Computer Assisted Mechanics and Engineering Sciences, 8: 397–407, 2001.

Comparison of Galerkin and collocation Trefftz formulations for plane elasticity

Vitor M.A. Leitão
Departamento de Engenharia Civil, Instituto Superior Técnico
Av. Rovisco Pais, 1049-001 Lisboa, Portugal

(Received June 30, 2000)

The purpose of this work is to compare and assess, more in terms of computational efficiency than in terms of accuracy, three alternative implementations of a boundary formulation based on the Trefftz method for linear elastostatics, namely a collocation-based and two Galerkin-based approaches. A finite element approach is used in the derivation of the formulation for the Galerkin-based alternative implementations. The coefficients of the structural matrices and vectors are defined either by regular boundary integral expressions or determined by direct collocation of the trial functions. Numerical tests are performed to assess the relative performance of the different alternative implementations.

1. Introduction

The basic feature of all Trefftz-based formulations is the use of solutions of the governing differential equation as trial functions anywhere in the domain and on the boundary, [1, 9]. As these solutions satisfy locally all the domain conditions, the role of the resulting discrete model is to approximate the boundary conditions of the problem. This is a feature shared with all boundary integral formulations. The formulations known as boundary element and boundary integral equation methods rely in the use of fundamental solutions which are singular by nature. This singular nature may cause numerical difficulties for the procedures usually found in these types of boundary methods, namely, the direct and the indirect approaches. In the direct approach sources are placed on the boundary which, due to the singular nature of the trial functions used, leads to singular and/or hypersingular boundary integrals. The indirect approach starts by defining a fictitious boundary (enclosing the actual one) for placing the sources thus avoiding the singular integrals although at the cost of needing a supporting criterion to decide on the optimal distance, which has a direct influence on the condition number of the solving system.

By contrast, Trefftz methods utilise, in general, regular solutions of the governing equations. Briefly, it can be said that the classical Trefftz method [14] consists in the superposition of a certain number of solutions of the governing differential equation, appropriately scaled by some unknown parameters. These unknown parameters (which define the contribution or "weight" of each function to the global solution) are then determined from the approximate enforcement of the boundary conditions.

The boundary conditions can be implemented using different methods, namely by the least-squares method, the Galerkin and the collocation methods. Depending on the technique being used, these alternative approaches may produce or not symmetric and sparse solving systems. In the collocation method (**CO**) the boundary conditions are enforced locally at a certain number of points. The major feature of this approach is that there are no integrations to carry out. However, it leads to non-symmetric and, in general, over-determined, systems of equations [10–12].

On the other hand, the Galerkin approach requires boundary integrations but leads naturally to symmetric and sparse systems of equations. The boundary conditions are enforced on average, in a weighed-residual form [2, 7, 8]. If the (static) boundary conditions are enforced at selected boundary points, by collocation, symmetry is preserved while reducing the computational cost of boundary integration [6].

Although the Trefftz approximation is a boundary solution method by nature, it is important to stress and emphasise the similarities with domain methods, namely, the finite element method. The main advantage of implementing the Trefftz method from a finite element standpoint is the possibility of combining the main features of the competing boundary element and finite element methods. The approximation bases are regular and the solving system is symmetric and sparse, like in the conforming finite element method but all structural matrices present boundary integral expressions, as in the conventional boundary element method.

The implementation of the Trefftz method within a hybrid finite element framework was first suggested by J. Jirousek, who has applied it to a variety of problems in structural mechanics. Basically, depending on the constraints placed on the domain approximation bases it is possible to define hybrid-mixed (no constraints), hybrid (either equilibrium or compatibility) and hybrid-Trefftz formulations (all domain conditions).

In this work, a hybrid-Trefftz finite element approach is used in the derivation of the formulation for the Galerkin-based alternative implementations. The theoretical framework for the hybrid-Trefftz model used here is presented in [2]. This model is derived directly from the fundamental equations governing the problem under analysis and mathematical programming is used to establish the associated variational statements and the conditions for the existence, uniqueness and multiplicity of the finite element solutions.

