4 - \varphi_2 - \varphi_5 &&= 0,\\
&4(2\varphi_1 + \varphi_4) - 3(3\varphi_1 + \varphi_3) &&= 0,\\
&3(2\varphi_2) - 2(2\varphi_1 + \varphi_4) &&\geq 0,\\
&\varphi_2 - \varphi_1 &&\geq 0,\\
&\varphi_1 &&= 0.
\end{aligned}$$

From the first, third and last equations we obtain that $\varphi_1 = \varphi_3 = \varphi_4 = 0$. The remaining equation and the inequalities now look like this:

$$\begin{aligned}
&0 - \varphi_2 - \varphi_5 = 0,\\
&6\varphi_2 \geq 0,\\
&\varphi_2 \geq 0.
\end{aligned} \tag{2}$$

This system has an infinite set of solutions, so this framework is not rigid (a deformation is shown in Fig. 6). But if we change the position of one of the diagonal cables as in Fig. 5b the system (2) will change in the following way:

$$\begin{aligned}
&0 - \varphi_2 - \varphi_5 = 0,\\
&6\varphi_2 \geq 0,\\
&\varphi_2 \leq 0.
\end{aligned}$$

This system has the only solution $(0,0,0,0,0)$, so this framework is rigid.

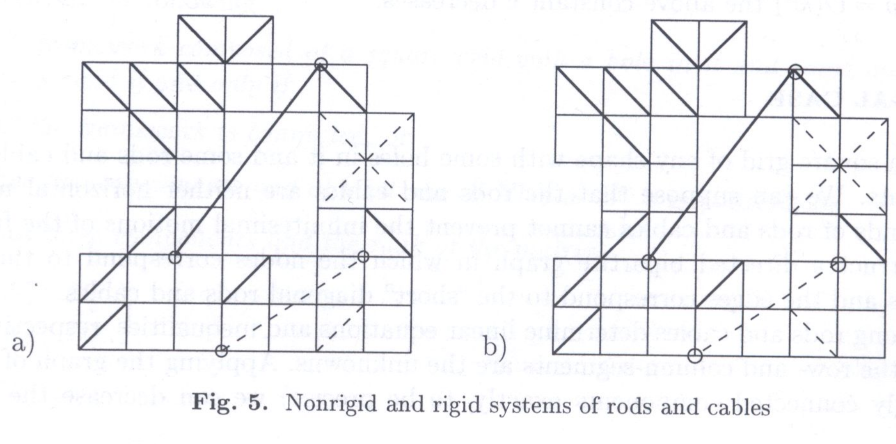

Fig. 5. Nonrigid and rigid systems of rods and cables

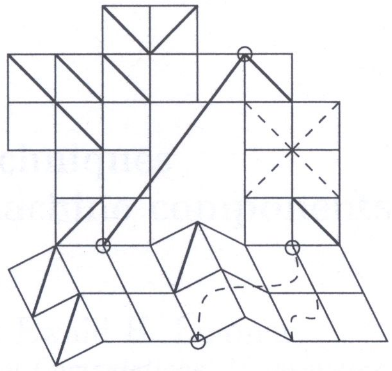

Fig. 6. The deformation of the previous nonrigid system

4. Conclusion

In this paper we could show that using the results of the graph theoretical model of square grids we have a very fast method to determine the rigidity of gridlike buildings if we can use only rigid rods, cables and struts to make them rigid.

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