Computer Assisted Mechanics and Engineering Sciences, 8: 45–78, 2001.

Shape optimization of mechanical structure by an adjoint variables method and genetic algorithm

Mohamad Younes and Alain Potiron
Laboratoire de Génie Mécanique et C.A.O.
ENSAM, 2, Boulevard du Ronceray, B.P. 3525, 49035 Angers Cedex, France

(Received July 15, 1999)

The shape optimization of machine elements or structures consists in searching the optimal form satisfying the imposed mechanical, technological and geometrical criteria. In this paper two methods, developed for shape optimization of uni and bidimensional mechanical structures are offered. The first one, known as the adjoint variables method, is based upon the evaluation of the sensitivity or the derivatives of the functional with respect to the evolution of the structure shape. It requires the use of a mathematical optimization code in order to converge towards the solution. The second method deals with Genetic Algorithms whose principle rises from the evolution of individuals living in nature. Within the framework of structures optimization, a new Genetic Algorithm has been developed. The analysis is carried out by the finite element method. The first part of this article is devoted to optimal shape research of unidimensional structures such as beams while the second treats the shape optimization of bidimensional parts. To show the effectiveness of each of the two methods, examples are presented, and the numerical results obtained show that a good convergence was obtained in each case.

1. INTRODUCTION

The effective and increasing improvement of computers and data-processing products, has greatly contributed to the fast evolution of the mechanical computer aided design (C.A.D.). Then, for designers seeking product optimization, new techniques can be implemented in order to improve the parts' performances and effectiveness and satisfy their technical qualities.

For this purpose, various methods and various numerical algorithms were proposed by the researchers in the field of machine elements design. The use of computer as a computational tool allowed for the development of powerful structure-analysis methods like the finite element method (F.E.M.). In the field of structure optimization other methods were applied and several teams of workers handled the subject, as mentioned below.

Haug et al. [12] have developed methods for sensitivity analysis, allowing for the derivative calculations of some mechanical functionals (like stresses, field displacement, etc.) induced by the variation of the design parameters. It is thus possible, by employing methods of gradients, to converge towards an optimal solution. As it will be seen in paragraph 3.1 below, the implementation of the method is greatly simplified when adjoint variables are used. In such related works [9, 10, 19], the shape variation of the structure is obtained by an imposed displacement of the boundary, and the calculations are developed by means of the material derivative concept as it is employed in fluid mechanics. Trompette et al. [20] continuing and developing these techniques have obtained excellent results in this field. Other work has been completed by Braibant [4] and Braibant and Fleury [5] in which an analytical method of derivative calculation is introduced in order to optimize bidimensional structures.

In addition to these nowadays traditional methods about this subject, other optimization approaches were proposed among which some are known under the name of methods of growth. These methods simulate the adaptation of the forms of biological structures, such as trees for instance, subjected to external loadings and constraints during their growth. The main idea consists in modifying the structure's shape by making it growing bigger or smaller. For this purpose, a deformation field can be defined in the volume and applied to the structure [2]. This fictitious loading can be generated by a field of temperature [8] or an hydrostatic-pressure field chosen arbitrarily.

It is in the work of Michalewicz [15] that the essential ideas of the methods relating to Genetic Algorithms (G.A.) and Evolution Strategies (E.S.) can be found. The author exposes the various stages of G.A's implementation and presents some applications. Within the more precise framework of mechanical applications, many authors have developed various methods of optimization in order to minimize, for example, the weight of structures [11, 13], or to optimize the dynamic performances of mechanical components as Baumal et al. [3] does with a vehicle suspension. However, it is only very recently that works based upon E.S, and specifically devoted to structural shape-optimization, were developed. In that field, we can distinguish those of Younes [23], Marcelin and Kallassy [14] and Sandgren et al. [18].

In order to investigate the problem of shape optimization of uni and bidimensional mechanical structures, two different ways are offered below. The first is based upon sensitivity methods with adjoint variables, while the second one is devoted to the use of evolution strategies. The results deduced from each of the two approaches are then compared so as to determine their own performances as best as possible.

2. Shape optimization of unidimensional structure

Shape optimization of unidimensional structures is connected here to the study of beams. Then, the structures can be modelled by their center line along which the cross-section varies. In this case, the geometrical representation is easily implemented and the structural analysis simply carried out by means of the finite element method. During the iterative optimization process, the initial finite element meshing of the structure is preserved because the transverse dimensions of the cross-section are the only ones that change. The goal of this study is to find an optimal shape of the beam ensuring a uniform distribution of stresses along the center line. The structure discretization by finite elements, and the limitation of their number, need a new formulation of the initial problem. Instead of seeking a continuous maximum normal-stress all along the center line of the beam, a mean stress will be defined on each element, which will remain constant and equal to a fixed value.

We then define the average maximum constraint σ_i of a finite element length l_i by

$$\sigma_i = \frac{1}{l_i} \int_0^{l_i} (\sigma_{xx})_{\max} \, d\xi . \tag{1}$$

It is this value which is taken as the local objective-function for all the N elements of the beam. The design parameters are the heights h_i of each finite elements cross-section of the discretized structure, the whole width of the beam retaining a constant value. Optimizing the structure, states now in the following form:

Minimize the total objective function

$$\Psi = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \tag{2a}$$

under the optimization constraints

$$\sigma_i - \sigma_0 = 0, \qquad i = 1, \dots, N . \tag{2b}$$

In what follows, two methods allowing for the resolution of the aforementioned problem will be detailed, namely the method of sensitivity with adjoint variables and an evolutionary algorithm.

3. Unidimensional optimization methods

Before implementing and developing them more precisely, we will point out the guiding principles of each of the two methods.

3.1. Methods of sensitivity

The starting point for the research of the optimal structure, is the initial shape of the part which we want to optimize. In the initial state, the design parameters have values consistent with the constraints of the problem. The mechanical analysis of the structure, subjected to the imposed loadings, is then carried out by the finite element method. Thus the nodal displacements in the mechanical case, or the nodal temperatures in the thermal case, will be obtained from which the calculation of the objective function and constraints are deduced.

Now, concerning the search for the optimal shape of a structure, it will be noticed that the majority of optimization methods involve mathematical calculation of the gradients of the objective-function and constraints, by means of the design parameters. This can be achieved by a sensitivity analysis of the structure's shape when the design parameters vary, leading to the convergence of the optimization process towards the optimal solution. In this paper, the optimization code which has been implemented is a modified version of A.D.S. [21, 22]. The algorithm is based upon an iterative process in which, at each stage, the design parameters take new values, allowing for the convergence towards the optimal solution when it exists. The algorithm of optimization is presented in Fig. 1.

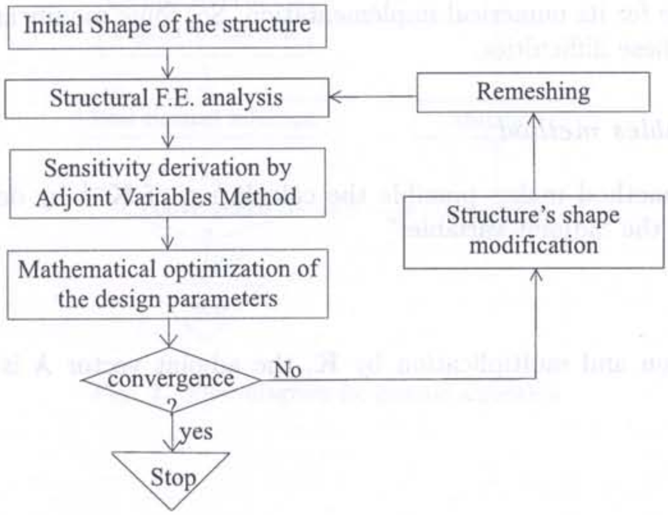

Fig. 1. General algorithm for shape optimization

Knowing that several methods of sensitivity calculation can be used, only the total-differentiation method associated with adjoint variables will be detailed since it has been implemented here. For the structure under study, we define a functional $\Psi(\mathbf{x}, \mathbf{u}(\mathbf{x}))$, the sensitivity of which will be computed referring to $\mathbf{x}$, the design-parameters vector. The field $\mathbf{u}(\mathbf{x})$ is a scalar or vectorial valued field which can be the displacement field, the temperature field, etc.

The functional Ψ describes either a state or a physical phenomenon to optimize, or a constraint to be satisfied whose evolution in the feasible field is controlled by the variation of its gradient with respect to $\mathbf{x}$.

3.1.1. Method of total differentiation

Since the functional $\Psi(\mathbf{x}, \mathbf{u}(\mathbf{x}))$ depends on the design-parameters vector $\mathbf{x}$ and on $\mathbf{u}(\mathbf{x})$, it's total derivative referring to $\mathbf{x}$ is given by the following formal expression

$$\frac{d\Psi}{d\mathbf{x}} = \frac{\partial\Psi}{\partial\mathbf{x}} + \frac{\partial\Psi}{\partial\mathbf{u}}\frac{d\mathbf{u}}{d\mathbf{x}} \tag{3}$$

If in the relation (3), the partial derivatives of Ψ with respect to $\mathbf{x}$ and $\mathbf{u}$ are easily evaluated, the only remaining difficulty lies in the calculation of $\frac{d\mathbf{u}}{d\mathbf{x}}$.

In the following, for the structural analyses, numerical methods will be used, based upon a discretization of the structure. Then, the state of balance will be put in the following matrix form

$$\mathbf{K}(\mathbf{x})\,\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}). \tag{4}$$

$\mathbf{U}(\mathbf{x})$ is the vector of the nodal unknown values of the displacement field, $\mathbf{K}(\mathbf{x})$ is the positive definite stiffness-matrix in the case of a linear behavior of the structure, and $\mathbf{F}(\mathbf{x})$ is the nodal-loading vector. Let us note that all three depend on the design-parameter vector $\mathbf{x}$.

After differentiation of Eq. (4) with respect to $\mathbf{x}$, the term $\frac{d\mathbf{u}}{d\mathbf{x}}$ is found to be

$$\frac{d\mathbf{U}}{d\mathbf{x}} = \mathbf{K}^{-1}\left(\frac{d\mathbf{F}}{d\mathbf{x}} - \frac{d\mathbf{K}}{d\mathbf{x}}\mathbf{U}\right). \tag{5}$$

Now replacing (5) into Eq. (3), the expression of $\frac{d\Psi}{d\mathbf{x}}$ will be

$$\frac{d\Psi}{d\mathbf{x}} = \frac{\partial\Psi}{\partial\mathbf{x}} + \frac{\partial\Psi}{\partial\mathbf{U}}\mathbf{K}^{-1}\left(\frac{d\mathbf{F}}{d\mathbf{x}} - \frac{d\mathbf{K}}{d\mathbf{x}}\mathbf{U}\right). \tag{6}$$

When handling big-size problems where the number of degrees of freedom is very large, this method is generally untractable for its numerical implementation. So, some modifications must be provided in order to overcome these difficulties.

3.1.2. Adjoint-variables method

The adjoint-variables method makes possible the calculation of $\mathbf{K}^{-1}$ by defining a new vector $\boldsymbol{\lambda}$ whose components are the "adjoint variables"

$$\boldsymbol{\lambda}^T = \frac{\partial\Psi}{\partial\mathbf{U}}\mathbf{K}^{-1}. \tag{7}$$

Then after transposition and multiplication by $\mathbf{K}$, the adjoint vector $\boldsymbol{\lambda}$ is solution of the linear equation

$$\mathbf{K}\boldsymbol{\lambda} = \frac{\partial\Psi}{\partial\mathbf{U}}^T. \tag{8}$$

Thus, $\boldsymbol{\lambda}$ is easily computed using the finite elements software, by loading the structure at each node by the corresponding components of $\frac{\partial\Psi}{\partial\mathbf{U}}^T$ appearing in the right side of (8). This latter vector will have been determined beforehand, starting from the analytical form of $\Psi(\mathbf{x}, \mathbf{U}(\mathbf{x}))$.

The total derivative of the functional Ψ, deduced from (3) via (7), will thus take the following form

$$\frac{d\Psi}{d\mathbf{x}} = \frac{\partial\Psi}{\partial\mathbf{x}} + \boldsymbol{\lambda}^T\left(\frac{d\mathbf{F}}{d\mathbf{x}} - \frac{d\mathbf{K}}{d\mathbf{x}}\mathbf{U}\right). \tag{9}$$

The aforementioned expression of the total derivative of the functional Ψ with respect to the design vector $\mathbf{x}$, allows for the implementation of optimization techniques based on methods of gradients.

3.2. Evolution strategies

The basic idea of the methods implementing E.S. consists in starting from an initial population of individuals, constituting a generation, and making it evolving during time. From it, a new generation will appear, made up partly by the best individuals of the preceding generation and partly by new elements who replace those which were eliminated.

3.2.1. Classical genetic algorithms

Classical genetic algorithms are models requiring a modification of the original problems. They derive from some of the mechanisms of evolution in nature, and they operate on binary strings analogous to chromosomes. They are applied in maximizing an unconstrained objective function, due to the inability of G.As to deal with non trivial constraints. They work according to the diagram given in Fig. 2 where the three principal operators of a genetic algorithm applying to a given population, appear as

- the selection operator,
- the crossover operator,
- the mutation operator.

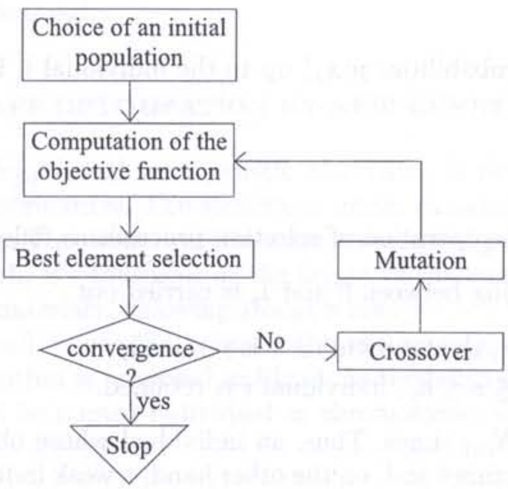

Fig. 2. Flow diagram for genetic algorithm

Algorithm procedure

In the first stage, an initial population is created made up of a finite number of individuals represented by chromosomes which are encoded by binary or real numbers. In the case of binary coding, the chromosomes will consist in series of genes equal to 0 or 1, and their lengths influence the required fitness of the solution. During the creation of new generations, the G.A. maintains a population of potential solutions constituted by the different individuals. When they evolve, the individuals undergo a simulated evolution during which those possessing the best "fitness" survive and reproduce, the others being eliminated. The simulated evolution strongly depend on the three aforementioned main operators that are: selection, crossover and mutation. Then according to these transformations, the new generation consists on some "children" or offsprings, and for the remaining, on "good" individuals of the precedent population. As mentioned by Michalewicz [15], «many evolution programs can be formulated for a given problem. Such programs may differ in many ways; they can use different data structures for implementing a single individual, "genetic" operators for transforming individuals, method for creating an initial population, methods for handling constraints of the problem, and parameters».