The domain is subdivided in regions or elements, which need not be bounded, simply connected or convex. The stress field is directly approximated in each element using a complete solution set of the governing Beltrami condition. This stress basis is used to enforce on average, in the Galerkin sense, the compatibility and elasticity conditions. The boundary of each element is, in turn, subdivided into boundary elements whereon the displacements are independently approximated. The same basis is used to enforce the static admissibility conditions. In general, these conditions are enforced, on average, in the Galerkin sense (**FG** or Full Galerkin). When Dirac functions are used for the displacement approximation, the static admissibility conditions, which reduce to the Neumann conditions as the stress approximation satisfies locally the domain equilibrium condition, are enforced by collocation thus minimising numerical integrations (**GC** or Galerkin Collocation) [6]. The resulting solving system is symmetric and sparse. The coefficients of the structural matrices and vectors are defined either by regular boundary integral expressions or determined by direct collocation of the trial functions.

The following sections will describe, in a concise manner, the basic aspects of the three alternative Trefftz formulations under analysis. Details can be found in [3–6, 11, 12]. Numerical tests are then performed to assess the relative performance, especially in terms of computational efficiency, of the different alternative implementations.

2. Fundamental relations

The system of equations governing the linear response of an elastic (linear, laminar or solid) body with domain V and boundary Γ, assigned to a Cartesian system of reference, can be stated as follows,

$$\mathbf{D}\,\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \qquad \text{in } V, \tag{1}$$

$$\boldsymbol{\varepsilon} = \mathbf{D}^*\,\mathbf{u} \qquad \text{in } V, \tag{2}$$

$$\boldsymbol{\varepsilon} = \mathbf{f}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) + \boldsymbol{\varepsilon}_0\,, \qquad \text{in } V \tag{3}$$

$$\mathbf{N}\,\boldsymbol{\sigma} = \mathbf{t}_\Gamma \qquad \text{on } \Gamma_\sigma, \tag{4}$$

$$\mathbf{u} = \mathbf{u}_\Gamma \qquad \text{on } \Gamma_u\,. \tag{5}$$

In the equilibrium and compatibility conditions (1) and (2), vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ list the independent components of the (generalised) stress and strain tensors and vectors $\mathbf{b}$ and $\mathbf{u}$ collect the (generalised) body-forces and displacements, respectively. The differential equilibrium and compatibility operators $\mathbf{D}$ and $\mathbf{D}^*$ are linear and adjoint in the context of geometrically linear analysis.

In description (3) of the constitutive relations, the (symmetric) flexibility matrix $\mathbf{f}$ collects the relevant elastic constants and the residual states are represented either by the (generalised) stresses or strains listed in vectors $\boldsymbol{\sigma}_0$ and $\boldsymbol{\varepsilon}_0$, respectively.

Appropriate combination of the domain conditions (1) to (3) yields the Navier description (6) for the differential system of equations governing the response on the domain under analysis, where $\mathbf{k} = \mathbf{f}^{-1}$ is the local stiffness matrix

$$\mathbf{D}\,\mathbf{k}\,\mathbf{D}^*\,\mathbf{u} + \mathbf{D}\,(\boldsymbol{\sigma}_0 - k\,\boldsymbol{\varepsilon}_0) + \mathbf{b} = \mathbf{0} \qquad \text{in } V. \tag{6}$$

The boundary conditions (4) and (5) apply to the entire limiting surface Γ of the domain V under analysis: $\Gamma = \Gamma_\sigma \cup \Gamma_u$ and $\emptyset = \Gamma_\sigma \cap \Gamma_u$. In the Neumann boundary condition (4), matrix $\mathbf{N}$ collects the components of the unit outward normal vector associated with the entries of the differential equilibrium matrix $\mathbf{D}$ and $\mathbf{t}_\Gamma$ is the (generalised) stress vector prescribed on portion Γ_σ of the enveloping surface. In the Dirichlet boundary condition (5), vector $\mathbf{u}_\Gamma$ defines the (generalised) displacements prescribed on the complementary domain Γ_u. The notation used above can accommodate mixed boundary conditions.

3. Discretization

Assume that domain V is discretized into elementary *regions* V^e and that its boundary, Γ^e, is in turn discretized into *sides*.

Two complementary regions, the static (Neumann) and kinematic (Dirichlet) boundaries Γ^e_σ and Γ^e_u, respectively, are distinguished on the boundary of a typical region ($\Gamma^e = \Gamma^e_\sigma \cup \Gamma^e_u$ and $\emptyset = \Gamma^e_\sigma \cap \Gamma^e_u$). For a *stress model*, the kinematic boundary, Γ^e_u, is the region of its boundary whereon the displacements are prescribed, $\Gamma^e_u = \Gamma^e \cap \Gamma_u$, and the static boundary, $\Gamma^e_\sigma = \Gamma^e_i \cup (\Gamma^e \cap \Gamma_\sigma)$, is the complementary region of the boundary where the displacements are not known. Hence, the static boundary of a stress element combines the inter-region boundaries with the static boundary of the mesh that region e may contain.

