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Optimisation of construction parameters of gyroscope system on elastic suspension

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In the paper are derived the equations of motion of a gyroscope on elastic suspension, mounted on a movable base. An algorithm for the selection of the optimum construction parameters and the matrices of amplifications of the gyroscope regulator is presented. The latter is aimed at the quickest transitory process damping and also at the minimisation of errors resulting from the friction in frame bearings, base angular motion and non-linearity of the impact.

1. Introduction

Improving the accuracy and reliability of gyroscope systems installed in aerial vehicles poses a major research and technical problem. Higher requirements are imposed on the accuracy of the results of theoretical studies being the basis for gyroscope design and construction [1, 2]. It is, therefore, necessary to carry out theoretical and simulation investigations into non-linear gyroscope models, which account for all possible interference and operating conditions. The results offer an explanation for non-linear phenomena and effects appearing in gyroscope dynamics. They also allow for proper selection of parameters, so that undesirable motion ranges could be avoided and the necessary gyroscope operation could be guaranteed.

The present paper deals with the dynamics, errors and control of a gyroscope applicable to aerial vehicle systems of orientation, stabilisation and navigation. A full, non-linear model of the motion of a gyroscope, mounted on a movable base (e.g. on board of an aerial vehicle), is analysed. Friction in frame bearings and the non-linearity impact at high angular frame deflections were accounted for in the model.

2. Derivation of the motion equations of a gyroscope on elastic suspension

In Fig. 1, the following co-ordinate systems are introduced:

$Ox_oy_oz_o$ – motionless absolute (inertial) system;

O_gxyz – movable system connected with a movable base;

$O_gx_1y_1z_1$ – movable system connected with an external frame;

$O_gx_2y_2z_2$ – movable system connected with the internal frame;

$O_gx_3y_3z_3$ – movable system connected with the gyroscope rotor;

$O_gx_2^oy_2^oz_2^o$ – movable system connected with the gyroscope axis.

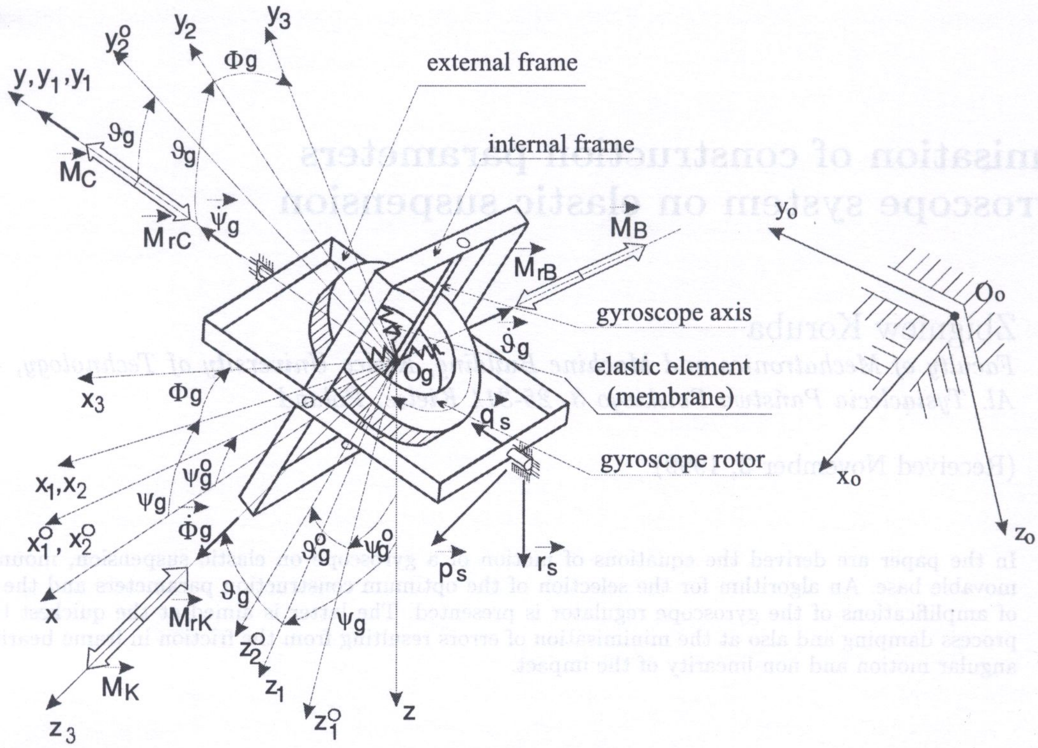

Fig. 1. General view of a gyroscope on elastic suspension together with the asumed coordinate systems, velocity vectors and moments of forces

Mutual angular position of the axes of co-ordinate systems will be specified by means of transformation matrices in the following manner.

1. The matrix of transformation from the system connected with the base to the system connected with the external frame – $O_g x_1 y_1 z_1$ system rotation with respect to $O_g xyz$ about the $O_g y_1$ axis by ψ_g angle

$$M_{pz} = \begin{bmatrix} \cos\psi_g & 0 & -\sin\psi_g \\ 0 & 1 & 0 \\ \sin\psi_g & 0 & \cos\psi_g \end{bmatrix}. \tag{1}$$

2. The matrix of transformation from the system connected with the external frame to the system connected with the internal frame – $O_g x_2 y_2 z_2$ system rotation with respect to $O_g x_1 y_1 z_1$ about the $O_g x_2$ axis by ϑ_g angle

$$M_{zw} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta_g & \sin\vartheta_g \\ 0 & -\sin\vartheta_g & \cos\vartheta_g \end{bmatrix}. \tag{2}$$

3. The matrix of transformation from the system connected with the external frame to the system connected with the internal frame – $O_g x_3 y_3 z_3$ system rotation with respect to $O_g x_2 y_2 z_2$ about the $O_g z_3$ axis by Φ_g angle

$$M_{wr} = \begin{bmatrix} \cos\Phi_g & \sin\Phi_g & 0 \\ -\sin\Phi_g & \cos\Phi_g & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

Hence the matrix of transformation from the system connected with the base to the system connected with the internal frame will be obtained as follows:

$$M_{pw} = M_{zw} \cdot M_{pz} = \begin{bmatrix} \cos\psi_g & 0 & -\sin\psi_g \\ \sin\vartheta_g \sin\psi_g & \cos\vartheta_g & \sin\vartheta_g \cos\psi_g \\ \cos\vartheta_g \sin\psi_g & -\sin\vartheta_g & \cos\vartheta_g \cos\psi_g \end{bmatrix} . \tag{4}$$