In the following, we will propose operators acting on an evolving population, in order that the corresponding simulation program converges and this is achieved when the individual with the best fitness is found.

a – The selection operator

The selection of the best individuals in a population is thus made possible. They will be preserved in the following generation. The principle of selection is based on random search, and works as follows.

Let us consider a population of N_{ind} individuals and let $f(\mathbf{x})$ be the objective function; $f(\mathbf{x}_i)$ is the corresponding value which the individual i confers to $f(\mathbf{x})$, and $\mathbf{x}_i$ is the vector of the design parameters. The total value f_{pop} of the objective function, corresponding to the whole population, is defined in the way of

$$f_{\text{pop}} = \sum_{i=1}^{N_{\text{ind}}} f(\mathbf{x}_i)\,. \tag{10}$$

Each individual in the population receives a measure of it's "fitness" and will be preserved in rising generation. For this purpose, a "survivance probability" $p(\mathbf{x}_i)$ is given by

$$p(x_i) = \frac{f(\mathbf{x}_i)}{f_{\text{pop}}}\,. \tag{11}$$

Now, the sum d_i of all the probabilities $p(\mathbf{x}_j)$ up to the individual i, is given by

$$d_i = \sum_{j=1}^{i} p(\mathbf{x}_j)\,. \tag{12}$$

Given these definitions, the operation of selection proceeds as follows

1. A random number r ranging between 0 and 1, is carried out
2. For $i = 1$: if $d_i \le r$, then individual 1 is retained,
   For $i = 2, N_{\text{ind}}$: if $d_{i-1} \le r \le d_i$, individual i is retained.

This process is repeated N_{ind} times. Thus, an individual whose objective function takes a high value may be selected several times and, on the other hand, a weak individual has a weak probability of being selected.

b – Crossover

Crossover allows for the reproduction of two individuals in the population by some shuffling of chromosomes, producing "offsprings" that contain a combination of information from each parent. There are several ways of carrying out the crossover of two chromosomes so that the genes of the children, are issued from genes of their parents. The traditional crossover operation generally, relates to the following procedure:

1. A random number r_c is pulled-off in order to decide if there will be marriage between a couple of individuals in the population. If r_c is smaller than a given crossover probability p_{co}, the marriage will be canceled, otherwise it will takes place.
2. A random integer n_c is pulled-off in order to determine a site of partition in the chromosomes of each parent. Then the two chromosomes exchange some own parts in the way suggested by Fig. 3.

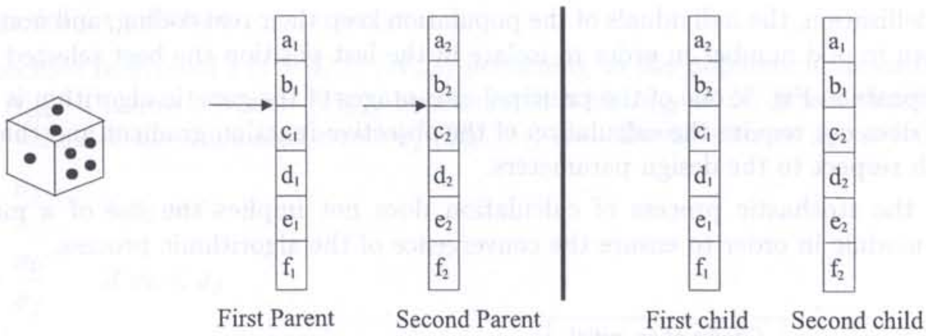

Fig. 3. Crossover between six genes of two individuals

c – Mutation

It concerns randomly all the individuals of a population except the best one. This latter can be affected neither by the crossing nor by the mutation. For a given individual of the evolving generation, the mutation consists on a random modification of a chromosome value.

Mutation rule:

Being given a mutation probability of value p_{mut} for the whole population, at each chromosome, a random value r_{mut} between 0 and 1, is attributed. If r_{mut} is larger than p_{mut}, the chromosome under consideration will be changed.

4. Unidimensional shape optimization by new genetic algorithm

In what follows, a methodology based on a genetic algorithm, is developed in order to optimize the shape of unidimensional structures. The structures under examination are beams and they are modelled by their average line. The design variables are geometrical parameters defining dimensions of the structure like it's length, it's thickness or the height of the cross-section. The problem deals with elastic linear isotropic materials, following Hooke's law.

Before exposing the method, a parallel between the mathematical parameters of optimization and those of the genetic algorithm is proposed in Fig. 4. In the language of the G.A.s, the vector $\mathbf{x}_i$ of the design parameters will be named individual or chromosome, while its component x_{ij} is the j-rd gene of the individual i.

	Mathematical optimization	Genetic Algorithm
Vector $\mathbf{x}_i$	Design - parameter Vector	Individual
x_{ij}	j ird component of $\mathbf{x}_i$	j ird gene of individual i

Fig. 4. Correspondence between mathematical optimization and G.A.

4.1. General principles of the algorithm

The main idea consists in developing a genetic algorithm accounting for the constraints associated with the shape optimization of the mechanical structures. This algorithm is derived according to the same principle as the traditional genetic algorithm. To achieve the sought goal, we will incorporate the constraints of the problem (2) into the selection operator in order to minimize the objective function.

In the modellization, the individuals of the population keep their real coding, and non binary, and they are chosen in odd number in order to isolate in the last position the best selected individual.

As that appears in Fig. 5, one of the principal advantages of the genetic algorithm is that its implementation does not require the calculation of the objective-function gradient and the constraints gradients with respect to the design parameters.

Moreover, the stochastic process of calculation does not implies the use of a mathematical optimization-module in order to ensure the convergence of the algorithmic process.

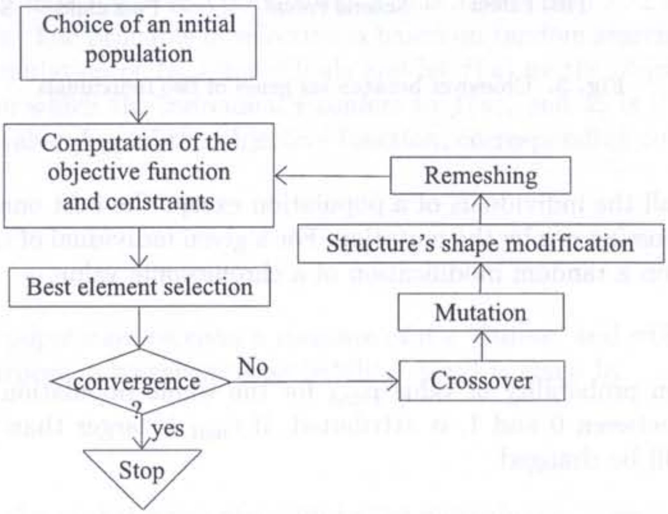

Fig. 5. G.A. for shape optimization of structures

On the basis of an initial population made-up of individuals, chosen randomly and coded as real numbers, the design parameters of the optimization problem are the individuals' genes for which we seek the optimal values. The objective function and the constraints will be evaluated for each chromosome pertaining to the evolving generation. The results are transmitted to the selection operator which will determine the best and the weakest individuals allowing for the selection the ideal chromosome. This latter will be preserved without crossover nor mutation during all time that the algorithm develops.

After the selection stage, a convergence criterion will stop the cycle of optimization if the fitness of the individual is reached. If the criterion is not satisfied, the research goes on by applying the crossover and mutation operators

A new generation is thus obtained with the appearance of new individuals with increased performances. Then, it follows from this an improvement of the structure's shape, requiring that the mesh of the finite elements model will be updated by a modification of the design parameters. The optimization process proceeds towards the optimal solution, until the convergence is achieved.

Now let us detail the appropriate genetic operators within the framework of the finite element method. The structure is discretized in N elements and the design variables of the optimization problem are the values of some parameters like the height, or the radius, of each finite element's cross-section.

In the optimization problem (2), the optimal solution must ensure the satisfaction of the constraints in the same time that the objective function is minimized. For numerical optimization problems several methods were proposed for handling constraints by genetic algorithms, most of them dealing with the use of penalty functions. In the following, a selective pressure is applied on the population, ensuring that the best individuals are preserved during evolution of the generation.

Method suggested for selection

For an unspecified individual i ($i = 1, \ldots, N_{\text{ind}}$) pertaining to the population, at each constraint j ($j = 1, \ldots, N_{\text{cst}}$) a fitness criterion $(P_{\text{fit}})_j$ is associated with $N_{\text{cst}} = N$. It is defined by the ratio between the values of the calculated stress σ_j (Eq. (1)) and allowable stress σ_0, as follows,

$$(P_{\text{fit}})_j = \frac{\sigma_j}{\sigma_0} \qquad \text{if } \sigma_j \leq \sigma_0 \tag{13}$$

$$(P_{\text{fit}})_j = \frac{\sigma_0}{\sigma_j} \qquad \text{if } \sigma_0 < \sigma_j \tag{14}$$

So, starting from the best genes of the chromosomes in the population, it is possible to select the best individual in the following way:

1. A benchmark of the N_{gen} genes of all the individuals is carried out by comparing their k-rd gene ($k = 1, \ldots, N_{\text{gen}}$) by means of their fitness criterion.
2. The gene which has the highest value of $(P_{\text{fit}})_k$ is selected.

This selection process leads to gather the N_{gen} better different genes of all the population where $N_{\text{gen}} = N$. As a result, the better offspring can be obtained by the assembly of the N_{gen} better genes. Then, this "super-individual" is kept unchanged during the operations of crossover and mutation applied to all the remaining individuals of the evolving generation.

b – Crossover and mutation

In the method that will be used, the classical crossover operation is preserved, and the chromosome is randomly splitted in only one point. A contrario, the mutation of genes depends on the value of their fitness criterion, and it proceeds as follows: If the value $(P_{\text{fit}})_k$ of a gene for a given individual is lower than a chosen mutation probability, this gene will be muted otherwise it is preserved in the new generation.

c – Illustration of the method performed for the study of beams

Let us consider the case of the shape optimization of a cantilever beam fixed on its origin and subjected to a shearing force F at it's end. After discretization of the structure by unidimensional finite elements, the design variables are the elementary heights h_k of the cross-sections (if rectangular) of each F.E. In the language of G.A., these heights are the genes of the individual and h_k is it's k-rd gene. The complete individual is composed by the whole set of the h_k (Fig. 6).

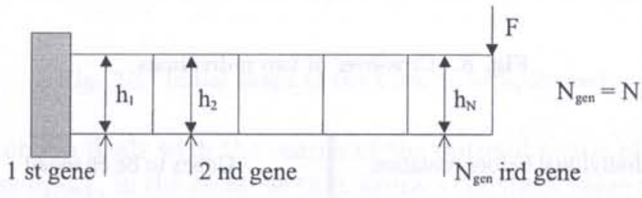

Fig. 6. Genes in mechanical optimization

Let us examine the case of a population of 5 individuals, each one being made up of 5 genes. For the structural study by the F.E.M., this corresponds to a discretization in 5 finite elements. The initial population of potential solutions is obtained using a random choice of genes giving the five structures sketched on the left of Fig. 7.

Above each finite element, the values of the mechanical stresses computed by the F.E. code are deferred, and below, the elementary fitness criteria computed by means of the relations (13)–(14) where $\sigma_0 = 2 \cdot 10^8$ N/m^2. At the current iteration, the optimal structure is represented by the best

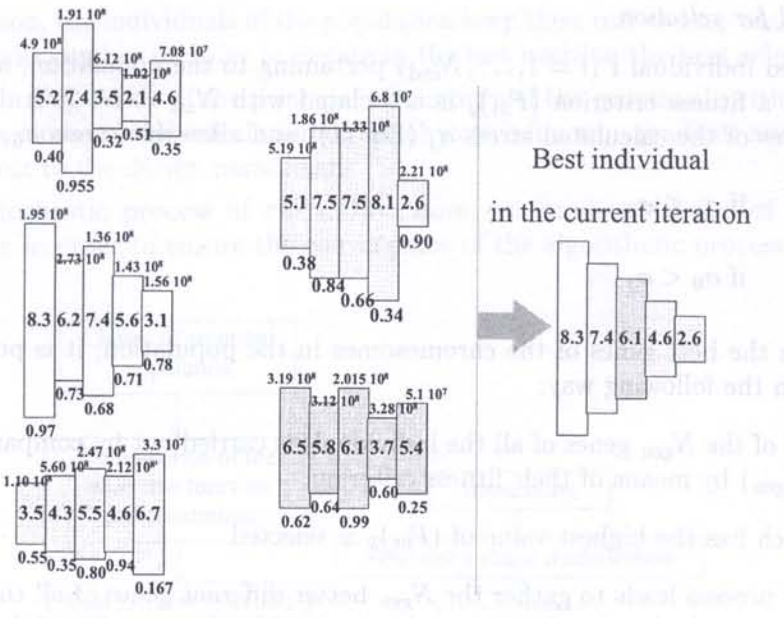

Fig. 7. Selection of the best individual

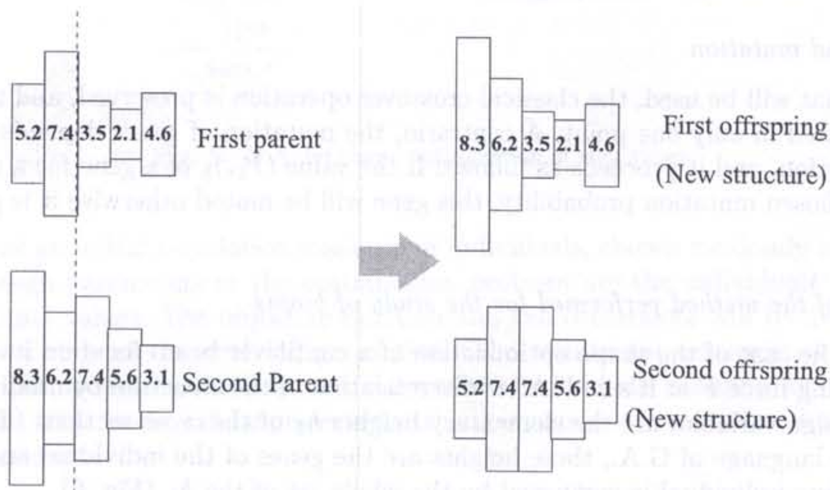

Fig. 8. Crossover of two individuals

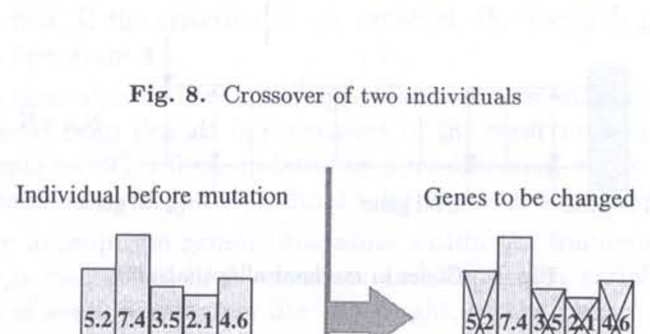

Fig. 9. Mutation of genes ($p_{\mathrm{mut}} = 0.9$)

individual made up of best genes. For example, it is found that the second individual has best first gene since this one has the greatest value of the fitness index ($(P_{\text{fit}})_1 = 0.97$).