The Trefftz approximation is, usually, of very high quality even when coarse meshes are used. As it is shown below, the domain approximation basis is polynomial and described by generalised (non-nodal) variables. This allows for a very loose definition of the geometry of the regions, which may not be bounded, simply connected or convex.

The geometry of the mesh is defined by the co-ordinates of the nodes strictly necessary to describe a given shape in parametric form, typically *n+1* nodes for a polynomial line of degree n in two-dimensional problems. The topology of the mesh is defined by assigning sides to the nodes and each region is identified by its connecting sides.

4. Indirect collocation — approximation

To solve the problem above represented by the Navier equation (6), the indirect collocation is, probably, the simplest technique. All that it requires is a complete set of homogeneous solutions plus a particular solution. It is possible to derive a complete system of functions for a bounded, simply connected, two-dimensional region using, for example, the complex representation of Muskhelishvili [13] (of biharmonic functions):

- for the displacement components in the complex plane, u and v, respectively,

$$2\mu(u + iv) = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \tag{7}$$

• for the projections of the stresses on the outward normal and on the tangent (n and t):

$$t_n - it_t = \varphi'(z) + \overline{\varphi'(z)} - e^{2i\alpha}\{\bar{z}\varphi''(z) + \psi'(z)\}, \tag{8}$$

where

$z = x + iy$ is the position vector in the complex plane of a boundary point with respect to some fixed position which is normally taken as the centroid of the region being analysed;

μ is the shear modulus;

$\kappa = 3 - 4\nu$ or $[(3-\nu)/(1+\nu)]$ for plane strain and plane stress, respectively;

ν is the Poisson ratio;

α is the angle between the normal and the x axis;

and the functions $\varphi(z)$ and $\psi(z)$ are arbitrary analytic functions. For convenience the following set is used,

$$\varphi(z) = iz^k \quad \text{and} \quad \psi(z) = 0; \qquad \varphi(z) = z^k \quad \text{and} \quad \psi(z) = 0;$$

$$\varphi(z) = 0 \quad \text{and} \quad \psi(z) = iz^k; \qquad \varphi(z) = 0 \quad \text{and} \quad \psi(z) = z^k,$$

where k is an integer.

When a sufficient number of these solutions is used, an approximation to a general plane elasticity problem in a simply connected region may be found by matching the boundary conditions in an appropriate manner.

Solutions exist as well for multiply connected regions and for regions with cuts or cracks [12]. If an analysis is required of a multiply connected domain then it is probably a better approach to discretize the domain with the use of more than one simply connected regions. It is possible, in this way, to decrease the possibility of numerical ill-conditioning which may arise when the degree of approximation required (which depends on the geometry of the domains under analysis) gets too high.

The use of more than one region is, therefore, recommended and it does not cause any implementation or numerical inconvenience. All it requires is a proper definition of the interface continuity conditions, that is, equilibrium and compatibility.

This is represented below for an interface between regions a and b,

$${}^a u(z_i) = {}^b u(z_i); \qquad {}^a v(z_i) = {}^b v(z_i); \qquad {}^a t_n(z_i) = {}^b t_n(z_i); \qquad {}^a t_t(z_i) = -{}^b t_t(z_i);$$

where u, v and t_n , t_t are, respectively, the displacement and the traction components at interface collocation point z_i .

5. Indirect collocation — solving system

A non-symmetric and, usually, overdetermined solving system is obtained by simply allocating at each collocation point two equations out of the set of four which is possible to establish, that is, two boundary displacements and two tractions given, respectively, by Eqs. (7) and (8).