Similarly, we will obtain a matrix of transformation from the system connected with the base to the system connected with the gyroscope rotor

$$M_{pr} = M_{wr} \cdot M_{pw} = \begin{bmatrix} \begin{matrix}\cos\psi_g \cos\Phi_g \\ +\sin\vartheta_g \sin\psi_g \sin\Phi_g\end{matrix} & \cos\vartheta_g \sin\Phi_g & \begin{matrix}-\sin\psi_g \cos\Phi_g \\ +\sin\vartheta_g \cos\psi_g \sin\Phi_g\end{matrix} \\ \begin{matrix}-\cos\psi_g \sin\Phi_g \\ +\sin\vartheta_g \sin\psi_g \cos\Phi_g\end{matrix} & \cos\vartheta_g \cos\Phi_g & \begin{matrix}\sin\psi_g \cos\Phi_g \\ +\sin\vartheta_g \cos\psi_g \sin\Phi_g\end{matrix} \\ \cos\vartheta_g \sin\psi_g & -\sin\vartheta_g & \cos\vartheta_g \cos\psi_g \end{bmatrix} . \tag{5}$$

In the case, when the axis is joined to the rotor by means of an elastic element, the gyroscope gains additionally two degrees of freedom and the respective transformation matrices (by analogy with the matrices M_{pz}, M_{zw} and M_{pw}) will be as follows,

$$M^o_{pz} = \begin{bmatrix} \cos\psi^o_g & 0 & -\sin\psi^o_g \\ 0 & 1 & 0 \\ \sin\psi^o_g & 0 & \cos\psi^o_g \end{bmatrix} ; \qquad M^o_{zw} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta^o_g & \sin\vartheta^o_g \\ 0 & -\sin\vartheta^o_g & \cos\vartheta^o_g \end{bmatrix} , \tag{6}$$

$$M_{po} = M^o_{pz} \cdot M^o_{zw} = \begin{bmatrix} \cos\psi^o_g & 0 & -\sin\psi^o_g \\ \sin\vartheta^o_g \sin\psi^o_g & \cos\vartheta^o_g & \sin\vartheta^o_g \cos\psi^o_g \\ \cos\vartheta^o_g \sin\psi^o_g & -\sin\vartheta^o_g & \cos\vartheta^o_g \cos\psi^o_g \end{bmatrix} , \tag{7}$$

where ψ^o_g, ϑ^o_g – angles which specify the gyroscope axis position in relation to the base.

Assumptions made:

1. The centres of mass of the rotor, internal and external frames, and the point of intersection of the rotor and frames rotation axes, overlap.
2. The axes of frames and the rotor constitute the principal central system of inertia axes.

Quantities given:

1. m_1, m_2, m_3 – masses of the external frame, internal frame (together with the axis) and the gyroscope rotor, respectively.
2. J_{x_1}, J_{y_1}, J_{z_1} – moments of inertia of the external frame in relation to the axes $O_g x_1$, $O_g y_1$, $O_g z_1$, respectively.
3. J_{x_2}, J_{y_2}, J_{z_2} – moments of inertia of the internal frame in relation to the axes $O_g x_2$, $O_g y_2$, $O_g z_2$, respectively.
4. J_{x_3}, J_{y_3}, J_{z_3} – moments of inertia of the rotor in relation to the axes $O_g x_3$, $O_g y_3$, $O_g z_3$, respectively.
5. $J^o_{x_2}$, $J^o_{y_2}$ – moments of inertia of the rotor in relation to the axes $O_g x^o_2$, $O_g y^o_2$, respectively.
6. Components of the angular base velocity vector (the base kinematic impact) $\vec{\omega}(p_s, q_s, r_s)$.

7. Components of the force acting on the rotor mass centre $\vec{F}(F_x\,,\,F_y\,,\,F_z)$ given in the system connected with the base O_gxyz.

8. Moments of forces:

   (a) the base affecting the eternal frame $\vec{M}_C(M_{Cx}\,,\,M_{Cy}\,,M_{Cz})$,

   (b) the external frame affecting the internal one $\vec{M}_B(M_{Bx_1}\,,\,M_{By_1}\,,\,M_{Bz_1})$,

   (c) the internal frame affecting the rotor $\vec{M}_K(M_{Kx_2}\,,\,M_{Ky_2}\,,\,M_{Kz_2})$.

9. Moments of friction forces in the external and internal frame bearings:

   (a) viscous

$$M_{rC} = M^V_{rC} = \eta_c \frac{\mathrm{d}\psi_g}{\mathrm{d}t}\,, \qquad M_{rB} = M^V_{rB} = \eta_b \frac{\mathrm{d}\vartheta_g}{\mathrm{d}t}\,,$$

   (b) solid

$$M_{rC} = M^T_{rC} = 0.5 \cdot T_{rc} \cdot d_c\,, \qquad M_{rB} = M^T_{rB} = 0.5 \cdot T_{rb} \cdot d_b\,,$$

   where

$$T_{rc} = \mu_c N_c \,\mathrm{sign}\left(\frac{\mathrm{d}\psi_g}{\mathrm{d}t}\right), \qquad T_{rb} = \mu_b N_b \,\mathrm{sign}\left(\frac{\mathrm{d}\vartheta_g}{\mathrm{d}t}\right);$$

   $\eta_c\,,\,\eta_b\,,\,\mu_c\,,\,\mu_b$ – friction coefficients in frame bearings;
   $N_c\,,\,N_b$ – standard reactions in bearings;
   $d_c\,,\,d_b$ – bearing pin diameters.

10. The moment of friction forces in the rotor bearing in the internal frame and aerodynamic drag $\vec{M}_{rK}$.

11. Interference signals in the form of moments of forces directly affecting the rotor $M_{zB}\,,\,M_{zC}$.

12. Rigidity coefficient κ of the elastic element connecting the axis with the rotor.

Quantities sought for:

1. The angles $\psi_g\,,\,\vartheta_g\,,\,\Phi_g$, with the aid of which the rotor position in relation to the O_gxyz system is specified.