Concerning the crossover operation, let us take for example the first two individuals. Having chosen the splitting position of the chromosome, their crossover leads to the two offsprings corresponding to two new structures as it can be seen in Fig. 8.

Now let us illustrate the mutation rule on the aforementioned example. If the mutation probability p_{mut} is taken equal to 0.9, the genes of the individuals undergo mutation when their fitness index P_{fit} is lower than p_{mut}. It is what appears in Fig. 9.

4.2. Applications

In what follows, we will expose some numerical results of the optimization of different beams, rising from the implementation of the modified genetic algorithm. The numerical methods of structural analysis are based on the F.E.M. and are specific to each application.

4.2.1. First example: Cantilever beam

This simple study has been chosen like first test of the method because the analytical solution of the optimized shape is well known. In it's initial shape, the beam has a uniform rectangular cross-section with an height h and a width b. The beam length L and width b remain constants during the process of optimization. The structure is clamped on it's origin and is loaded on the other hand by a shearing force F, according to Fig. 10.

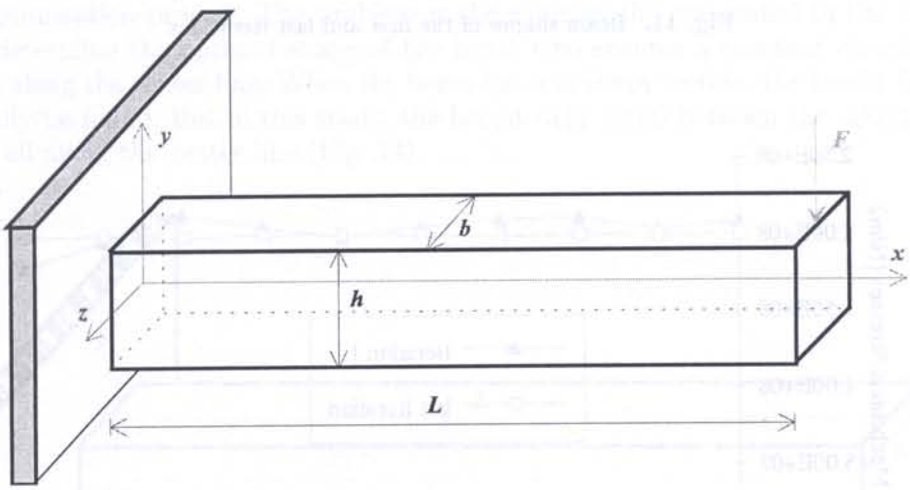

Fig. 10. Initial shape of the beam to be optimized

The optimization problem deals with the search of the optimal shape of the beam which ensures that the maximum stress $\sigma_{\xi\xi\max}$ in the cross-section, keeps a constant value along the center line. The structural analysis was implemented using proper F.E. programs based upon the theory of Strength of Material. The beam has been discretized by N finite elements, so that the design variables consist in the N heights h_k of the elements.

The numerical application, relates to a beam with length $L = 0.4$ m and width $b = 0.02$ m, subjected to a force $F = 12000$ N on it's end. The constitutive material is steel with a Young's modulus $E = 2 \cdot 10^{11}$ Pa and a Poisson's ratio $\nu = 0.3$. The structure is discretized in $N = 10$ finite elements and the admissible stress limits $\sigma_{\xi\xi\max}$ to the value $\sigma_0 = 2 \cdot 10^8$ Pa.

The initial population includes 49 individuals ($N_{\text{ind}} = 49$). The maximum iteration-number in the G.A. has been chosen as 50 while crossover and mutation probabilities are worth respectively $p_{\text{co}} = 0.5$ and $p_{\text{mut}} = 0.9$.

The results which have been obtained by the G.A. implementation on a micro-computer are given in Fig. 11. The two shapes corresponding to the first and the last iterations can be compared and it will be noted that the first iteration shape is very close to that of the required optimum beam. This is due to the effectiveness of the aforementioned selection process in which, at each iteration stage, the best genes in the generation are retained.

Moreover, the stresses evolutions corresponding to the first and last iterations show, in Fig. 12, that in the former case, the distribution is not yet homogeneous as it would be.

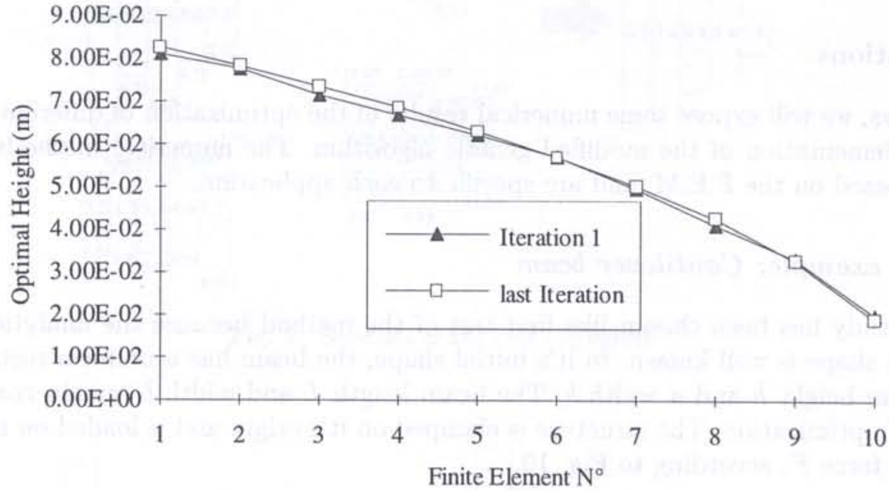

Fig. 11. Beam shapes of the first and last iterations

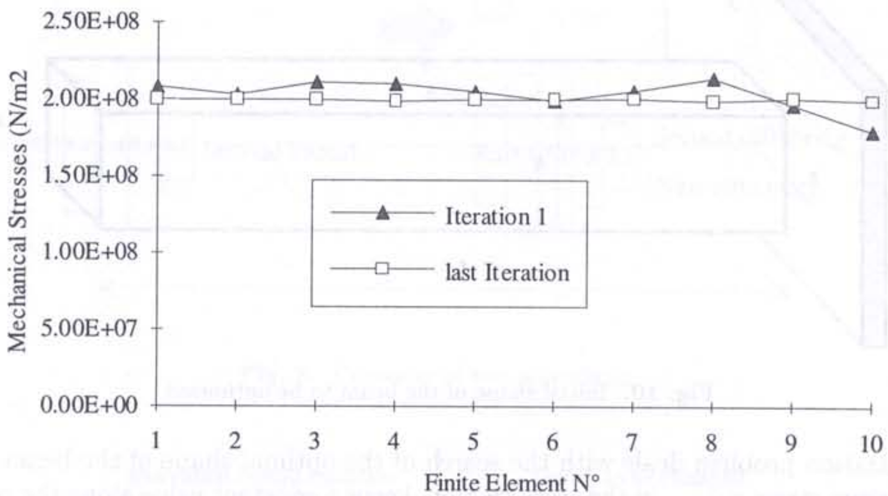

Fig. 12. Stresses distributions in the first and last beam shapes

In order to represent the stresses evolutions during the successive iterations, a relative error for each finite element will be defined by the ratio $|(\sigma_{\xi\xi \max} - \sigma_0)/\sigma_0|$. For the best individual selected in each generation and concerning two consecutive iterations, the convergence can be fast as it can be observed in Fig. 13.

The aim of this first example was to have a benchmark of our approach, and it has been found that the G.A. method was sufficiently accurate. Then, it has been decided to investigate a more complex problem corresponding to a beam lying on an elastic foundation.

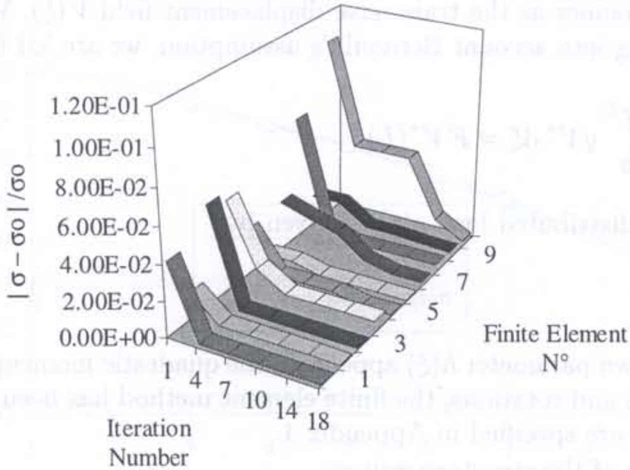

Fig. 13. Evolution of the stresses relative-error

4.2.2. Beam on elastic foundation

It is proposed here, to apply the genetic-algorithm process to the case of a beam lying on an elastic foundation, and subjected to a shear force at it's end. The structural analysis requires a particular preliminary study in order to compute the displacements of the cross-sections of the beam. In that example, the cross-section is rectangular and the elastic foundation is characterized by a unit stiffness R (in $\mathrm{N/m^2}$). The width b of the section and the length L of the beam are led constant during the optimization process. The problem is the same as the one stated in the first example. We want to determine the optimal shape of the beam who ensures a constant distribution of the normal stress along the center line. When the beam has a uniform section, the height-function $h(\xi)$, can analytically be found. But in this study, the height $h(\xi)$ varies between the origin and the end of the beam, all along the center line (Fig. 14).

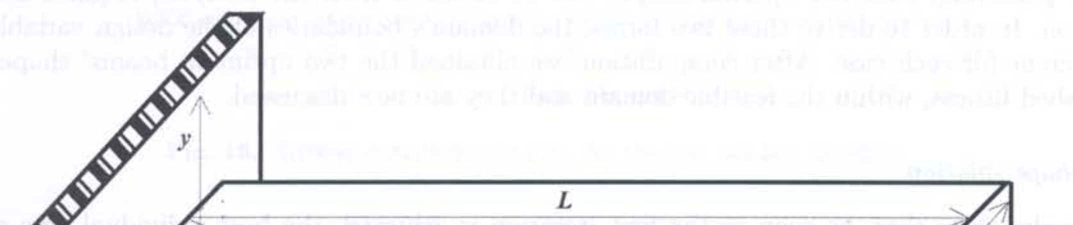

Fig. 14. Cantilever beam on an elastic foundation

It is then untractable to obtain an analytical expression for the optimal $h(\xi)$, and only a numerical study can be intended, and for that reason, the F.E.M. will be chosen. The static equation of balance for the cantilever beam supported by an elastic foundation, is found by application of the principle of the virtual-work. For this purpose, a virtual displacement field $V^*(\xi)$, kinematically admissible, is defined in the same manner as the transverse displacement field $V(\xi)$. Within the framework of beams' theory and taking into account Bernoulli's assumption, we are led to the following equality,

$$\int_0^L EI_z V_{\xi^2} V^*_{\xi^2} \, \mathrm{d}\xi - \int_0^L q\, V^* \, \mathrm{d}\xi = F\, V^*(L) \tag{15}$$

where the continuously distributed load $q(\xi)$, is given by

$$q(\xi) = -R\, V(\xi). \tag{16}$$

The structural unknown parameter $h(\xi)$ appears in the quadratic moment I_z. In order to compute the nodes' displacements and rotations, the finite element method has been implemented, and these particular developments are specified in Appendix 1.

The balance equation of the structure writes:

$$(\mathbf{K}_g + \mathbf{G}_g)\, \mathbf{U}_g = \mathbf{F}_g\,. \tag{17}$$

The index g reads for the whole structure. $\mathbf{U}_g$ is the global displacement vector accounting for all the nodes of the structure, and $\mathbf{F}_g$ is the vector of generalized forces. $\mathbf{K}_g$ et $\mathbf{G}_g$ are respectively the global stiffness matrix of the beam and the elastic foundation matrix.

The design parameters being the N finite elements' heights h_i, it will be expected that the optimization algorithm provides a single final form which optimizes the objective function and satisfies the imposed constraints.

In the case of the beam lying on an elastic foundation, with the objective function defined by (2), the numerical calculation led to two optimal solutions. One is similar to the optimal shape of the beam already studied while the second gives a completely different shape. Let us consider the same numerical data and the same parameters of the G.A. as those of the Cantilever beam. The structure made up of the same material as previously, lies on an elastic foundation of stiffness $R = 1{\cdot}10^6$ N/m^2.

The possibility that two optimal shapes can be obtained from the analysis, requires a special attention. In order to derive these two forms, the domain's boundaries of the design variables will be different for each case. After computation, we obtained the two optimum beams' shapes with the wished fitness, within the feasible domain and they are now discussed.

First shape solution

The results show that, as soon as the first iteration is achieved, the best individual gets genetic components close to the optimal values as that appears in Fig. 15. Thus, it is shown that the effectiveness of the selection operator is highly related to the convergence of the G.A.