The resulting system of equations is of the following form:

$$\begin{bmatrix} \Theta_d^0(z_1) & \Theta_d^1(z_1) \\ \Theta_d^0(z_2) & \Theta_d^1(z_2) \\ 0 & \Theta_t^1(z_3) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \overline{u}(z_1) \\ \overline{u}(z_2) \\ \overline{t_n}(z_3) \end{bmatrix} \quad \Rightarrow \quad \Theta c = b. \tag{9}$$

In the expression above the first and second equations represent the enforcement of the u component of the displacement boundary condition at collocation point z_1 , and z_2 , respectively, while the third equation represents the enforcement of the normal component of the traction boundary condition at collocation point z_3. The matrix term $\Theta_d^0(z_1)$ represents the rigid body terms at the first collocation point and $\Theta_d^1(z_1)$ is the linear term of the boundary displacements description. All the other (higher order) terms would be expressed in a similar manner. The traction equation only differs from the displacement one in the sense that the tractions corresponding to constant (rigid body) displacements are, obviously, zero. The unknown coefficients c_0 and c_1 are obtained with the use of least-squares type solvers or other suitable techniques.

6. Galerkin — approximation

The stress model of the hybrid-Trefftz formulation is based on the direct approximation of the stresses in the domain of the region and of the displacement components on its Neumann boundary,

$$\boldsymbol{\sigma} = \mathbf{S}_V\,\mathbf{X} + \boldsymbol{\sigma}_P \qquad \text{in } V^e, \tag{10}$$

$$\mathbf{u} = \mathbf{U}_\Gamma \mathbf{q} \qquad \text{on } \Gamma_\sigma^e. \tag{11}$$

In the definitions above, matrices $\mathbf{S}_V$ and $\mathbf{U}_\Gamma$ collect stress and boundary displacement modes and the corresponding weighting vectors, $\mathbf{X}$ and $\mathbf{q}$, define *generalised stresses* and *generalised boundary displacements*, respectively. Vector σ_P is used to model particular solutions that will be, from now on, disregarded for the sake of simplicity.

Besides completeness and linear independence, no constraints are placed *a priori* on the selection of the boundary approximation basis $\mathbf{U}_\Gamma$. However, in the Trefftz method for elastostatics, the stress approximation basis $\mathbf{S}_V$ is constrained to be associated with the (elastic) displacement field $\mathbf{U}_V$ that solves locally the Navier equation (6), as given in Eq. (7).

The domain approximation (10) is strictly region dependent, as a set of weights $\mathbf{X}$ is associated to each region and different approximation bases $\mathbf{S}_V$ are implemented in each region. Approximation (11) is strictly side dependent, as a set of weights $\mathbf{q}$ is associated to each side of the region(s) and different approximation bases $\mathbf{U}_\Gamma$ can be implemented in each side.

The boundary approximation basis $\mathbf{U}_\Gamma$ is taken as a subset of linear independent terms of the mapping of the complementary solution basis $\mathbf{U}_V$ on each side of the region(s).

Depending on the type of boundary displacement approximation basis chosen, continuous (polynomial based) or "concentrated" (Dirac based), it is possible to obtain a continuous (Galerkin) representation or a boundary collocation representation.

The boundary collocation representation [6] simplifies the computation of the compatibility matrix and leads to a boundary displacement approximation matrix where sets of Dirac functions are placed at collocation points which are, in the tested implementation, interior points of the boundary thus leading to a discontinuous approximation.

7. Galerkin — elementary equations

The equations governing the discrete hybrid-Trefftz stress model summarised in Table 1 are derived from the dual transformations of the basic approximations (10) and (11), which define the following generalised strains, $\mathbf{e}$, and tractions, $\mathbf{Q}_\Gamma$,

$$\mathbf{e} = \int \mathbf{S}_V^t\,\boldsymbol{\varepsilon}\,\mathrm{d}V^e, \tag{12}$$

$$\mathbf{Q}_\Gamma = \int \mathbf{U}_\Gamma^t\,\mathbf{t}_\Gamma\,\mathrm{d}\Gamma_\sigma^e, \tag{13}$$

Table 1. Basic equations for the hybrid-Trefftz stress model

Equilibrium	$\mathbf{Q}_\Gamma = \mathbf{A}^t\mathbf{X},$	(14)
Compatibility	$\mathbf{e} = \mathbf{A}\,\mathbf{q},$	(15)
Elasticity	$\mathbf{e} = \mathbf{F}\,\mathbf{X}.$	(16)