2. The angles $\psi^o_g\,,\,\vartheta^o_g$, with the aid of which the gyroscope axis position in relation to the O_gxyz system is specified.

3. Angular velocities

$$\dot{\psi}_g = \frac{\mathrm{d}\psi_g}{\mathrm{d}t}\,, \qquad \dot{\vartheta}_g = \frac{\mathrm{d}\vartheta_g}{\mathrm{d}t}\,, \qquad \dot{\Phi}_g = \frac{\mathrm{d}\Phi_g}{\mathrm{d}t}$$

4. Angular velocities

$$\dot{\psi}^o_g = \frac{\mathrm{d}\psi^o_g}{\mathrm{d}t}\,, \qquad \dot{\vartheta}^o_g = \frac{\mathrm{d}\vartheta^o_g}{\mathrm{d}t}\,.$$

The rotor angular velocity vector amounts to

$$\vec{\omega}_g = \frac{\mathrm{d}\vec{\psi}_g}{\mathrm{d}t} + \frac{\mathrm{d}\vec{\vartheta}_g}{\mathrm{d}t} + \frac{\mathrm{d}\vec{\Phi}_g}{\mathrm{d}t} \tag{8}$$

whereas the gyroscope axis angular velocity vector is

$$\vec{\omega}_g^o = \frac{d\vec{\psi}_g^o}{dt} + \frac{d\vec{\vartheta}_g^o}{dt}. \tag{9}$$

The projection of the rotor angular velocity vector components on individual axes of the coordinate systems can be determined as follows,

$$\begin{bmatrix} \dot{\psi}_{gx_1} \\ \dot{\psi}_{gy_1} \\ \dot{\psi}_{gz_1} \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\psi}_g \\ 0 \end{bmatrix}; \quad \begin{bmatrix} \dot{\psi}_{gx_2} \\ \dot{\psi}_{gy_2} \\ \dot{\psi}_{gz_2} \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\psi}_g \cos\vartheta_g \\ -\dot{\psi}_g \sin\vartheta_g \end{bmatrix}; \quad \begin{bmatrix} \dot{\psi}_{gx_3} \\ \dot{\psi}_{gy_3} \\ \dot{\psi}_{gz_3} \end{bmatrix} = \begin{bmatrix} \dot{\psi}_g \cos\vartheta_g \sin\Phi_g \\ \dot{\psi}_g \cos\vartheta_g \cos\Phi_g \\ -\dot{\psi}_g \sin\vartheta_g \end{bmatrix};$$

$$\begin{bmatrix} \dot{\vartheta}_{gx_1} \\ \dot{\vartheta}_{gy_1} \\ \dot{\vartheta}_{gz_1} \end{bmatrix} = \begin{bmatrix} \dot{\vartheta}_g \cos\psi_g \\ 0 \\ -\dot{\vartheta}_g \sin\psi_g \end{bmatrix}; \quad \begin{bmatrix} \dot{\vartheta}_{gx_2} \\ \dot{\vartheta}_{gy_2} \\ \dot{\vartheta}_{gz_2} \end{bmatrix} = \begin{bmatrix} \dot{\vartheta}_g \\ 0 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} \dot{\vartheta}_{gx_3} \\ \dot{\vartheta}_{gy_3} \\ \dot{\vartheta}_{gz_3} \end{bmatrix} = \begin{bmatrix} \dot{\vartheta}_g \cos\Phi_g \\ -\dot{\vartheta}_g \sin\Phi_g \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} \dot{\Phi}_{gx_3} \\ \dot{\Phi}_{gy_3} \\ \dot{\Phi}_{gz_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\Phi}_g \end{bmatrix}.$$

In a similar manner, the projection of the vector components of the gyroscope axis angular velocity will have the following form,

$$\begin{bmatrix} \dot{\psi}^o_{gx_1} \\ \dot{\psi}^o_{gy_1} \\ \dot{\psi}^o_{gz_1} \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\psi}^o_g \\ 0 \end{bmatrix}; \quad \begin{bmatrix} \dot{\psi}^o_{gx_2} \\ \dot{\psi}^o_{gy_2} \\ \dot{\psi}^o_{gz_2} \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\psi}^o_g \cos\vartheta^o_g \\ -\dot{\psi}^o_g \sin\vartheta^o_g \end{bmatrix};$$

$$\begin{bmatrix} \dot{\vartheta}^o_{gx_1} \\ \dot{\vartheta}^o_{gy_1} \\ \dot{\vartheta}^o_{gz_1} \end{bmatrix} = \begin{bmatrix} \dot{\vartheta}_g \cos\psi^o_g \\ 0 \\ -\dot{\vartheta}_g \sin\psi^o_g \end{bmatrix}; \quad \begin{bmatrix} \dot{\vartheta}^o_{gx_2} \\ \dot{\vartheta}^o_{gy_2} \\ \dot{\vartheta}^o_{gz_2} \end{bmatrix} = \begin{bmatrix} \dot{\vartheta}^o_g \\ 0 \\ 0 \end{bmatrix}.$$

Hence:

1. Vector components of the external frame angular velocity in the $O_g x_1 y_1 z_1$ system:

$$\begin{bmatrix} \omega_{gx_1} \\ \omega_{gy_1} \\ \omega_{gz_1} \end{bmatrix} = M_{pz} \cdot \begin{bmatrix} p_s \\ q_s \\ r_s \end{bmatrix} + \begin{bmatrix} \dot{\psi}_{gx_1} \\ \dot{\psi}_{gy_1} \\ \dot{\psi}_{gz_1} \end{bmatrix} = \begin{bmatrix} p_s \cos\psi_g - r_s \sin\psi_g \\ \dot{\psi}_g + q_s \\ p_s \sin\psi_g + r_s \cos\psi_g \end{bmatrix}.$$

2. Vector components of the internal frame angular velocity in the $O_g x_2 y_2 z_2$ system:

$$\begin{bmatrix} \omega_{gx_2} \\ \omega_{gy_2} \\ \omega_{gz_2} \end{bmatrix} = M_{pw} \cdot \begin{bmatrix} p_s \\ q_s \\ r_s \end{bmatrix} + \begin{bmatrix} \dot{\vartheta}_{gx_2} \\ \dot{\vartheta}_{gy_2} \\ \dot{\vartheta}_{gz_2} \end{bmatrix} + \begin{bmatrix} \dot{\psi}_{gx_2} \\ \dot{\psi}_{gy_2} \\ \dot{\psi}_{gz_2} \end{bmatrix}$$

$$= \begin{bmatrix} p_s \cos\psi_g - r_s \sin\psi_g + \dot{\vartheta}_g \\ (p_s \sin\psi_g + r_s \cos\psi_g)\sin\vartheta_g + (q_s + \dot{\psi}_g)\cos\vartheta_g \\ (p \sin\psi_g + r_s \cos\psi_g)\cos\vartheta_g - (q_s + \dot{\psi}_g)\sin\vartheta_g \end{bmatrix}.$$