As it will be observed in Fig. 16, the stresses corresponding to the optimal beam, are quite constant and equal to the limiting stress value in all elements, whereas for the first iteration, some large oscillations of the stresses-values appear leading to a bad solution.

Second beam's shape solution

The second optimal form is obtained by modifying the limits of variations for the design parameters. These limits are continuously restricted until the convergence is obtained. On iterations totaling 50, it has been found that the value of genes corresponding to the first iteration are close to those of the best individual (Fig. 17).

Now let us consider the stresses' evolution in each F.E. occurring in the first and the last iterations. Even though the genes between the two solutions are close to each other, it can be observed that the stresses are far from the admissible stress for some finite elements. Despite some oscillations of the mean stresses which occur during iterations, Fig. 18, the convergence is always reached.

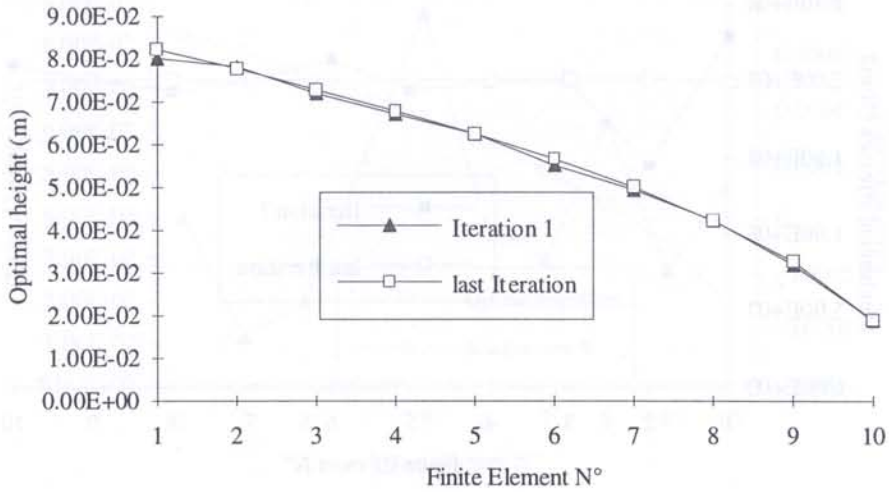

Fig. 15. Optimal solution: corresponding shape for the first iteration

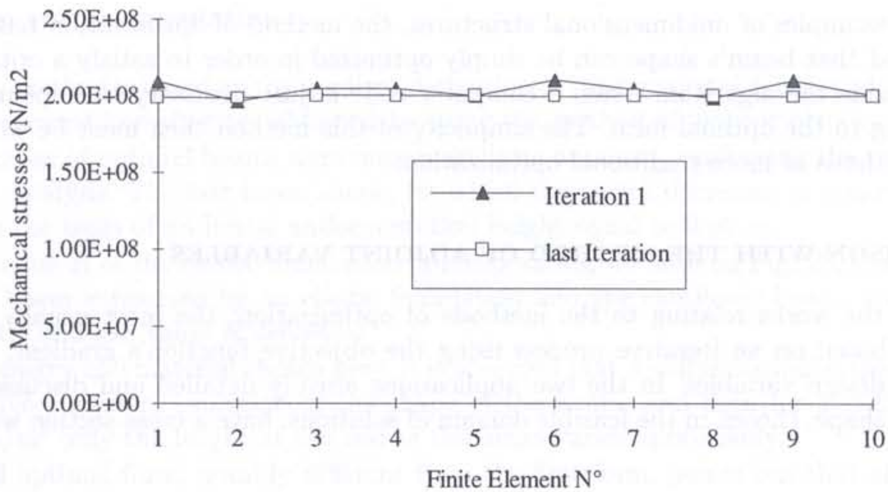

Fig. 16. Stresses distribution in F.E., for the first and last iterations

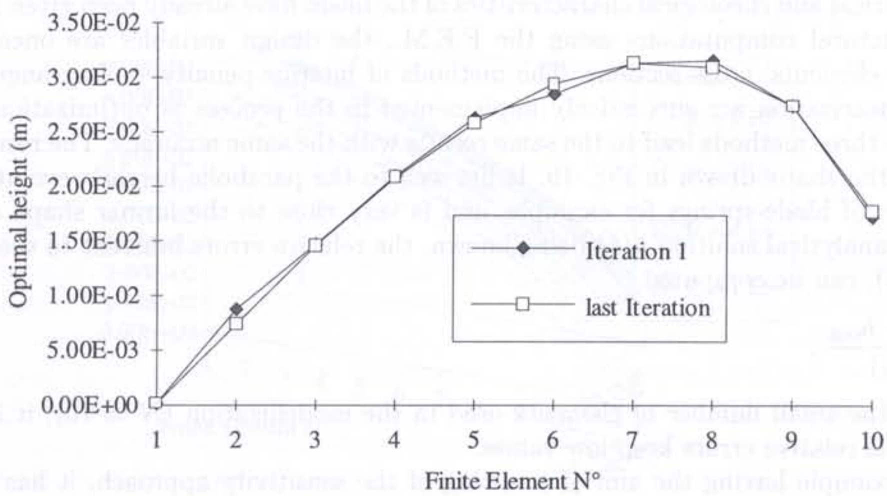

Fig. 17. Beam on elastic foundation. Second optimal beam's shape

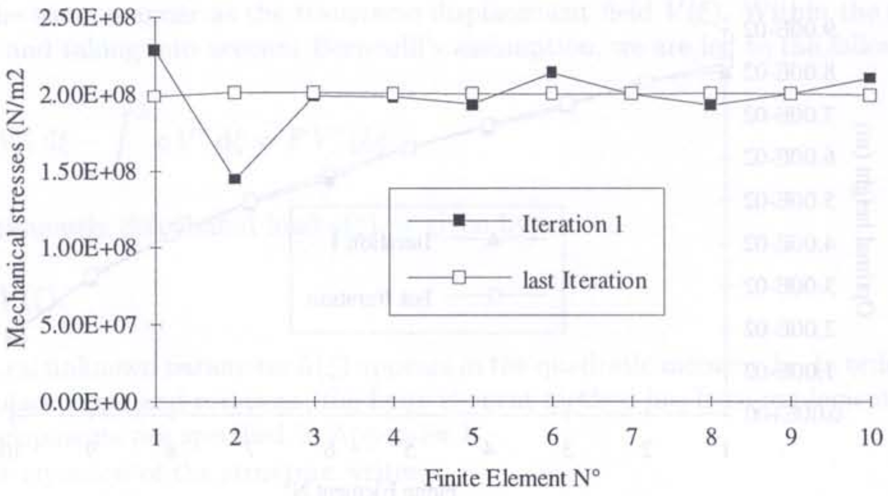

Fig. 18. Stresses distribution in F.E., for the first and last iterations

Conclusion

Through two examples of unidimensional structures, the method of optimization based on the use of G.A. showed that beam's shape can be simply optimized in order to satisfy a criterion of equal resistance. Beside the algorithm itself, a computer code is just necessary to implement the G.A. process leading to the optimal form. The simplicity of this method then must be worst compared with other methods of more traditional optimization.

5. Comparison with the method of adjoint variables

In almost all the works relating to the methods of optimization, the most usually implemented are generally based on an iterative process using the objective function's gradient, computed by means of the design variables. In the two applications already detailed and discussed below, the initial beam's shape, chosen in the feasible domain of solutions, have a cross-section with a constant height.

5.1. Cantilever beam

All the geometrical and rheological characteristics of the beam have already been given in Section 4.2.

In the structural computations using the F.E.M., the design variables are once more the N heights of the elements' cross-sections. The methods of interior penalty, or Lagrange's multipliers and convex linearization are successively implemented in the process of optimization. It has been found that the three methods lead to the same results with the same accuracy. The resulting optimal structure has the shape drawn in Fig. 19. It fits well to the parabolic height's evolution known in the calculation of blade-springs for example, and is very close to the former shape deduced from the G.A. The analytical solution $h(\xi)$ being known, the relative errors inherent to the method, and defined by (18), can be computed,

$$\varepsilon = \frac{h(\xi_i) - h_{\text{opt}}}{h(\xi_i)} . \tag{18}$$

Despite of the small number of elements used in the modellization ($N = 10$), it is observed in Fig. 19 that the relative errors keep low values.

This first example having the aim of a testing of the sensitivity approach, it has been decided to investigate the problem corresponding to a beam lying on an elastic foundation, as it has been studied formerly in Section 4.2.2.

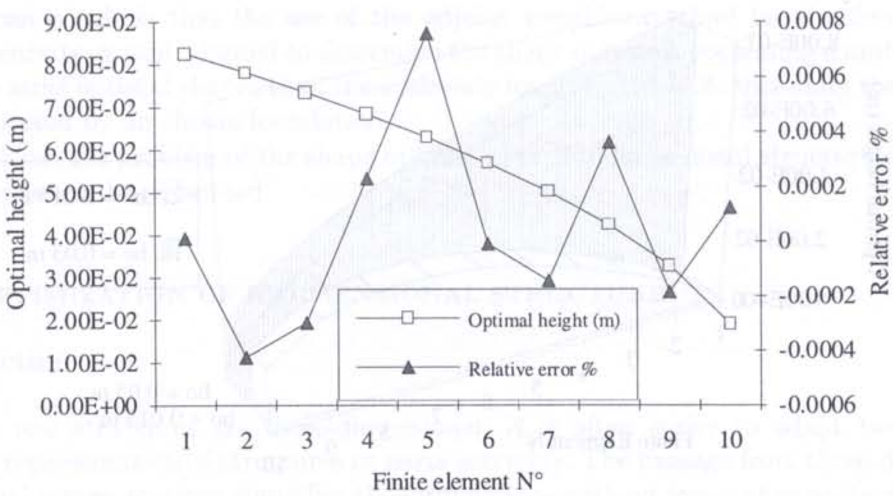

Fig. 19. Optimal shape of the cantilever beam

5.2. Beam on elastic foundation

The sensitivity method is based on the gradient derivation of the functional Ψ using adjoint variables. The results presented hereafter are obtained by using the method of interior penalty.

The two shapes of optimal beams were once more time obtained, confirming the results deduced from the G.A. analysis. The first beam shape, for which the height decreases in a monotonous way, is obtained on the basis of an initial uniform-section height equal to 0.05 m.

When the value R of the elastic-foundation stiffness varies, we note on Fig. 20, that the optimal shapes of the beam supported by an elastic foundation and the cantilever beam, are very similar when the elastic stiffness value R decreases.

For the geometry and material chosen here, it is observed that a stiffness smaller than $1 \cdot 10^6$ N/m^2 does not involve any more evolution of the shape of the beam, and than for a stiffness R larger than $1 \cdot 10^7$ N/m^2 only the height at the end of the beam varies appreciably.

The second optimal form, notably different from the first form, points out that of a «fish». For low values of the stiffness R, the parabolic-shape of beams of equal-strength is reached. The optimal shape of the beam, characterized by its height $h(\xi)$, evolves oppositely to the stiffness of the elastic foundation. The height does not vary any more in a monotonous way but grows and decreases,

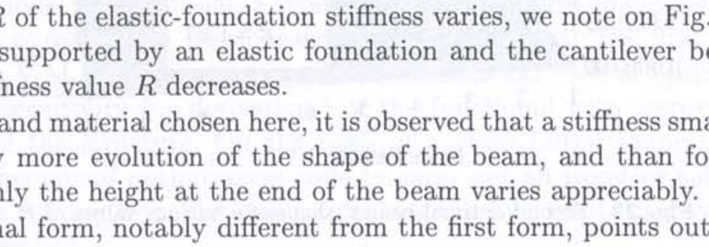

Fig. 20. First optimal shapes for various values of R

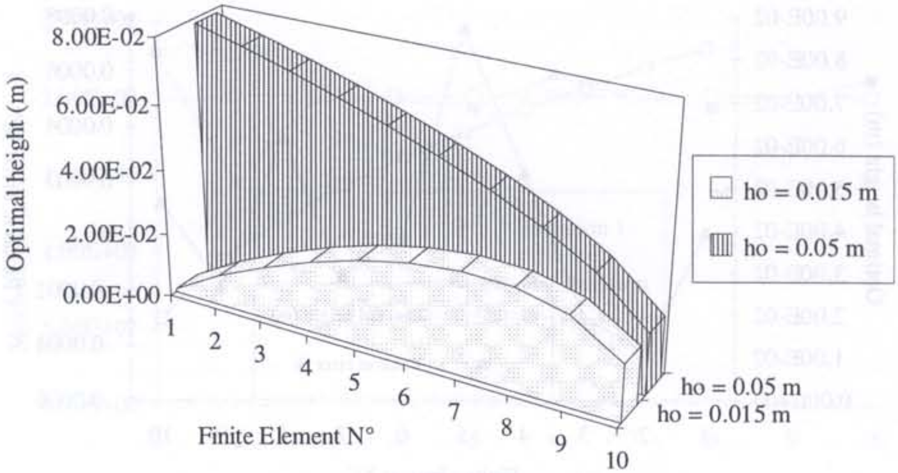

Fig. 21. Optimal shapes of beams supported by an elastic foundation

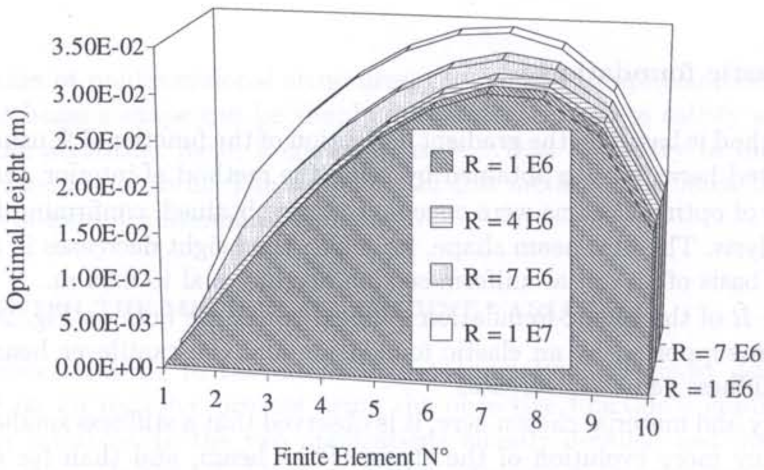

Fig. 22. Second optimal-beams' shapes for various values of R

passing through a maximum. This surprising solution is obtained when the initial structure has a constant height h_0 chosen to be equal to 0.015 m.