The following definitions are found for the compatibility matrix and for the flexibility matrix,

$$\mathbf{A} = \int \mathbf{T}_V^t\,\mathbf{U}_\Gamma\,\mathrm{d}\Gamma_\sigma^e\,, \tag{17}$$

$$\mathbf{F} = \int \mathbf{S}_V^t\,\mathbf{f}\,\mathbf{S}_V\,\mathrm{d}V^e = \int \mathbf{T}_V^t\,\mathbf{U}_V\,\mathrm{d}\Gamma^e. \tag{18}$$

Following a straightforward Galerkin approach it can be easily confirmed that the equilibrium (14) and elasticity (16) equations represent the $\mathbf{U}_\Gamma$- and $\mathbf{S}_V$-weighed residual enforcement of the local Neumann and constitutive conditions (4) and (3), respectively, for the assumed stress field (10),

$$\int \mathbf{U}_\Gamma^t\,\{\mathbf{t} - \mathbf{t}_\Gamma\}\,\mathrm{d}\Gamma_\sigma^e = 0, \tag{19}$$

$$\int \mathbf{S}_V^t\{\boldsymbol{\varepsilon} - [\mathbf{f}(\boldsymbol{\sigma})]\}\,\mathrm{d}V^e = 0. \tag{20}$$

Similarly, the compatibility equation (14) can be interpreted as the $\mathbf{S}_V$-weighed residual enforcement of the compatibility condition (2), which is integrated by parts to incorporate the Dirichlet condition (5) and implement the assumed boundary displacement approximation (11),

$$\int \mathbf{S}_V^t\,\{\boldsymbol{\varepsilon} - \mathbf{D}^*\mathbf{u}\}\,\mathrm{d}V^e = 0. \tag{21}$$

8. Galerkin — solving system

The symmetric, boundary integral finite element solving system (22) is obtained combining Eqs. (14)–(16), to eliminate the generalised strains as explicit variables,

$$\begin{bmatrix} \mathbf{F} & -\mathbf{A} \\ \mathbf{A}^t & \mathbf{O} \end{bmatrix} \begin{Bmatrix} \mathbf{X} \\ \mathbf{q} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ -\mathbf{Q}_\Gamma \end{Bmatrix}. \tag{22}$$

The solving system for the assembled mesh presents the same structure of the elementary system (22). The assembly procedure is based on simple direct allocation instructions, instead of the double summation typical of the conventional (conforming) finite element formulation and can be performed either directly on the elementary system (22) or on the supporting equations given in Table 1.

To assemble the elasticity condition, it is sufficient to collect the elementary relations (16) for all building elements/regions. The resulting elasticity matrix is block-diagonal and symmetric and vector $\mathbf{X}$ lists the generalised stresses for all regions.

9. Numerical implementation

The most relevant aspects in the numerical implementation of both solving systems (9) and (22) are commented in [3–6, 11, 12]. One of the main aspects of any implementation of a numerical technique is the type of solver used.

For the Galerkin based approaches the sparsity of the system is fully exploited with the use of routine *MA47* of the Harwell Subroutine Library that is a direct solver.

The collocation-based approach uses routine *MA44AD*, a least squares solver based on the SVD algorithm, of the Harwell Subroutine Library. This solver does not exploit sparsity and is, therefore, much less suitable than the *MA47* solver for the case of multiple regions.

10. Numerical tests

In [3, 6, 12] tests are presented that illustrate the convergence patterns of the solutions obtained with the three alternative Trefftz models, their sensitivity to incompressibility and mesh distortion and their relative accuracy in the estimation of stresses. From those results it is possible to conclude that all the alternatives perform quite well in terms of accuracy. What remains to be done is the comparison of the approaches in terms of computational efficiency.

To assess the computational efficiency of the Trefftz models, the (total and partial) time taken to assemble and solve the system and the amount of required storage, are represented for the test cases considered. In this work two problems are analysed:

- a square plate for which an exact solution is available;
- an L-shaped plate discretized with multiple regions.

Square plate with exact solution

The first example analysed is that of a square plate of unit length discretized in a single region subjected to Neumann (on two parallel sides) and Dirichlet (on the other two sides) boundary conditions associated with the following cubic stress field,

$$\{\sigma_x\,,\sigma_y\,,\sigma_{xy}\} = \{-0.32y(3x^2+4y^2),\ +0.32y(18x^2-y^2),\ -0.96x(2x^2-y^2)\}.$$

The exact solution for the strain energy is $U_0 = 384(89+64\nu)/21875\,E$, where E is the Young modulus and ν the Poisson ratio, and is very accurately represented by each of the alternative Trefftz models described.