3. Vector components of the rotor angular velocity in the $O_g x_3 y_3 z_3$ system:

$$\begin{bmatrix} \omega_{gx_3} \\ \omega_{gy_3} \\ \omega_{gz_3} \end{bmatrix} = M_{pr} \cdot \begin{bmatrix} p_s \\ q_s \\ r_s \end{bmatrix} + \begin{bmatrix} \dot{\psi}_{gx_3} \\ \dot{\psi}_{gy_3} \\ \dot{\psi}_{gz_3} \end{bmatrix} + \begin{bmatrix} \dot{\vartheta}_{gx_3} \\ \dot{\vartheta}_{gy_3} \\ \dot{\vartheta}_{gz_3} \end{bmatrix} + \begin{bmatrix} \dot{\Phi}_{gx_3} \\ \dot{\Phi}_{gy_3} \\ \dot{\Phi}_{gz_3} \end{bmatrix} \Rightarrow$$

$$\omega_{gx_3} = p_s(\cos\psi_g\cos\Phi_g + \sin\vartheta_g\sin\psi_g\sin\Phi_g) + (q_s+\dot\psi_g)\cos\vartheta_g\sin\Phi_g + \dot\vartheta_g\cos\Phi_g$$
$$+ r_s(\cos\psi_g\sin\vartheta_g\sin\Phi_g - \sin\psi_g\cos\Phi_g),$$
$$\omega_{gy_3} = -p_s(\cos\psi_g\sin\Phi_g - \sin\vartheta_g\sin\psi_g\cos\Phi_g) + (q_s+\dot\psi_g)\cos\vartheta_g\cos\Phi_g - \dot\vartheta_g\sin\Phi_g$$
$$+ r_s(\cos\psi_g\sin\vartheta_g\cos\Phi_g + \sin\psi_g\sin\Phi_g),$$
$$\omega_{gz_3} = p_s\sin\psi_g\cos\vartheta_g - (q_s+\dot\psi_g)\sin\vartheta_g + r_s\cos\psi_g\cos\vartheta_g + \dot\Phi_g\,.$$

4. Vector components of the gyroscope axis velocity in the $O_gx_1y_1z_1$ system

$$\begin{bmatrix}\omega^o_{gx_1}\\ \omega^o_{gy_1}\\ \omega^o_{gz_1}\end{bmatrix} = M^o_{pz}\cdot\begin{bmatrix}p_s\\ q_s\\ r_s\end{bmatrix} + \begin{bmatrix}\dot\psi^o_{gx_1}\\ \dot\psi^o_{gy_1}\\ \dot\psi^o_{gz_1}\end{bmatrix} = \begin{bmatrix}p_s\cos\psi^o_g - r_s\sin\psi^o_g\\ \dot\psi^o_g + q_s\\ p_s\sin\psi^o_g + r_s\cos\psi^o_g\end{bmatrix}.$$

5. Vector components of the gyroscope axis velocity in the $O_gx^o_2y^o_2z^o_2$ system

$$\begin{bmatrix}\omega^o_{gx_2}\\ \omega^o_{gy_2}\\ \omega^o_{gz_2}\end{bmatrix} = M_{po}\cdot\begin{bmatrix}p_s\\ q_s\\ r_s\end{bmatrix} + \begin{bmatrix}\dot\vartheta^o_{gx_2}\\ \dot\vartheta^o_{gy_2}\\ \dot\vartheta^o_{gz_2}\end{bmatrix} + \begin{bmatrix}\dot\psi^o_{gx_2}\\ \dot\psi^o_{gy_2}\\ \dot\psi^o_{gz_2}\end{bmatrix}$$
$$= \begin{bmatrix}p_s\cos\psi^o_g - r_s\sin\psi^o_g + \dot\vartheta^o_g\\ (p_s\sin\psi^o_g + r_s\cos\psi^o_g)\sin\vartheta^o_g + (q_s+\dot\psi^o_g)\cos\vartheta^o_g\\ (p_s\sin\psi^o_g + r_s\cos\psi^o_g)\cos\vartheta^o_g - (q_s+\dot\psi^o_g)\sin\vartheta^o_g\end{bmatrix}.$$

The gyroscope motion equations will be derived with the use of 2^{nd} kind Lagrangian equations. For that purpose, we will determine the system kinetic energy E_k (it is equal to the sum of the kinetic energy of the external and internal frames, the rotor and the axis), as well as the system of the potential energy E_p:

$$E_k = \frac{1}{2}\left[J_{x_1}\omega^2_{gx_1} + J_{y_1}\omega^2_{gy_1} + J_{z_1}\omega^2_{gz_1}\right] + \frac{1}{2}\left[J_{x_2}\omega^2_{gx_2} + J_{y_2}\omega^2_{gy_2} + J_{z_2}\omega^2_{gz_2}\right]$$
$$+ \frac{1}{2}\left[J_{x_3}\omega^2_{gx_3} + J_{y_3}\omega^2_{gy_3} + J_{z_3}\omega^2_{gz_3}\right] + \frac{1}{2}\left[J^o_{x_2}(\omega^o_{gx_2})^2 + J^o_{y_2}(\omega^o_{gy_2})^2\right], \tag{10}$$

$$E_p = \frac{1}{2}\kappa(\psi_g - \psi^o_g)^2 + \frac{1}{2}\kappa(\vartheta_g - \vartheta^o_g)^2. \tag{11}$$

Lagrangian function will thus be equal to

$$L = E_k - E_p\,. \tag{12}$$

If we take into account the fact that the generalised co-ordinate Φ_g is cyclic, the gyroscope motion equation will assume the following form:

Equations of the gyroscope axis motion

$$J^o_{y_2}\frac{\mathrm{d}}{\mathrm{d}t}(\omega^o_{gy_2}\cos\vartheta^o_g) + J^o_{x_2}\omega^o_{gx_2}\omega^o_{gz_1} - J^o_{y_2}\omega^o_{gy_2}\omega^o_{gx_1}\sin\vartheta^o_g - \kappa\left(\psi_g - \psi^o_g\right) = M_C - M_{rC}\,, \tag{13}$$

$$J^o_{x_2}\dot\omega^o_{gx_2} - J^o_{y_2}\omega^o_{gy_2}\omega^o_{gz_2} - \kappa(\vartheta_g - \vartheta^o_g) = M_B - M_{rB}\,. \tag{14}$$

Equations of the gyroscope rotor motion

$$J_{y_1}\dot\omega_{gy_1} - (J_{x_1} - J_{z_1})\omega_{gx_1}\omega_{gz_1} + (J_{x_2} + J_{x_3})\omega_{gx_2}\omega_{gz_1} + \kappa(\psi_g - \psi^o_g)$$
$$- \left[J_{z_2}\dot\omega_{gz_2} + (J_{y_2} + J_{y_3})\omega_{gx_2}\omega_{gy_2} + J_{z_3}\frac{\mathrm{d}}{\mathrm{d}t}(\omega_{gz_2} + \dot\Phi_g)\right]\sin\vartheta_g$$
$$+ \left[(J_{y_2} + J_{y_3})\dot\omega_{gy_2} - J_{z_2}\omega_{gx_2}\omega_{gz_2} - J_{z_3}(\omega_{gz_2} + \dot\Phi_g)\omega_{gx_2}\right]\cos\vartheta_g = M_{zC}\,, \tag{15}$$

$$(J_{x2}+J_{x3})\dot{\omega}_{gx2}+(J_{z2}-J_{y2}-J_{y3})\omega_{gy2}\omega_{gz2}+J_{z3}(\omega_{gz2}+\dot{\Phi}_g)\omega_{gy2}+\kappa(\vartheta_g-\vartheta_g^o)=M_{zB}\,, \tag{16}$$

$$J_{z3}\frac{\mathrm{d}}{\mathrm{d}t}(\omega_{gz2}+\dot{\Phi}_g)=M_K-M_{rK}\,. \tag{17}$$

In case of a motionless base, the equations will take the form:

$$J^o_{y2}\frac{\mathrm{d}^2\psi^o_g}{\mathrm{d}t^2}\cos^2\vartheta_g-J^o_{y2}\dot{\psi}^o_g\dot{\vartheta}^o_g\sin 2\vartheta^o_g-\kappa(\psi_g-\psi^o_g)=M_C-M_{rC}\,, \tag{18}$$

$$J^o_{x2}\frac{\mathrm{d}^2\vartheta^o_g}{\mathrm{d}t^2}+\frac{1}{2}J^o_{y2}\dot{\psi}^o_g\dot{\vartheta}^o_g\sin 2\vartheta^o_g-\kappa(\vartheta_g-\vartheta^o_g)=M_B-M_{rB}\,, \tag{19}$$

$$\begin{aligned}&\left[J_{y1}+J_{z2}+(J_{y2}+J_{y3}-J_{z2})\cos^2\vartheta_g\right]\frac{\mathrm{d}^2\psi_g}{\mathrm{d}t^2}-(J_{y2}+J_{y3}-J_{z2})\dot{\psi}_g\dot{\vartheta}_g\sin 2\vartheta_g\\&\quad+J_{z3}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\Phi}_g-\dot{\psi}_g\sin\vartheta_g)\sin\vartheta_g-J_{z3}(\dot{\Phi}_g-\dot{\psi}_g\sin\vartheta_g)\dot{\vartheta}_g\cos\vartheta_g+\kappa(\psi_g-\psi^o_g)=M_{zC}\,,\end{aligned} \tag{20}$$

$$\begin{aligned}&(J_{x2}+J_{x3})\frac{\mathrm{d}^2\vartheta_g}{\mathrm{d}t^2}+\frac{1}{2}(J_{y2}+J_{y3}-J_{z2})\dot{\psi}^2_g\sin 2\vartheta_g+\kappa(\vartheta_g-\vartheta^o_g)\\&\quad-J_{z3}(\dot{\Phi}_g-\dot{\psi}_g\sin\vartheta_g)\dot{\vartheta}_g\cos\vartheta_g=M_{zB}\,,\end{aligned} \tag{21}$$

$$J_{z3}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\Phi}_g-\dot{\psi}_g\sin\vartheta_g)=M_K-M_{rK}\,. \tag{22}$$

If, on the other hand, we disregard the inertia of frames and introduce the notations $J_{x3}=J_{y3}=J_{gk}$, $J^o_{x2}=J^o_{y2}=J^o_{gk}$ (the rotor and the axis are axially symmetric), $J_{z3}=J_{go}$, the equations will be as follows,

$$J^o_{gk}\frac{\mathrm{d}\omega^o_{gy3}}{\mathrm{d}t}\cos\vartheta^o_g+J_{gk}(\omega^o_{gz1}-\omega^o_{gy3}\sin\vartheta^o_g)\omega^o_{gx3}-\kappa(\psi_g-\psi^o_g)=M_C-M_{rC}\,, \tag{23}$$

$$J^o_{gk}\omega^o_{gx3}-J^o_{gk}\omega^o_{gy3}\omega^o_{gz3}-\kappa(\vartheta_g-\vartheta^o_g)=M_B-M_{rB}\,, \tag{24}$$

$$\begin{aligned}&J_{gk}\frac{\mathrm{d}\omega_{gy2}}{\mathrm{d}t}\cos\vartheta_g+J_{gk}\omega_{gx2}(\omega_{gz1}+\omega_{gy2}\sin\vartheta_g)+J_{go}(\omega_{gz2}+\dot{\Phi}_g)\sin\vartheta_g\\&\quad-J_{go}(\omega_{gz2}+\dot{\Phi}_g)\omega_{gx2}\cos\vartheta_g+\kappa(\psi_g-\psi^o_g)=M_{zC}\,,\end{aligned} \tag{25}$$

$$J_{gk}\frac{\mathrm{d}\omega_{gx2}}{\mathrm{d}t}-J_{gk}\omega_{gy2}\omega_{gz2}+J_{go}(\omega_{gz2}+\dot{\Phi}_g)\omega_{gy2}+\kappa(\vartheta_g-\vartheta^o_g)=M_{zB}\,, \tag{26}$$

$$J_{go}\frac{\mathrm{d}}{\mathrm{d}t}(\omega_{gz2}+\dot{\Phi}_g)=M_K-M_{rK}\,. \tag{27}$$