It appears thus that the choice of the initial solution selected in the feasible domain, imposes the optimal shape of the beam

In the case of the aforementioned application, the two shapes of the optimal beams are those sketched in Fig. 21.

Moreover, in order to compare the evolution of the two forms resulting from a bifurcation of the solutions according to the stiffness R, we present Fig. 22, the corresponding "fish-shapes" of the beams for various values of R. This latter figure will be compared to Fig. 21 when the evolutions of the two solutions are examined for differences.

Observing this figure, an opposite conclusion with that presented in the first case can be set up: when the stiffness of the foundation increases the optimal shape of the beam grows bigger, the transverse dimensions of the cross-sections develop except at the end of the beam.

It is surprising to note that the height at the embedding side, is always very small except when R reaches very large values, and that it remains always smaller than the height at the loading end.

These results never mentioned in the literature at our knowledge, being surprising, we modelled these beams using the Cadsap F.E. [6] software and carried out the calculation of the normal stresses. The numerical results thus obtained confirm that the "fish-shapes" beams are of equal strength and that they give the same stresses' values than the classical "parabolic" beams.

Then, we can conclude that the use of the adjoint variables method for the derivation of the functional sensitivity, is well adapted to determine the shape of beams possessing a uniform strength. It leads to the same optimal shapes than those already found by the G.A. including the "fish shapes" of beams supported by an elastic foundation.

In what follows, the problem of the shape optimization of bidimensional structure will be tackled and two methods will be presented.

6. Shape optimization of bidimensional structure

6.1. Introduction

Although the real structures are three-dimensional, it is often easier to adapt two-dimensional models to the representation of structures or parts assembly. The passage from three-dimensional to two-dimensional representations simplifies the formulations without losses of important information, in order to obtain quickly usable results by the designers.

Within the framework of the shape optimization of bidimensional structures, the geometry of the structure constitutes the design variable of the problem. It evolves during the optimization process until the convergence to an optimal solution is reached.

A geometrical representation of the free boundary being essential, among all the possible modelizations, the mathematical approach of the contour geometry by B-splines curves, appears very interesting. The particularity of this representation, compared to other types of geometrical modelization, is that it is defined by a set of poles controlling the shape of the curve. Particularly, it is possible to obtain a local modification of the curve and to choose independently its polynomial degree and the number of poles [1].

The geometrical model being chosen, a strategy of the structure shape-optimization has now to be defined. For that purpose, two possible methods were studied. The first method, the Adjoint Variables Method (A.V.M.), requires the use of a mathematical code of optimization carrying out a calculation of the sensitivity (or derivatives) of the functional with respect to the evolution of the external shape of the structure. On the basis of a given initial shape of the structure, it is possible via the mathematical optimization code to carry out an iterative calculation, based upon the numerical results coming from the sensitivity computation, until the convergence towards the optimal solution is reached.

The regularity of the finite elements mesh is one of the essential factors influencing the accuracy of numerical calculations, especially in the stages of structural sensitivity analysis. A disturbed sensitivity calculation of the objective functional and constraints, compared to the design variables of the problem, led to the divergence of the optimization algorithm and to an erroneous solution [17].

In this work, an interfacing between the programs of mathematical optimization and the finite elements code Cadsap was realized. It takes account of the decoding of the structure and reactualizes the mesh for each iteration since the shape of the structure evolves during the optimization process.

The second method, which was implemented, relates to the Genetic Algorithms (G.A.). These ones are optimization algorithms based on evolution and selection criteria and are analogues to those of the living organisms in nature.

On the basis of an initial population constituted of series of individuals, the generation evolves during time. A new generation will appears, made up of the best individuals of the preceding generation associated to a new series of individuals.

In the follows, a genetic algorithm will be developed in order to solve some problems of bidimensional structural optimization, in which new operators of selection, crossover and mutation will be proposed. From a mathematical point of view, the optimization problem is solved by taking account of imposed equality constraints.

Before presenting the two methods applied to some mechanical parts and their numerical results, the goal of this study must be recalled.

6.2. Optimization problem

The shape optimization of mechanical bidimensional-structures remains one of the principal objectives of mechanical design. The goal is to satisfy the constraints imposed on a part, in order to obtain the optimum of an objective function written in the form of a functional.

In the following, a problem of optimal mechanical-strength will be solved, using the Finite Element Method. It is stated as follows:

Find the ideal shape of the structure which ensures a uniform distribution of the von Mises stresses in the part. After than the domain has been discretized, the study will be restricted to the adjacent zone of the evolving boundary of the part which has to be optimized.

In order to bring back the formulation to an optimization problem, the functional Ψ_i representing the average von Mises stress in the finite element number i (of D_i domain) is defined by

$$\Psi_i = \iint_{D_i} g\, m_i \, \mathrm{d}D, \tag{19}$$

m_i is a ponderation function, defined on D_i by

$$m_i = \frac{1}{\iint_{D_i} \mathrm{d}D} \quad \text{inside } D_i\,, \qquad m_i = 0 \quad \text{outside } D_i. \tag{20}$$

In a two-dimensional state of stresses, g is defined by the von Mises equivalent stress given by

$$g = \sqrt{(\sigma_{11})^2 + (\sigma_{22})^2 + 3(\sigma_{12})^2 - \sigma_{11}\sigma_{22}}\,. \tag{21}$$

For the optimized part, the total objective function Ψ will be the arithmetical mean of the elementary functionals Ψ_i and reads:

$$\Psi = \frac{1}{N}\sum_{i=1}^{N} \Psi_i\,. \tag{22}$$

N is the number of the finite elements adjacent to the variable boundary to be optimized.

The structural optimization problem becomes then

Minimize

$$\Psi = \frac{1}{N}\sum_{i=1}^{N} \Psi_i \tag{23a}$$

under the equality constraints

$$\Psi_i - \sigma_0 = 0\,, \qquad i = 1, \ldots, N\,; \tag{23b}$$

σ_0 is the imposed stress.

7. Bidimensional shape optimization methods

7.1. Adjoint Variables Method

The boundary method and *the domain method* form part of the most recent methods available in the field of shape optimization. They are based on the sensitivity calculation of a functional associated with the mechanical problem, with respect to shape modification. These methods are carried out by means of material derivative concepts and variational formulation as they are developed by Choï and Haug [10] and Haug et al. [12]. Then, Trompette et al. [20] used them for resolving the shape optimization problem of mechanical structures.

Each one of these two methods has advantages and disadvantages. The main advantage of the first one is that the calculation is restricted only to the finite elements belonging to the boundary which must be optimized. So, its numerical implementation is greatly facilitated. Let M be a point inside D and has coordinate x and y. The displacement field and the strain and the stress fields are noted $\mathbf{u}(\mathbf{x})$, $\varepsilon_{ij}(\mathbf{u})$ and $\sigma_{ij}(\mathbf{u})$. In the field of linear elasticity, the sensitivity formulation of Ψ_i includes only surface integrals taking the form shown by Haug et al. [12],

$$\begin{aligned}\Psi_i' = &-\iint_S \Big[\sum_{i,j=1}^{3} \sigma_{ij}(\mathbf{u})\,\varepsilon_{ij}(\boldsymbol{\lambda})\Big]\,(\mathbf{V}^T\mathbf{n})\,\mathrm{d}S + m_i \iint_S [g(\sigma(\mathbf{u})) - \Psi_i](\mathbf{V}^T\mathbf{n})\,\mathrm{d}S \\ &+ \iint_{S_0} \sum_{i,j=1}^{3} [\sigma_{ij}(\mathbf{u})n_j\,(\nabla\boldsymbol{\lambda}_i^T\mathbf{n}) + \sigma_{ij}(\boldsymbol{\lambda})n_j\,\nabla u_i^T\mathbf{n})](\mathbf{V}^T\mathbf{n})\,\mathrm{d}S \\ &+ \iint_{S_1\cup S_2} \Big[\sum_{i=1}^{3} F_i\lambda_i\Big]\,(\mathbf{V}^T\mathbf{n})\,\mathrm{d}S + \iint_{S_2} \sum_{i=1}^{3} [\nabla(T_i\lambda_i)^T\mathbf{n} + H(T_i\lambda_i)](\mathbf{V}^T\mathbf{n})\,\mathrm{d}S\end{aligned} \tag{24}$$

where $F = \{F_1 \;\; F_2 \;\; F_3\}$ is the external force, $T = \{T_1 \;\; T_2 \;\; T_3\}$ the traction force, $\mathbf{n}$ is the unit vector normal to the free boundary, $\mathbf{V}$ is the vectorial field imposing the "domain deformation" and H is the curvature of the external boundary S, $H = \mathrm{div}(\mathbf{n})$. $\boldsymbol{\lambda}$ is an adjoint displacement, solution of the equation

$$\iiint_D \Big[\sum_{i,j=1}^{3} \sigma_{ij}(\boldsymbol{\lambda})\,\varepsilon_{ij}(\boldsymbol{\lambda}^*)\Big]\,\mathrm{d}D = \iiint_D \Big[\sum_{i,j=1}^{3} g_{\sigma_{ij}}(\mathbf{u})\sigma_{ij}(\boldsymbol{\lambda}^*)\Big]\,m_k\,\mathrm{d}D. \tag{25}$$

The boundary S of the domain D which is divided into three parts:

– the fixed boundary S_0 along which the displacements are imposed.

– the free boundary S_1 subjected to changes piloted by $\mathbf{V}$.

– the boundary S_2 submitted to the external forces density $\mathbf{T}$.

$\boldsymbol{\lambda}^*$ is a kinematically acceptable field, associated with $\boldsymbol{\lambda}$.

Nevertheless, a major disadvantage appears in this method: the lack of numerical precision of the sensitivity value, due to the inaccurate calculation of the mechanical stresses on the moving boundary S_1. For some types of finite elements, for example three nodes triangle or four nodes rectangle, this approach leads to inaccurate result.

When the problem formulation is handled by mean of the domain method, these disadvantages are avoided since the sensitivity expressions are volume integrals which calculations are carried out for the whole set of the finite elements of the structure.

Meanwhile, the computation of the volume integrals on the totality of the structure generates a greater complexity and a large computing time. Moreover, the number of the terms appearing in the sensitivity expression is larger than the preceding used in the boundary method. Effectively in this last case, some terms are eliminated while passing from a volume integration to boundary integration.

The methodology used in this work consists in combining the advantages of the two methods and rejecting the disadvantages of them.

The approach consists in using the basic idea of the two methods. The integrals on the free boundary, needed for the calculation of sensitivity by the boundary method, will be transformed into volume integrals and calculated at the Gauss points of the finite elements adjacent at this boundary. This is possible by applying the divergence theorem.

7.2. New Genetic Algorithm

The difficulties arising in structural shape-optimization by means of the adjoint-variables method is due to the sensitivity calculation and the necessity to use a mathematical optimization code. When the analysis is carried out by means of genetic algorithms, the calculation don't need these two mathematical modules and this is one of the most important advantages of this method. Of course, the mesh updating remains the same one here.

In the following, a new methodology based on a genetic algorithm, is developed in view of optimizing the shape of bidimensional structures. The design variables are geometrical parameters defining the shape of the structure. In the language of the G.A.s, the vector $\mathbf{x}_i$ of the design parameters will be named individual or chromosome, while its component x_{ij} is the j-rd gene of the individual i.

The research of the best of them is based upon particular operators, the goal of which is to select or modify, either by random process or by crossover, the individuals genes. The genes are constituted by some characteristics of the individuals which are for example in the case of the mechanical studies, a dimension, a material characteristics, etc. After several transformations of the populations, it is generally possible to decide which of individuals satisfy the criteria allowing for the choice of the "ideal" individual. So, the G.A. must account for the evolution of the populations on which act a number of external constraints imposed by the environment. In the mechanical studies, these constraints come from the use which will be made of the optimized parts. Following Hooke's law, the problem deals with mechanical parts made up of elastic linear isotropic materials.

In the developed algorithm, the individuals of the population kept their real coding. An odd number of individuals is selected in order to put, in last position in the queue, the best selected individual. This one should not undergo any operation of crossover or mutation.

7.2.1. General characters of the algorithm

The principal stages of the algorithm process as well as the key points of this method must now be pointed out.

a – Selection

Let us recall that an individual being made up of genes which are the design variables, the selection operator acts in order to determine the best of them in a generation.

Thus, for any unspecified individual i ($i = 1, \dots, N_{\text{ind}}$) pertaining to the population, a probability called "existence probability" $(P_{\text{ex}})_j$, is associated to each constraint Ψ_j, $j = 1, \dots, N$ (Eq. (19)). It is defined by the ratio between the values of the calculated constraint Ψ_j and admissible stress σ_0 and conversely

$$(P_{\text{ex}})_j = \frac{\Psi_j}{\sigma_0} \qquad \text{if } \Psi_j \le \sigma_0 \,, \tag{26a}$$

$$(P_{\text{ex}})_j = \frac{\sigma_0}{\Psi_j} \qquad \text{if } \sigma_0 < \Psi_j \,. \tag{26b}$$

So, for each individual pertaining to the population, the objective function and the constraints Ψ_i are evaluated. The existence probability is computed, in the form of (26a) and (26b), and is associated to each constraint according to the aforementioned procedure.

In the second stage, the lower value of $(P_{\text{ex}})_j$ for each individual is defined. For an individual of the generation, a second probability $\mathrm{P_{ind}}$ is defined and called the "individual performance". P_{ind} is the minimal value of the probabilities $(P_{\text{ex}})_j$ ($j = 1, \dots, N$) and is given by

$$P_{\text{ind}} = \Big[\min{(P_{\text{ex}})_j}\Big]_{\text{ind}} \,, \qquad \text{ind} = 1, \dots, N_{\text{ind}} \,. \tag{27}$$

The best selected individual is that one which has the greatest performance value P_{ind} since it gives the nearest constraints to the admissible stress. It is placed in last position in order to avoid mutation and crossovers with other individuals of the population.

b – Crossover

For the purpose of crossover, the random choice of the individuals components (or genes) of the initial population is carried out. Knowing that, the existence probability (P_{ex}) of the very weak individuals takes sometimes an important value, the implementation of a specific approach forcing the convergence towards an optimal solution is required.