The results, see [3, 6], show that the exact solution is recovered when the appropriate degrees of freedom are accommodated by the stress and boundary displacement approximations, namely when the stress approximation includes the cubic term and the displacement approximation includes the quartic term.

To recover the exact solution is therefore necessary that the number of stress parameters (unknowns) be, at least, 15, corresponding to $3+4{*}n_\sigma$ with $n_\sigma = 3$. The number of unknown displacements should be, at least, 20, corresponding to 2 (sides) *2 (directions) $*n_q$ where $n_q = 5$ is the number of collocation points or the degree of the displacement approximation +1, respectively, for the Galerkin-collocation and the full Galerkin approaches.

The indirect collocation approach requires, at least, a total of $3+4{*}n_\sigma$ with $n_\sigma = 3$ unknowns for the traction equations while for the displacement equation an extra set of three rigid body displacements are also required.

For this approach the system of equations is, usually, overdetermined, that is, the number of equations is larger than the number of unknowns. In all tests carried out here, and based on past experience, the number of equations is taken to vary between 10 to 50% higher than that of unknowns.

Although exact values are recovered with a very low number of unknowns, a much higher number of unknowns was considered so that comparisons could be made.

The first test was then carried out assuming $n_\sigma = 26$ and $n_q = 15$ and the results were as in Table 2.

Table 2. Robinson's test; dimensions of the system and CPU time (in 1/100-th of second) on an IBM RISC workstation

	unknowns	equations	Galerkin based		Collocation	Solving
			create **F**, Eq. (22)	create **A**, Eq. (22)	create Θ, Eq. (9)	
FG	197	197	143	99	.	5
GC	197	197	127	2	.	5
CO	109	128	.	.	2	3

It can be observed from Table 2 that for the Galerkin based approaches most of the time is consumed in creating the system matrix (for convenience, the system matrix is decomposed into two parts corresponding to the creation of the flexibility matrix **F**, and the compatibility matrix **A**). Notice, in particular, the decrease in the time taken to create matrix **A** with the **GC** approach when compared to the **FG** approach due to the need, in the latter approach, to numerically integrate all the terms of the matrix.

The **CO** approach is, by far, the fastest alternative for the same degree of approximation because no integrations are required and the matrix is obtained by straightforward allocation. There is also a (not relevant) difference in the time taken to compute the **F** matrix for which the only explanation is the fact that different implementations (of the same formulation) were used.

The results above may be misleading as only one region is considered. As referred in the previous section, when multiple regions are considered the least-squares solver used for the **CO** approach performs less efficiently in comparison to the sparse linear solver used by the other approaches.

L-shaped plate

A second test with multiple regions was then carried out. Consider the L-shaped plate shown in Fig. 1 discretized into three square regions. The tangential displacement is allowed at the supported sides.

In previous works [3, 6, 12], this test has been used to assess the estimates produced by the different Trefftz models for the stress components at point $P(-a/10, +a/10)$ in the vicinity of the wedge singularity. The material constants used are $E = 1$ and $\nu = 0.1$.

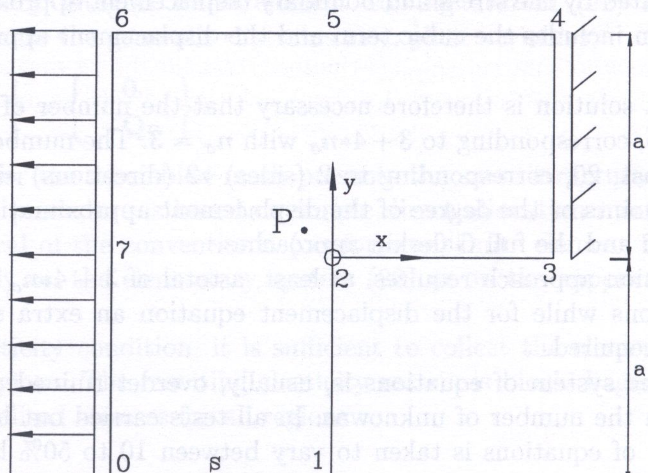

Fig. 1. L-shaped plate in tension

At this point it should be stressed that this test is, in fact, a difficult challenge for any formulation due to the singularity at the inner corner. From the results previously published [3, 6, 12], it is possible to conclude that all the different Trefftz models perform very well in particular when the approximation bases also include singular terms (this was not considered in the comparison presented here).