3. Linearization of the motion equations of a gyroscope on elastic suspension

The gyroscope axis and rotor motion can be affected by the moments M_B and M_C being the gyroscope motion input due to external forces. Another kind of impact is provided by the base angular motion, specified by angular velocities $p_s(t)$, $q_s(t)$ and $r_s(t)$ and representing the motion parametric input. The equations are strongly non-linear and the methods of their analytic solution are not known. We will deal with the impact of the gyroscope parameter selection, non-linearity and the base angular velocity on the gyroscope axis and rotor motion. All the analyses will be carried out by means of numerical methods. In order to select the optimum gyroscope parameters, Eqs. (23)–(27) will be linearized.

Let us take the following simplifying assumption,

1. The moments of inertia of the external and internal frames are disregarded, i.e.

$$J_{x1}=0;\qquad J_{y1}=0;\qquad J_{z1}=0;\qquad J_{x2}=0;\qquad J_{y2}=0;\qquad J_{z2}=0.$$

2. The gyroscope rotor and axis are axially symmetric, thus $J_{x3} = J_{y3} = J_{gk}$, $J^o_{x2} = J^o_{y2} = J^o_{gk}$. Additionally we will notify $J_{z3} = J_{go}$.

3. The base angular motion velocities p_s, q_s, r_s are very low and the gyroscope axis performs small deflections, i.e.

$$p_s \ll 1, \qquad q_s \ll 1, \qquad r_s \ll 1, \qquad \psi^o_g \ll 1, \qquad \vartheta^o_g \ll 1, \qquad \psi_g \ll 1, \qquad \vartheta_g \ll 1.$$

Then

$$\begin{aligned} &\omega_{gx_1} = p_s\,, \qquad \omega_{gy_1} = \dot{\psi}_g + q_s\,, \qquad \omega_{gz_1} = r_s\,, \qquad \omega_{gx_2} = \omega_{gx_3} = p_s + \dot{\vartheta}_g\,, \\ &\omega_{gy_2} = \omega_{gy_3} = \omega_{gy_1} = \dot{\psi}_g + q_s\,, \qquad \omega_{gz_2} = r_s\,, \qquad \omega_{gz_3} = r_s + \dot{\Phi}_g\,. \end{aligned}$$

4. Friction in the bearings is of the viscous type and the moments of friction forces in the external and internal frame are assumed to have the form

$$M_{rC} = \eta_c \frac{\mathrm{d}\psi_g}{\mathrm{d}t}\,, \qquad M_{rB} = \eta_b \frac{\mathrm{d}\vartheta_g}{\mathrm{d}t}\,,$$

5. The moment of forces driving the rotor is equal to the moment of friction forces in the rotor bearings and the aerodynamic drag

$$M_K = M_{rK}\,.$$

Then Eq. (27) will take the form

$$\omega_{gz_2} + \frac{\mathrm{d}\Phi_g}{\mathrm{d}t} = \mathrm{const} = n,$$

where n [rad/s] is assumed to be a known quantity.

Taking into account the above-mentioned assumptions and the notations introduced, the linearized equations of the gyroscope motion will take the form

$$J^o_{gk}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\psi}^o_g + q) + \eta_c\dot{\psi}^o_g - \kappa(\psi_g - \psi^o_g) = M_C\,, \tag{28}$$

$$J^o_{gk}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\vartheta}^o_g + p) + \eta_b\dot{\vartheta}^o_g - \kappa(\vartheta_g - \vartheta^o_g) = M_B\,, \tag{29}$$

$$J_{gk}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\psi}_g + q) - J_{go}n(\dot{\vartheta}_g + p) + \kappa(\psi_g - \psi^o_g) = M_{zC}\,, \tag{30}$$

$$J_{gk}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\vartheta}_g + p) + J_{go}n(\dot{\psi}_g + q) + \kappa(\vartheta_g - \vartheta^o_g) = M_{zB}\,. \tag{31}$$

Let us introduce dimensionless time

$$\tau = \Omega \cdot t \tag{32}$$

where

$$\Omega = \frac{J_{go} \cdot n}{J_{gk} + J^o_{gk}}\,.$$

The independent variable τ, changing the time scale, makes the numerical analysis easier as it equalises the value of the components of equations and enables the introduction of numerically greater integration scale. As a result, numerical errors will decrease.

After substituting (28)–(31) with (32) and introducing appropriate re-arrangements, the linearized Eqs. (28)–(31) will take the form

$$\frac{d^2\nu}{d\tau^2} = \upsilon\frac{d\sigma}{d\tau} + \upsilon\frac{d\psi_g^o}{d\tau} + b_b\frac{d\vartheta_g^o}{d\tau} - (\kappa_p+\kappa_o)\nu + \bar{M}_{zB} - \bar{M}_B + p, \tag{33}$$

$$\frac{d^2\sigma}{d\tau^2} = -\upsilon\frac{d\nu}{d\tau} - \upsilon\frac{d\vartheta_g^o}{d\tau} + b_c\frac{d\psi_g^o}{d\tau} - (\kappa_p+\kappa_o)\sigma + \bar{M}_{zC} - \bar{M}_C - q, \tag{34}$$

$$\frac{d^2\vartheta_g^o}{d\tau^2} = -b_b\frac{d\vartheta_g^o}{d\tau} + \kappa_o\nu + \bar{M}_B - \frac{dp}{d\tau}, \tag{35}$$

$$\frac{d^2\psi_g^o}{d\tau^2} = -b_c\frac{d\psi_g^o}{d\tau} + \kappa_o\sigma + \bar{M}_C - \frac{dq}{d\tau}, \tag{36}$$

where

$$\nu = \vartheta - \vartheta_g^o, \qquad \sigma = \psi - \psi_g^o, \qquad \upsilon = \frac{J_{gk}^o + J_{gk}}{J_{gk}},$$

$$b_B = \frac{\eta_b}{J_{gk}^o\Omega}, \qquad b_C = \frac{\eta_C}{J_{gk}^o\Omega}, \qquad \kappa_o = \frac{\kappa}{J_{gk}^o\Omega^2}, \qquad \kappa_p = \frac{\kappa}{J_{gk}\Omega^2},$$

$$M_B = \frac{M_B}{J_{gk}^o\Omega^2}, \qquad M_C = \frac{M_C}{J_{gk}^o\Omega^2}, \qquad M_{zB} = \frac{M_{zB}}{J_{gk}\Omega^2}, \qquad M_{zC} = \frac{M_{zC}}{J_{gk}\Omega^2}.$$