This approach mixes the traditional process of crossover and another process. So in the same time, a crossover and a mutation leading to the disappearance of the weakest individuals. It proceeds as follows.

Being given their performance P_{ind}, the individuals are classified by decreasing order, from the best to the weakest. An individual is considered weak if its performance P_{ind} is lower than a specified value P_{imp} and in this case it will be muted. Then, the best individuals are crossed two by two, producing several children who will take the places of the weakest individuals.

When the crossover of two individuals produces four children, the chromosomes are distributed in the following way:

First child: The genes of the pair positions are those of the first parent and the genes of the odd positions are those of the second parent.

Second child: He is obtained by reversing the operation relating to the first child.

Third child: The first and the last genes are those of the first parent while the other genes are those of the second parent.

Fourth child: He is obtained by reversing the operation relating to the first child.

For example, the crossover of two parents with 6 genes $(a_1, b_1, c_1, d_1, e_1, f_1)$ and $(a_2, b_2, c_2, d_2, e_2, f_2)$ gives 4 children whose genes are presented in Fig. 23.

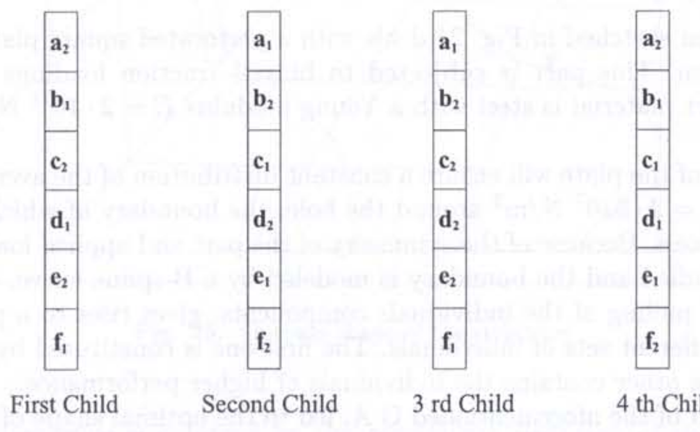

Fig. 23. Set of children

c – Modification of the variables limits

There are several types of design variables, which can be used and called or not by the program. The variables of the first kind are numerical variables with integer values, real or double precision values, etc.

In this studies, all the variables are a real and double precision to obtain a good calculation accuracy because it has been found that a small variation of the numerical values can lead to an important disturbance of the numerical results.

During the random pulling, the design variables get arbitrary values and they can be selected very far from the optimal values. Regarding to the computer time consuming, this is a non negligible disadvantage.

Then, this can be avoid by changing the variables limits in order to reduce continuously variation intervals, "a good" pulling of the parameters and favoring a rapid convergence towards the optimal solution.

d – Criterion of convergence

The convergence criterion of the algorithm is based upon the limitation of the maximum number of iterations in the case where the stop criterion below is not yet satisfied.

– Stop criterion:

The calculation stops if all the stresses in the elements satisfy the following inequality,

$$\left|\frac{\Psi_i - \sigma_0}{\sigma_0}\right| \le \varepsilon, \qquad i = 1, \ldots, N. \tag{28}$$

ε is a small value which can be taken equal to 0.01.

7.2.2. Application

The optimal shape of the bidimensional structure under consideration is devoted to investigate the genetic algorithm reliability, after what it will be applied to various studies. The same discretization of the structure and the same design variables as taken in the case of the shape optimization by the adjoint variables method are used.

A population of 99 individuals with a crossover probability P_{crois} of 0.5 and a P_{imp} value of 0.95 are chosen. The number of iterations involved is fixed at 25.

a – Optimization of a perforated plate

The present application sketched in Fig. 24 deals with a perforated square plate of 100×100 mm^2 with a thickness 5 mm. This part is subjected to biaxial traction loadings $\sigma_x = 15$ MPa and $\sigma_y = 10$ MPa. The part material is steel with a Young modulus $E = 2 \cdot 10^{11}$ N/m^2 and a Poisson's ratio $\nu = 0.3$.

The optimal shape of the plate will ensure a constant distribution of the average von Mises stress (Ψ_i , $i = 1, \ldots, N$), $\sigma_0 = 2 \cdot 510^7$ N/m^2 around the hole, the boundary of which will change during the computational process. Because of the symmetry of the part and applied loadings, just a quarter of the plate will be studied and the boundary is modeled by a B-spline curve.

The initial random pulling of the individuals components, gives rises to a population including the existence of two different sets of individuals. The first one is constituted by individuals of weak performances while the other contains the individuals of higher performance.

The implementation of the aforementioned G.A. led to the optimal shape of Fig. 25 and satisfies all the constraints of the optimization problem. The distribution of the von Mises stresses on the boundary elements is quite homogeneous as shows in Fig. 25.

It will be noted that the average von Mises stress value is equal to σ_0 in all the finite elements of the free boundary.

Concerning the best individual obtained at the end of the first iteration, the evolution of the stress acting inside the element near the boundary is given in Fig. 26. Its inaccuracy compared to the initial solution comes mainly from the stress value in the second finite element which takes a value very far away from the admissible stress.

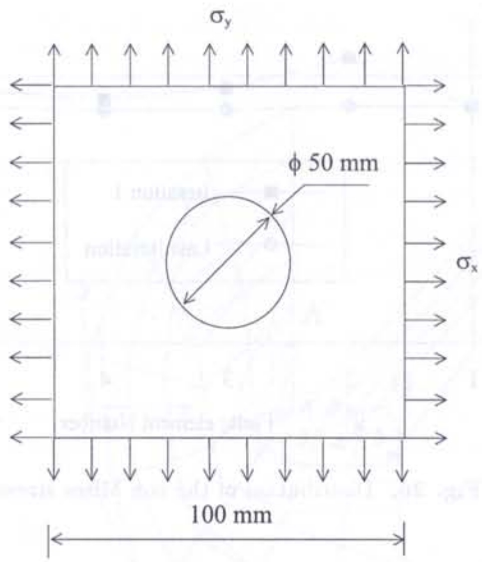

Fig. 24. Perforated plate in biaxial traction

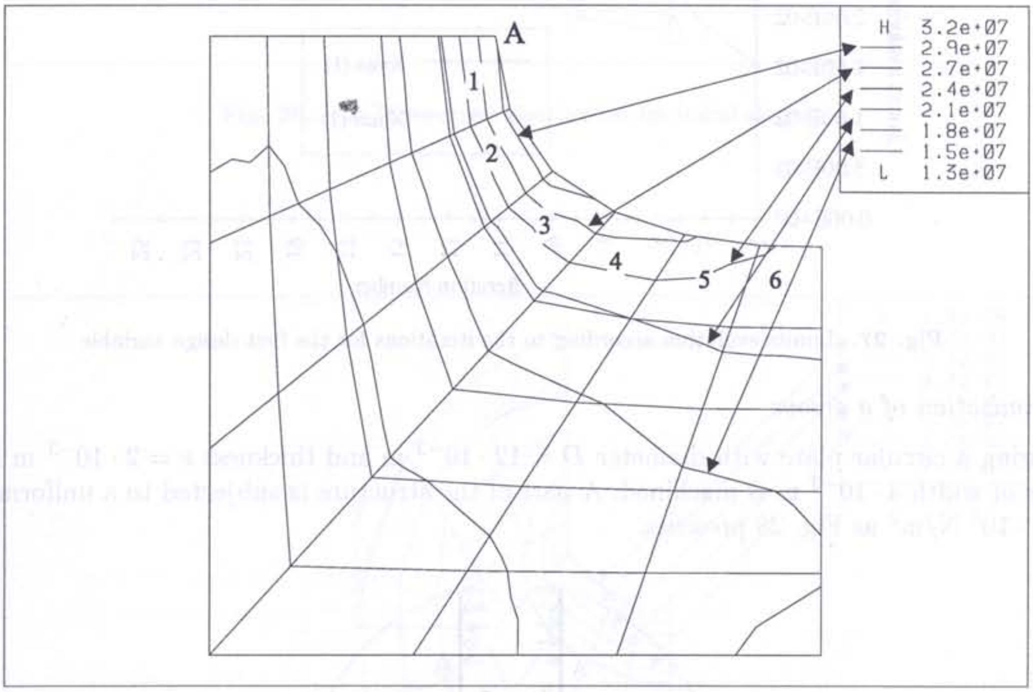

Fig. 25. Optimal shape of the structure

The algorithm convergence is enforced by the continuous evolution of the limits values of the design variables which decreases the research domain. This is illustrated below in the case of the first design variable, corresponding to the x-coordinate of node A located on the changing boundary. During the iterations, the upper and lower limits converges towards the same value restricting the feasible domain to a small one as it can be noted in Fig. 27.

The reliability of the G.A. procedure has been verified by comparing the optimal shape of the hole to solutions provided in the literature. In [5], the solution which has been found is quite identical to the one provided by the use of the proposed G.A.

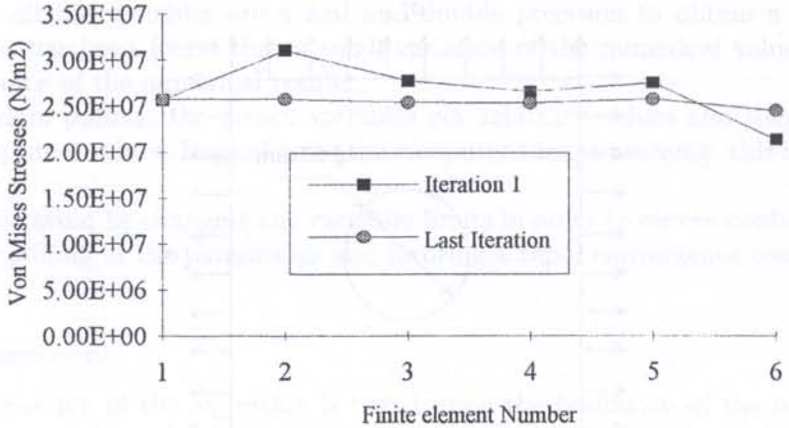

Fig. 26. Distribution of the von Mises stress

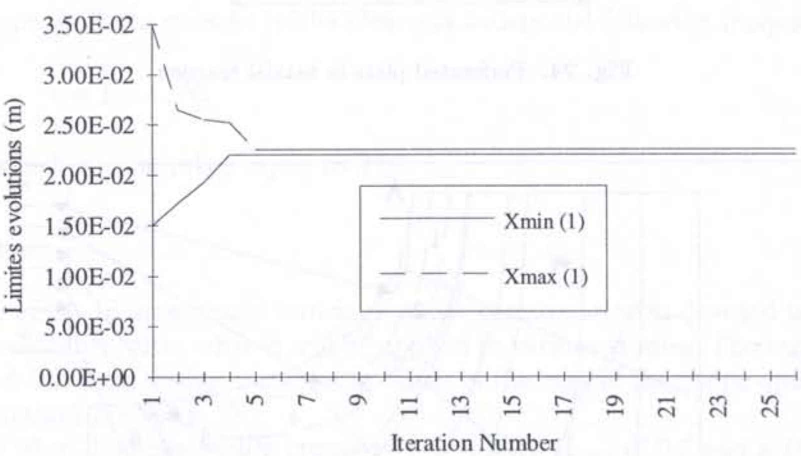

Fig. 27. Limits evolution according to the iterations for the first design variable

b – Optimization of a groove

Considering a circular plate with diameter $D = 12 \cdot 10^{-3}$ m and thickness $e = 2 \cdot 10^{-3}$ m in which a groove of width $4 \cdot 10^{-3}$ m is machined. A part of the structure is subjected to a uniform loading $\sigma_x = 15 \cdot 10^6$ N/m^2 as Fig. 28 presents.

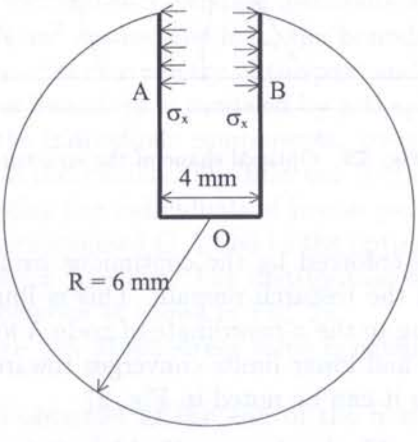

Fig. 28. Groove to be optimized

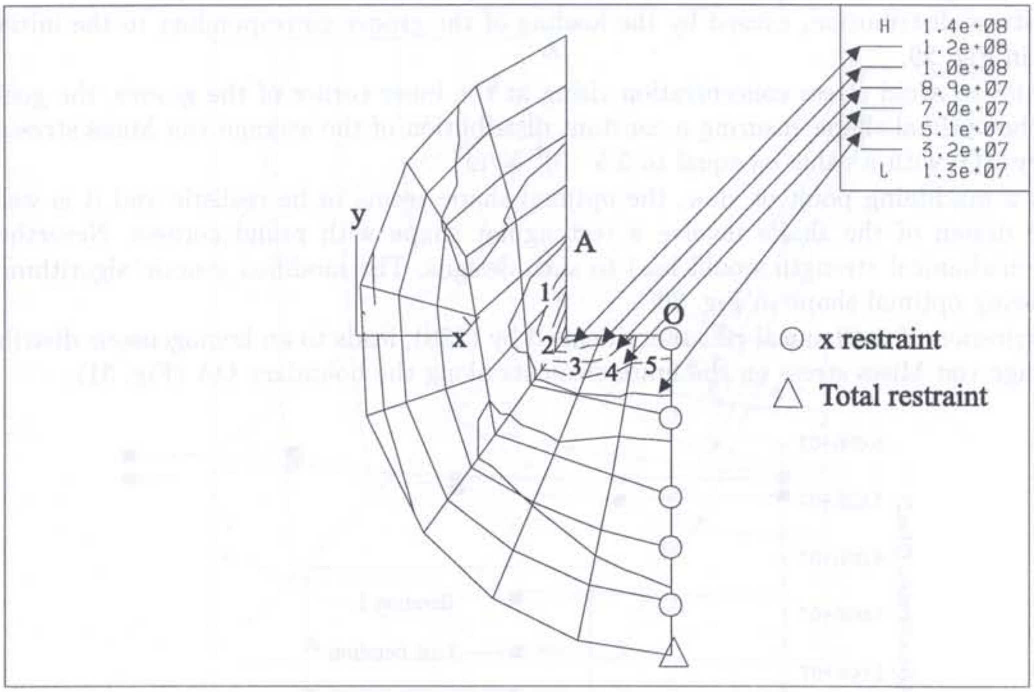

Fig. 29. von Mises stress distribution for initial shape

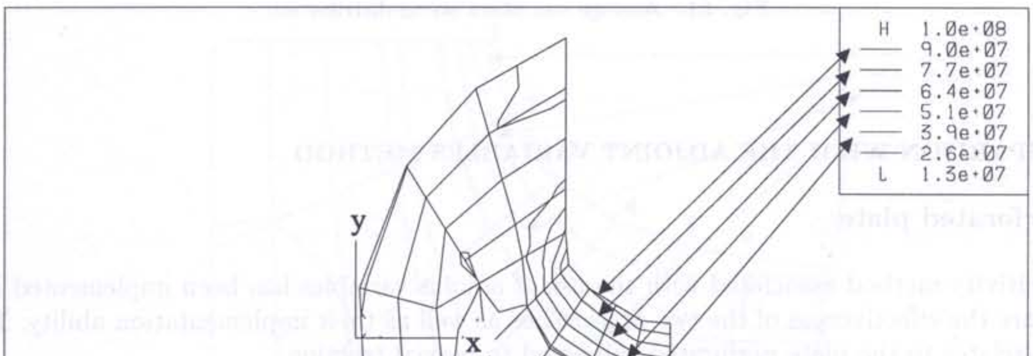

Fig. 30. Optimal shape of the groove ($\sigma_0 = 5.5 \cdot 10^7$ N/m^2)

The study is carried out in the case of plane stress state. Due to the geometric symmetry and applied loading, half of the part will be treated for optimization of the contour AOB.