The non-dimensional traction distributions (the actual tractions divided by the applied load "t") shown in Figs. 2 and 3 are obtained with the Galerkin-collocation approach. Similar results are obtained with the other approaches. In this case the stress approximation was assumed to be of degree fifteen in each region, $n_\sigma = 15$, and the boundary displacement approximation was assumed to be of degree six, $n_q = 6$, yielding $N = 315$ degrees of freedom. The traction distributions are shown here to reproduce the very good agreement with the Neumann boundary conditions. The values 0–8 on the horizontal axis of Figs. 2 and 3 represent the "s/a" coordinate as represented in Fig. 1 where "s" is the length starting from the left bottom corner.

In Table 3 all the CPU times and dimensions of the systems of equations obtained with the different techniques are represented.

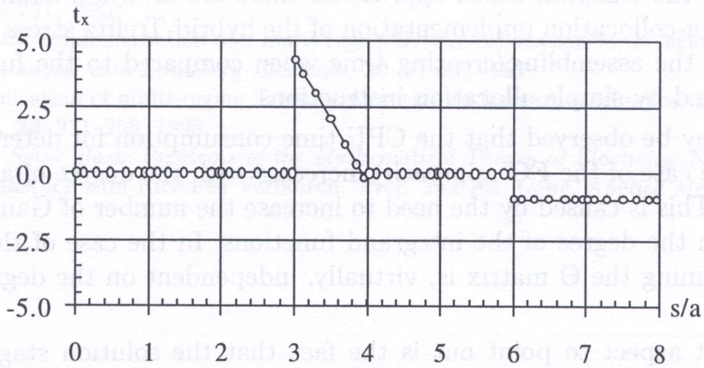

Fig. 2. Non-dimensional boundary traction t_x in the L-shaped plate

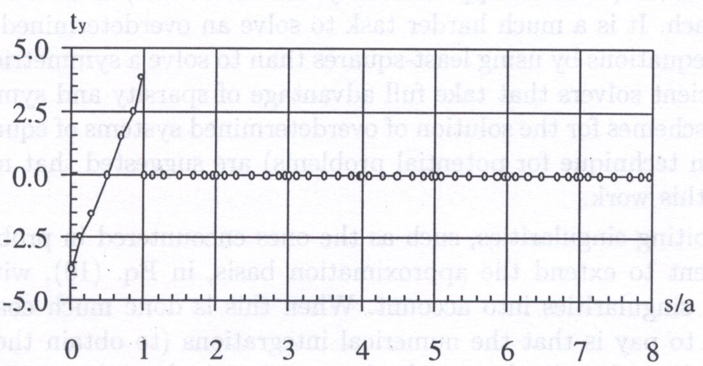

Fig. 3. Non-dimensional boundary traction t_y in the L-shaped plate

Table 3. L-shaped plate. Dimensions of the system and CPU time (in 1/100-th of second) on an IBM RISC workstation

	unknowns	equations	Galerkin based		Collocation	Solving
			create **F**, Eq. (22)	create **A**, Eq. (22)	create Θ, Eq. (9)	
FG	315	315	111	78	.	3
GC	315	315	91	3	.	3
CO	198	288	.	.	4	19

It can be observed from Table 3 that, again, for the Galerkin based approaches most of the time is consumed in creating the system matrix. As expected, the same decrease in the time taken to create matrix **A** with the **GC** approach when compared to the **FG** approach observed previously is seen for this test as well. The **CO** approach is still the fastest alternative for the same degree of approximation but, now that the system shows a high degree of sparsity, it is evident that there are advantages in the use of a solver that fully exploits sparsity such as the *MA47* solver.

11. Closure

The main conclusion that can be extracted from the work presented here is that computational efficiency is strongly dependent on the type of Trefftz model considered. Of the three alternative Trefftz models described here the collocation approach was, for the tests carried out, the most efficient one in terms of time taken to carry out the basic tasks of assembling and solving the system matrix. This is so due to the fact that, for the **CO** approach, there are no numerical integrations to carry out whereas for the Galerkin based approaches there are always a number of integrations to compute. The Galerkin-collocation implementation of the hybrid-Trefftz stress model, **GC**, achieves a strong reduction in the assembling/creating time when compared to the full Galerkin approach as matrix **A** is obtained by simple allocation instructions.