4. Selection of the optimum gyroscope parameters

The gyroscope parameters should be selected in such a way as to ensure its stability and the damping of the transient process in the shortest time. Let us first introduce the following notations,

$$x_1 = \nu, \qquad x_2 = \frac{d\nu}{d\tau}, \qquad x_3 = \sigma, \qquad x_4 = \frac{d\sigma}{d\tau}, \qquad x_5 = \frac{d\vartheta_{gk}^o}{d\tau}, \qquad x_6 = \frac{d\psi_{gk}^o}{d\tau}. \tag{37}$$

After taking into account (37), the system (33)–(36) will be as follows,

$$\mathbf{x}' = \mathbf{A}\cdot\mathbf{x} \tag{38}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T;$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -(\kappa_p+\kappa_o) & 0 & 0 & \upsilon & b_b & \upsilon \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\upsilon & -(\kappa_p+\kappa_o) & 0 & -\upsilon & b_c \\ \kappa_o & 0 & 0 & 0 & -b_b & 0 \\ 0 & 0 & \kappa_o & 0 & 0 & -b_c \end{bmatrix}. \tag{39}$$

In order to determine stable and optimum parameters, a modified Golubiencev method [5, 6] will be applied. The algorithm based on this method has the following form. Let us introduce a new variable

$$\mathbf{x}(\tau) = \mathbf{y}(\tau)\cdot e^{\delta(\tau)}, \tag{40}$$

at the same time

$$\delta = \operatorname{Tr}\mathbf{A} = -\frac{1}{6}(b_b + b_c). \tag{41}$$

After the transformation we will obtain

$$\mathbf{y}' = \mathbf{G} \cdot \mathbf{y} \tag{42}$$

where

$$\mathbf{G} = \begin{bmatrix} -\delta & 1 & 0 & 0 & 0 & 0 \\ -(\kappa_p + \kappa_o) & -\delta & 0 & \upsilon & b_b & \upsilon \\ 0 & 0 & -\delta & 1 & 0 & 0 \\ 0 & -\upsilon & -(\kappa_p + \kappa_o) & -\delta & -\upsilon & b_c \\ \kappa_o & 0 & 0 & 0 & -b_b - \delta & 0 \\ 0 & 0 & \kappa_o & 0 & 0 & -b_c - \delta \end{bmatrix}. \tag{43}$$

The characteristic equation of matrix $\mathbf{G}$ whose $\operatorname{Tr}\mathbf{G} = 0$ is transformed to the characteristic polynominal of the form

$$\omega^6 + g_2\omega^4 - g_3\omega^3 + g_4\omega^2 - g_5\omega + g_6 = 0. \tag{44}$$

We seek such values κ_o, κ_p, υ, b_b, b_c for the matrix $\mathbf{G}$ so that the characteristic equation (44) would have roots, which are only imaginary or equal zero [5]. For that purpose, the characteristic equation (44) coefficients g_2, g_3, g_4, g_5, g_6 (the coefficient $g_1 = \operatorname{Tr}\mathbf{G} = 0$) should be determined as the sums of all possible determinant combinations of principal minors of the degrees 2, 3, 4, 5 and 6, subsequently, of the matrix $\mathbf{G}$ described by (43). At the same time, it is necessary to carry out the maximisation of the absolute value of the matrix $\mathbf{A}$ trace described by the expression (41).

In order to reduce the errors in the gyroscope axis motion, which result primarily from non-linearity impact and friction forces in frame bearings, a linear-square regulator will be applied [9]. The law on the regulator control has the form

$$\mathbf{u} = -\mathbf{K} \cdot \mathbf{x}. \tag{45}$$

The conjugation matrix $\mathbf{K}$ occurring in Eq. (45) is determined from the following relation,

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \tag{46}$$

where $\mathbf{B}^T$ is the control transposed matrix

$$B^T = \begin{bmatrix} 0 & \frac{1}{J_{gk}\Omega^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{J_{gk}\Omega^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{J_{gk}^o\Omega^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{J_{gk}^o\Omega^2} \end{bmatrix},$$

and $\mathbf{P}$ is the solution to the Riccati algebraic equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - 2\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0}. \tag{47}$$

Weight matrixes $\mathbf{R}$ and $\mathbf{Q}$ occurring in Eqs. (46) and (47) are selected in such a way as to get the quickest damping of transient processes and interference connected with the angular motion of the base. It is obvious that the technical limitations of the control execution device must be taken into account.

The software covering the algorithm responsible for the gyroscope parameter optimisation based on the modified Golubiencev method and conjugation $\mathbf{K}$ matrix determination was written in Matlab–Simulink [8].

5. Results of investigations

Numerical investigations into a gyroscope on elastic suspension were carried out for the following initial data,

$$J_{go} = 5 \cdot 10^{-4}\,\mathrm{kgm}^2; \qquad J_{gk} = 2.5 \cdot 10^{-4}\,\mathrm{kgm}^2; \qquad n = 300\,\mathrm{rad/s}; \qquad \upsilon = 1.5.$$

In all cases the gyroscope was affected by the input in the form of the following initial conditions,

$$\vartheta(0) = 0; \qquad \frac{\mathrm{d}\vartheta(0)}{\mathrm{d}\tau} = \frac{50}{\Omega}\left[\frac{\mathrm{rad}}{\mathrm{s}}\right]; \qquad \psi(0) = 0; \qquad \frac{\mathrm{d}\psi(0)}{\mathrm{d}\tau} = -\frac{50}{\Omega}\left[\frac{\mathrm{rad}}{\mathrm{s}}\right]; \tag{48}$$

$$\vartheta_o(0) = 0; \qquad \frac{\mathrm{d}\vartheta_o(0)}{\mathrm{d}\tau} = 0; \qquad \psi_o(0) = 0; \qquad \frac{\mathrm{d}\psi_o(0)}{\mathrm{d}\tau} = 0. \tag{49}$$

We assume that the base vibrations are of harmonic character:

$$p_s = \frac{p_{so}}{\Omega}\sin\nu\tau; \qquad q_s = \frac{q_{so}}{\Omega}\cos\nu\tau; \qquad r_s = \frac{r_{so}}{\Omega}\sin\nu\tau; \tag{50}$$

and that $p_{so} = q_{so} = r_{so} = 2\,\mathrm{rad/s}$; $\nu = 10\,\mathrm{rad/s}$ as well as they appear at the time instant $t = 0.2$ s and operate for the time $t = 0.1$ s.