The stress distribution, caused by the loading of the groove corresponding to the initial shape, is given in Fig. 29.

In order to avoid stress concentration rising at the inner corner of the groove, the goal will be to find the optimal shape ensuring a constant distribution of the average von Mises stresses at the boundary AO, with a value σ_0 equal to $5.5 \cdot 10^7$ N/m^2.

From a machining point of view, the optimal shape seems to be realistic and it is well known that the design of the shafts involve a rectangular shape with round corners. Nevertheless the optimal mechanical strength would lead to such designs. The modified genetic algorithm leads to the following optimal shape in Fig. 30.

The criterion of mechanical resistance defined by (23b), leads to an homogeneous distribution of the average von Mises stress on the finite elements along the boundary OA (Fig. 31).

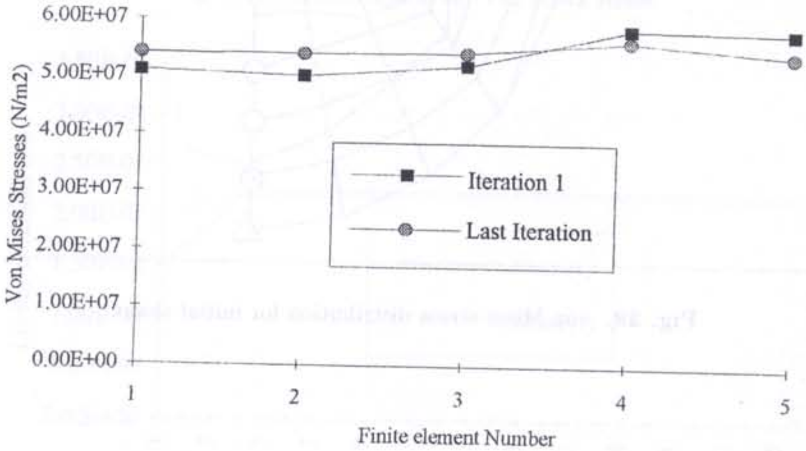

Fig. 31. Average von Mises stress distribution

8. Comparison with the adjoint variables method

8.1. Perforated plate

The sensitivity method associated with the use of adjoint variables has been implemented in order to compare the effectiveness of the two approaches as well as their implementation ability. The first example relates to the plate perforated subjected to biaxial traction.

The boundary is modeled by a B-spline curve which poles of are initially distributed on a circle with a diameter of 0.05 m. The distribution of the von Mises stress in the initial shape is recalled in Fig. 32.

To avoid the stress concentration on the boundary of the hole on B, the optimal shape must ensure a constant distribution of the average von Mises stress inside the finite elements adjacent to the boundary AB.

For the optimal shape corresponding to the last iteration in the optimization algorithm, in the central zone of each finite element the average stresses don't never exceed the value of $2.5 \cdot 10^7$ N/m^2 which was fixed in this study. This optimal shape is shown in Fig. 33 and it has to be compared with the solution given in Section 7.2.2 by the use of G.A.

In addition, the effectiveness of the shape modification can be judged by comparing the values of the average von Mises stress along the finite elements of the free boundary in the cases of the initial and optimized part (Fig. 34).

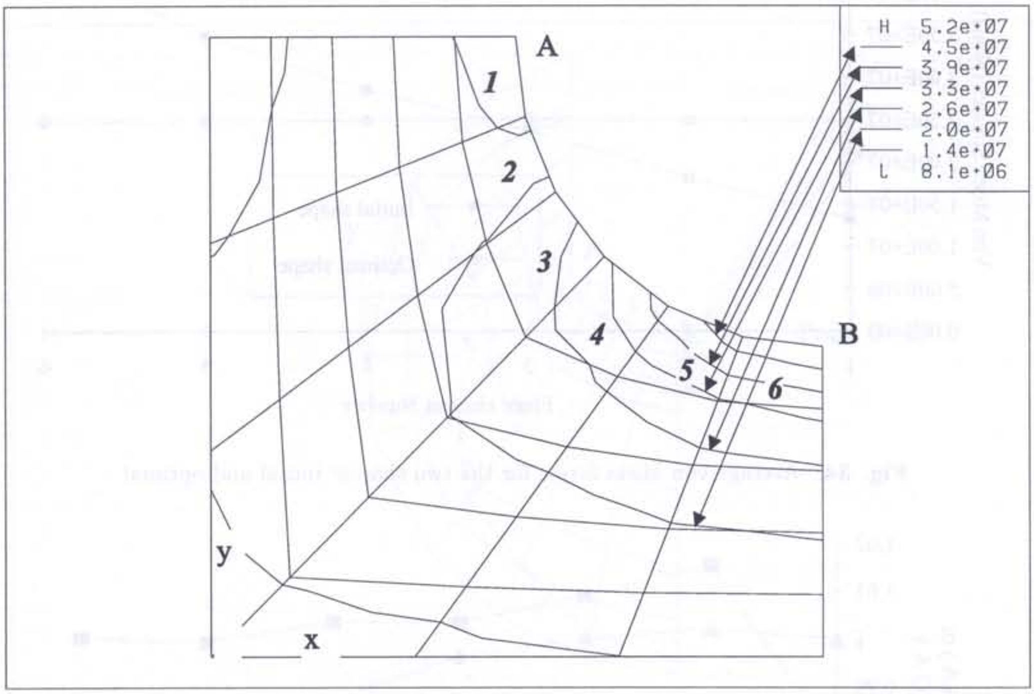

Fig. 32. Isovalues of the von Mises stress for the initial shape

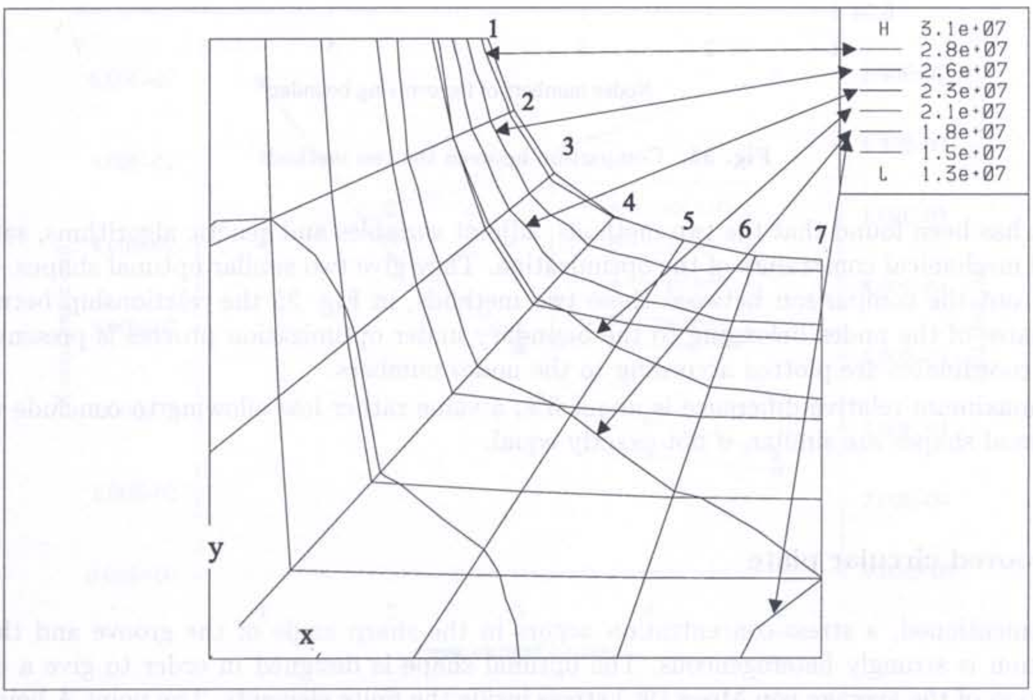

Fig. 33. Isovalues of the von Mises stress for the optimal shape

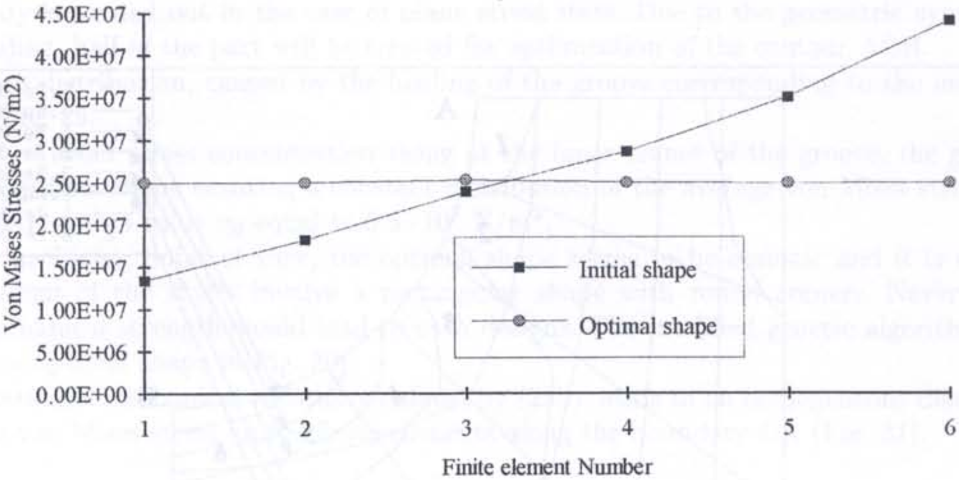

Fig. 34. Average von Mises stress for the two shapes: initial and optimal

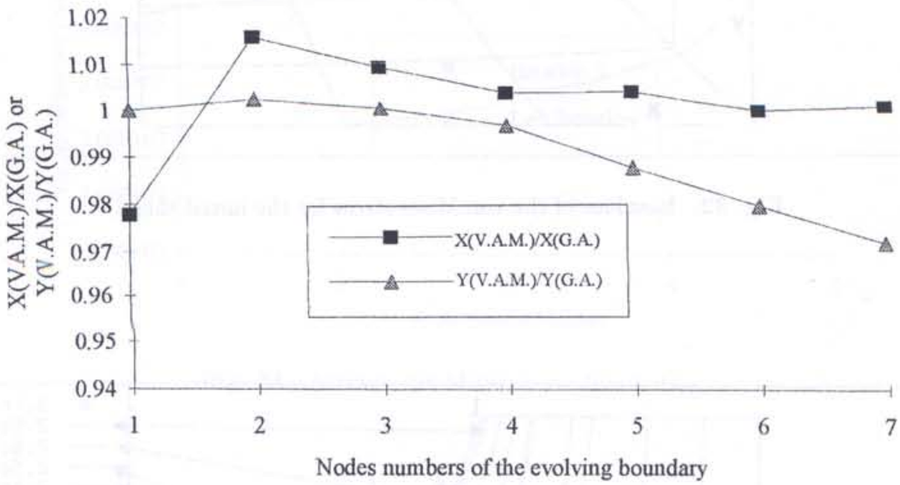

Fig. 35. Comparison between the two methods

So, it has been found that the two methods, adjoint variables and genetic algorithms, satisfy the imposed mechanical constraints of the optimization. They give two similar optimal shapes. In order to carry out the comparison between these two methods, in Fig. 35 the relationship between the co-ordinates of the nodes belonging to the boundary under optimization process is presented. The ratio of coordinates are plotted according to the nodes numbers.

The maximum relative difference is about 3%, a value rather low allowing to conclude that the two optimal shapes are similar, if not exactly equal.

8.2. Grooved circular plate

As aforementioned, a stress-concentration occurs in the sharp angle of the groove and the stress distribution is strongly heterogeneous. The optimal shape is designed in order to give a constant distribution of the average von Mises (Ψ_i) stress inside the finite elements. The point A being fixed, the A.V.M. algorithm gives the following final shape of Fig. 36.

In Fig. 36, the isovalues of the von Mises stress show a certain uniformity on the optimal boundary due to the good reliability of the algorithm.