Furthermore, it may be observed that the CPU time consumption for determining the **F** matrix (and also for **A** in the case of the **FG** approach) increases with the degree of approximation (of the stresses) considered. This is caused by the need to increase the number of Gauss points proportionally to the increase in the degree of the integrand functions. In the case of the **CO** approach, the CPU time for determining the Θ matrix is, virtually, independent on the degree of approximation considered.

Another important aspect to point out is the fact that the solution stage is much faster (at least for the tests presented here and for a whole range of others which are not presented here for brevity) than the assembling stage for the Galerkin-based approaches, **GC** and **FG**. On the contrary, the solution stage is slower (or takes approximately the same time) than the assembling stage for the collocation approach. It is a much harder task to solve an overdetermined and, therefore, non-symmetric, system of equations by using least-squares than to solve a symmetric system of equations by using fast and efficient solvers that take full advantage of sparsity and symmetry. In a previous work [11], alternative schemes for the solution of overdetermined systems of equations (in the context of a Trefftz collocation technique for potential problems) are suggested that may be more efficient than the one used in this work.

For problems exhibiting singularities, such as the ones encountered in problems of fracture mechanics, it is convenient to extend the approximation basis, in Eq. (10), with a set of solutions that already take the singularities into account. When this is done much coarser meshes may be utilized but the price to pay is that the numerical integrations (to obtain the **F** and **A** matrices) are more time consuming (than in the standard approximation) as the new stress approximation functions (the Williams fields) are of singular nature. It is expected that, even more so in these cases, there will be advantages in promoting the reduction of the number of numerical integrals to compute.

Acknowledgement

This work is part of the research activity developed at ICIST, Instituto Superior Técnico, and has been supported by Junta Nacional de Investigação Científica e Tecnológica, through the research project PRAXIS/2/2.1/CEG/33/94.

References

[1] R.A. Adey, N. Kamiya, E. Kita, eds. Trefftz method: 70 years, special issue. *Adv. Engng. Software*, **24**, 1995.
[2] J.A.T. Freitas. Formulation of elastostatic hybrid-Trefftz stress elements. *Comp. Meth. Appl. Mech. Engng*, **153**: 127–151, 1998.
[3] J.A.T. Freitas, C. Cismaşiu. Numerical implementation of hybrid-Trefftz displacement elements. *Comput. Struct.*, in press, 1999.
[4] J.A.T. Freitas, Z.-Y. Ji. Hybrid-Trefftz boundary integral formulation for the simulation of singular stress fields. *Int. J. Numer. Meth. Engng.*, **39**: 281–308, 1996.
[5] J.A.T. Freitas, Z.-Y. Ji. Hybrid-Trefftz equilibrium model for crack problems. *Int. J. Numer. Meth. Engng.*, **39**: 569–584, 1996.
[6] J.A.T. Freitas, V.M.A. Leitão. A symmetric collocation Trefftz boundary integral formulation. *Computational Mechanics*, in press, 1999.
[7] J. Jirousek, A. Wróblewski. T-elements: State of the art and future trends. *Arch. Comp. Meth. Engng.*, **3/4**: 323–434, 1996.
[8] J. Jirousek, A.P. Zieliński. Survey of Trefftz-type element formulations. *Comp. Struct.*, **63**: 225–242, 1997.
[9] M. Kleiber, H. Mang, A.P. Zieliński, eds. Trefftz method: recent developments and perspectives. Special issue, *CAMES*, **4**(3/4), 1997.
[10] J. Kołodziej. Review of application of boundary collocation methods in mechanics of continuous media. *SM Archives*, **12**(4): 187–231, 1987.
[11] V.M.A. Leitão. On the implementation of a multi-region Trefftz-collocation formulation for 2-D potential problems. *Engineering Analysis with Boundary Elements*, **20**: 51–61, 1997.
[12] V.M.A. Leitão. Application of multi-region Trefftz-collocation to fracture mechanics. *Engineering Analysis with Boundary Elements*, **22**: 251–256, 1998.
[13] N.I. Muskhelishvili. *Some Basic Problems of the Mathematical Theory of Elasticity.* Noordhoff, 1963.
[14] E. Trefftz. Ein Gegenstück zum Ritzchen Verfahren. *Proc. 2nd Int. Cong. Applied Mechanics*, Zurich, 1926.