Figure 2 presents angular vibrations of the gyroscope rotor with non-optimum parameters. Figure 3 shows the impact of the optimum parameters on the transitory damping process both for the rotor and the gyroscope axis. It is clearly seen that the gyroscope does not demonstrate asymptotic stability – it does not keep the axis angular position in space as pre-set by initial conditions (48) and (49). What is more, the impact of the base vibrations, described by the expressions (50), on the axis additional getting off is clearly noticeable. In order to obtain asymptotic stability of the gyro-

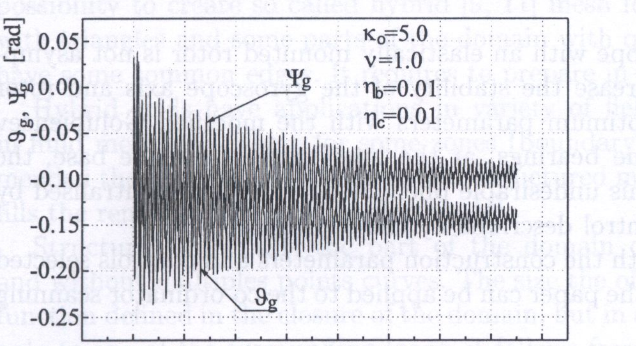

Fig. 2.

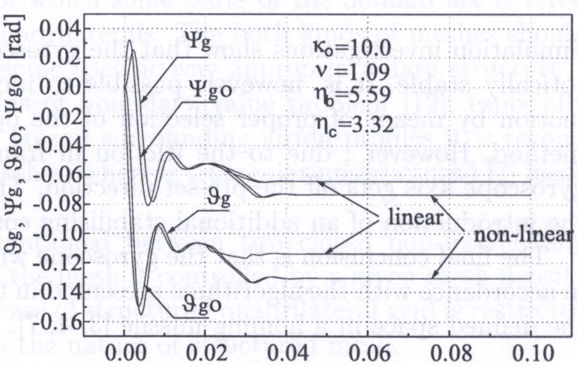

Fig. 3.

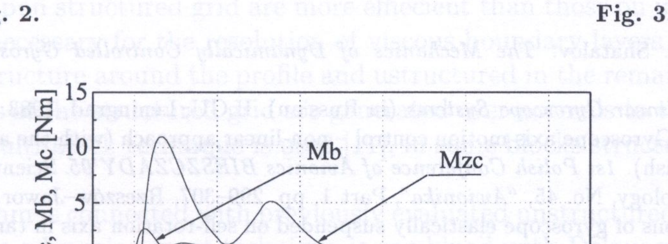

Fig. 4

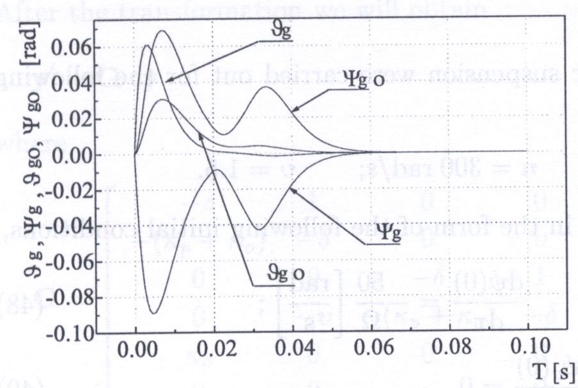

Fig. 5.

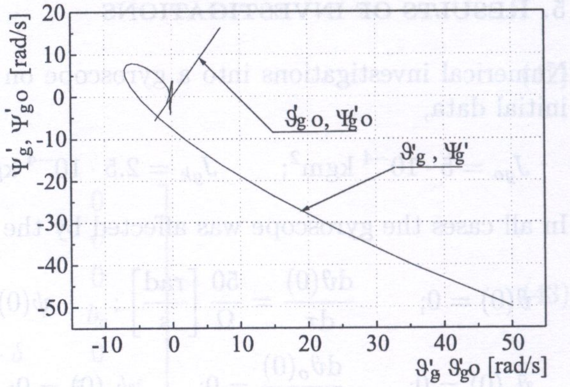

Fig. 6.

scope under consideration (thus keeping by its axis the pre-set position in space), the conjugation matrix **K** described by the expression (46) was applied. Its elements possess the following value,

$$K = \begin{bmatrix} -48.66 & 78.35 & -1.72 & 79.03 & 70.48 & 3.09 & 0.74 & -1.48 \\ -1.72 & -85.93 & -63.34 & 48.98 & -0.18 & -0.36 & 62.55 & -12.76 \\ 67.96 & -1.95 & 0.11 & 0.23 & -67.35 & -31.91 & 0.00 & 0.00 \\ 0.32 & 0.64 & 71.70 & 5.55 & 0.00 & 0.00 & -65.98 & -13.50 \end{bmatrix}$$

The course of changes of the control values in time is shown in Fig. 4. Additional effects of the damping of transitory processes after the application of linear-square regulator are presented in Figs. 5 and 6.

6. Conclusions

Simulation investigations show that the gyroscope with an elastically mounted rotor is not asymptotically stable. It is, however, possible to increase the stability of the gyroscope axis and rotor motion by means of proper selection of the optimum parameters with the modified Golubiencev method. However , due to the friction in frame bearings, at the interference from the base, the gyroscope axis gets off the pre-set direction. This undesirable phenomenon could be neutralised by the introduction of an additional stabilising control described by the expression (45).

The final conclusion is that the gyroscope with the construction parameters and controls selected in accordance with the algorithms presented in the paper can be applied to the co-ordinator scanning the defined space in a homing missile [3, 4, 7].

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