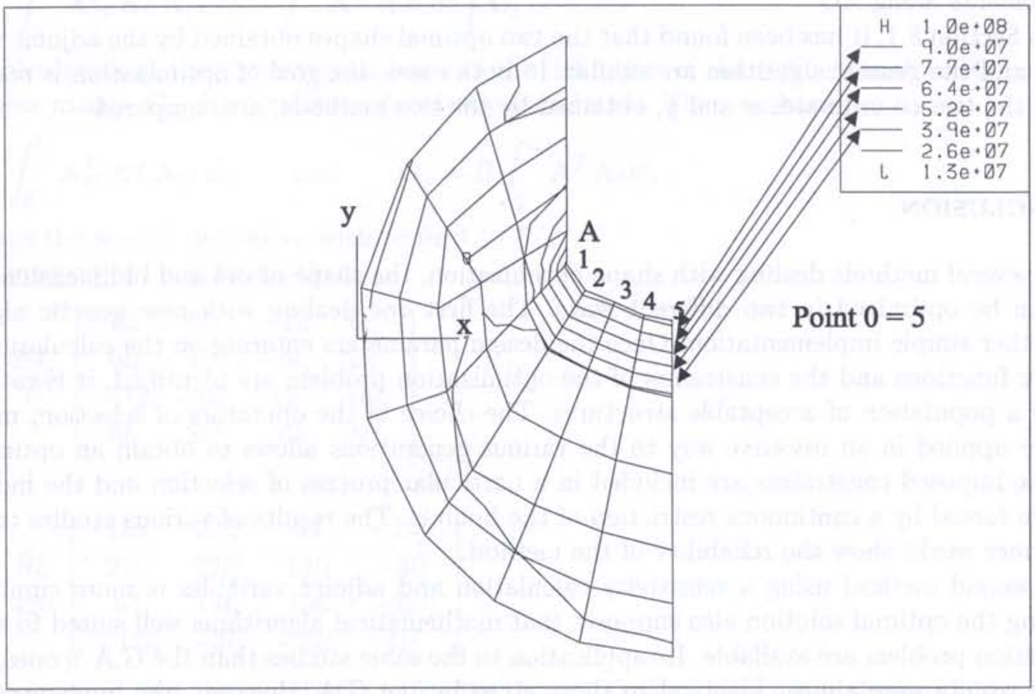

Fig. 36. The optimal shape of the groove when $\sigma_0 = 5.5 \cdot 10^7$ Pa

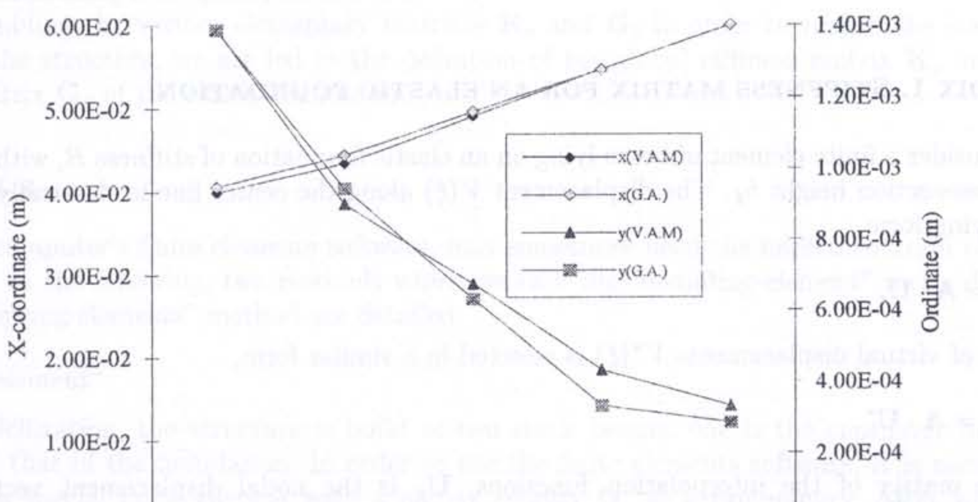

Fig. 37. Comparison between A.V.M. and G.A.

The bottom of the groove is very affected by the shape modification and the sharp angle is replaced by a round-off. The required value $\sigma_0 = 5.5 \cdot 10^7$ N/m^2 is obtained in the center of the finite elements along AO.

As in Section 8.1, it has been found that the two optimal shapes obtained by the adjoint variables method and the genetic algorithm are similar. In both cases, the goal of optimization is reached. In Fig. 37, the two co-ordinates x and y, obtained by the two methods, are compared.

9. Conclusion

Among several methods dealing with shape optimization, the shape of uni and bidimensional structures can be optimized in two different ways. The first one dealing with new genetic algorithm, has a rather simple implementation. Once the design parameters entering on the calculation of the objective functions and the constraints of the optimization problem are identified, it is rather easy to build a population of acceptable structures. The choice of the operators of selection, mutation, crossover applied in an iterative way to the various generations allows to obtain an optimal solution. The imposed constraints are included in a particular process of selection and the individuals evolution forced by a continuous restriction of the bounds. The results of various studies compared with former works show the reliability of the method.

The second method using a sensitivity calculation and adjoint variables is more cumbersome. Obtaining the optimal solution also supposes that mathematical algorithms well suited to solve the optimization problem are available. Its application to the same studies than the G.A.'s ones, showed that the results were almost identical to those given by the G.A. However, the implementation is more complex and the mechanical and mathematical developments more complicated especially in the bidimensional case.

Thus at least two possibilities are offered to the designer to find a solution to the problems of parts-shape-optimization, allowing for a good mechanical strength, or for a weight reduction and manufacturing costs.

In conclusion of this work, it appears that the G.A.s' method for the shape optimization of uni and bidimensional structures, is a reliable tool with a relative simplicity of implementation.

Appendix 1. Stiffness matrix for an elastic foundation

Let us consider a finite element of beam lying on an elastic foundation of stiffness R, with a length l_i and a cross-section height h_i. The displacement $V(\xi)$ along the center line is classically written in the following form,

$$V(\xi) = \mathbf{A} \cdot \mathbf{U}_e . \tag{A1}$$

The field of virtual displacements $V^*(\xi)$ is selected in a similar form,

$$V^*(\xi) = \mathbf{A} \cdot \mathbf{U}_e^* . \tag{A2}$$

$\mathbf{A}$ is the matrix of the interpolation functions, $\mathbf{U}_e$ is the nodal displacement vector, $\mathbf{U}_e = \{V_1\ \theta_1\ V_2\ \theta_2\}^T$ and $\mathbf{U}_e^*$ – the vector of the virtual displacements of nodes, $\mathbf{U}_e^* = \{V_1^*\ \theta_1^*\ V_2^*\ \theta_2^*\}^T$ where V_1, V_2 and θ_1, θ_2 are respectively the displacements and rotations associated with each of the two nodes of the element.

The virtual work done by the internal forces and the distributed forces involved by the elastic foundation is written as

$$\int_0^{l_i} V_{\xi^2}^{*T}\, EI\, V_{\xi^2}\, \mathrm{d}\xi + \int_0^{l_i} V^{*T}\, R\, V\, d\xi. \tag{A3}$$

Substituting $V(\xi)$ and $V^*(\xi)$ by their expressions (A1) and (A2), (A3) reads

$$\mathbf{U}_e^{*T}\left\{\int_0^{l_i}\mathbf{A}_{\xi^2}^T\,EI\,\mathbf{A}_{\xi^2}\,\mathrm{d}\xi+\int_0^{l_i}\mathbf{A}^T\,R\,\mathbf{A}\,d\xi\right\}\mathbf{U}_e\,. \tag{A4}$$

The calculation of the two integrals leads to the definition of the stiffness matrix $\mathbf{K}_e$ as well as the stiffness matrix $\mathbf{G}_e$ characterizing the elastic foundation. Then we have

$$\mathbf{K}_e=\int_0^{l_i}\mathbf{A}_{\xi^2}^T\,EI\,\mathbf{A}_{\xi^2}\,dx\qquad\text{and}\qquad\mathbf{G}_e=R\int_0^{l_i}\mathbf{A}^T\mathbf{A}\,\mathrm{d}\xi. \tag{A5}$$

$(\cdot)_{\xi^2}$ means the second derivative with respect to ξ.

After integration, it is found that

$$\mathbf{K}_e=\frac{EI_z}{l_i^3}\begin{bmatrix}12 & 6l_i & -12 & 6l_i\\ 6l_i & 4l_i^2 & -6l_i & 2l_i^2\\ -12 & -6l_i & 12 & -6l_i\\ 6l_i & 2l_i^2 & -6l_i & 4l_i^2\end{bmatrix} \tag{A6}$$

and

$$\mathbf{G}_e=\frac{Rl_i}{420}\begin{bmatrix}156 & 22l_i & 54 & -13l_i\\ 22l_i & 22l_i^2 & 13l_i & -3l_i^2\\ 54 & 13l_i & 156 & -22l_i\\ -13l_i & -3l_i^2 & -22l_i & 4l_i^2\end{bmatrix}. \tag{A7}$$

It can be observed that the matrix $\mathbf{G}_e$ looks like the mass matrix of a beam element which is given by

$$\mathbf{M}_e=\frac{\rho h_i l_i}{420}\begin{bmatrix}156 & 22l_i & 54 & -13l_i\\ 22l_i & 22l_i^2 & 13l_i & -3l_i^2\\ 54 & 13l_i & 156 & -22l_i\\ -13l_i & -3l_i^2 & -22l_i & 4l_i^2\end{bmatrix}. \tag{A8}$$

ρ is the mass per unit volume of the material.

By assembling the various elementary matrices $\mathbf{K}_e$ and $\mathbf{G}_e$ in order to realize the static equilibrium of the structure, we are led to the definition of the global stiffness matrix $\mathbf{K}_g$ and global stiffness matrix $\mathbf{G}_g$ of the elastic foundation.

APPENDIX 2

The use of computer's finite elements software, may sometimes need the implementation of specific procedures. In the following, two methods which we call: the "bounding-element" method and the boundary "spring-elements" method are detailed.

"Bounding-element"

In this modellization, the structure is build of two stuck beams: one is the cantilever beam and the other is that of the foundation. In order to use the finite elements software, it is necessary to implement a special finite element with a matrix rigidity $\mathbf{G}_e$ as a preliminary. After what, it is attached on the finite elements of beams, as suggested by Fig. A1.

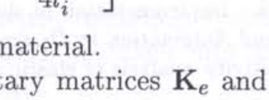

Fig. A1. Bounding-element

Boundary "Spring-Elements"

The second method consists in a modellization of the elastic foundation by boundary "spring-elements" available in Cadsap [6] according to Fig. A2.

For a finite element, the total stiffness of the foundation is distributed half-part on each node.

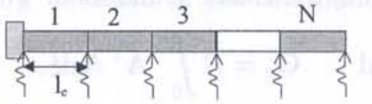

Fig. A2. Boundary "Spring-Elements"

REFERENCES

[1] V. Anand. *Computer Graphics and Geometric Modeling for Engineers.* John Wiley and Sons, 1993.

[2] H. Azegami. A proposal of a shape-optimization method using a constitutive equation of growth (in the case of a static elastic body). *JSME International Journal Series I*, **33**(1): 64–71, 1991.

[3] A.E. Baumal, J.J. Mcphee, P.H. Calamai. *Genetic Algorithm Optimization for active vehicule suspension design*, pp. 455–464. The Canadian Society for Mechanical Engineering CSME, Canada, 1996.

[4] V. Braibant. *Optimisation de Forme des Structures en vue de la Conception Assistée par Ordinateur.* Ph.D. thesis, 1986.

[5] V. Braibant, C. Fleury. Shape Optimal Design using B-spline. *Computer Methods in Applied Mechanics and Engineering*, **44**: 247–267, 1984.

[6] Cadsap, Algor, Inc., 150 Beta Drive Pittsburgh. PA 15238-2932 USA. Distributed by Cadlm, 9, rue Raoul Dautry, 91190 Gif sur Yvette, France.

[7] C. Caux, H. Pierreval, M.C. Portmann. Les algorithmes génétiques et leur application aux problèmes d'ordonnancement. *RAIRO AP II*, 1–33, 1994.

[8] J.L. Chen, W.C. Tsai. Shape optimization by using simulated biological growth approaches. *AIAA Journal*, **31**(11): 2143–2147, 1993.

[9] K.K. Choï, J.L.T. Santos, M.C. Frederick. Implementation of design sensitivity analysis with finite element codes. *J. of Mechanisms Transmissions and Automation in Design*, **109**: 385–391, 1987.

[10] K.K. Choï, E.J. Haug. Shape design sensitivity analysis of elastic structures. *Journal of Structural Mechanics*, **11**(2): 231–269, 1983.

[11] D.E. Goldberg, M. Asce, M.P. Samtani. Engineering optimization via genetic algorithm. *Procceeding Ninth Conference on Electronic Computation*, 471–482. ASCE, New York, 1986.

[12] J. Haug, K. Choï, V. Komkov. *Design Sensitivity of Structural Systems*, Mathematics in science and engineering, v. 177. Academic Press, New York, 1986.

[13] R. Le Riche, C. Knopf-Lenoir, R. Haftka. Optimisation de Structures Composites par Algorithmes Génétiques. *Colloque PRIMECA sur la conception optimale*, 4 juillet, 31–37, 1995.

[14] J.-L. Marcelin, A. Kallassy. Vers une optimisation intégrée des structures ou des systèmes mécaniques. *Revue de CFAO et d'information graphique*, **11**: 289–306, 1996.

[15] Z. Michalewicz. *Genetic Algorithm + Data Structure = Evolution Programs.* Springer-Verlag, 1992.

[16] J.-M. Renders. *Algorithmes génétiques et réseaux de neurones.* Hermes, 1995.

[17] S. Saigal, S. Mukherjee. *Sensitivity analysis and optimization with numerical methods.* The American Society of Mechanical Engineers, AMD-Vol. 115, USA, 1990.

[18] E. Sandgren, E. Jensen, J.W. Welton. Topological design of structural components using genetic optimization methods. In: S. Saigal, S. Mukherjee (eds.), *Sensitivity Analysis and Optimization with Numerical Methods.* The American Society of Mechanical Engineers, AMD-Vol. 115, 31–43, USA, 1990.

[19] J. Sokolowski, J.P. Zolezio. *Introduction to Shape Optimization.* Springer-Verlag, 1992.

[20] P. Trompette, C. Fleury, C. Knopf-Lenoir. Optimisation des structures. *Approche de l'ingénieur IPSI*, Paris 9–11 septembre, 1987.

[21] G.N. Vanderplaats. *Numerical optimization techniques for engineering design with applications.* Mc Graw–Hill, 1984.

[22] G.N. Vanderplaats. *ADS – A Fortran Program for Automated Design Synthesis. Version 3.1.* Engineering Design Optimization, INC., 1987.

[23] M. Younes. *Shape Optimization of Bidimensional Parts by means of Adjoint Variables Methods and Genetic Algorithms* (in French). Ph.D. thesis, Ecole Centrale de Nantes-France, 1997.