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T-functions for 2-D creeping flow on domains with circular cylinders, corners, and possessing symmetry

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The purpose of the paper is to propose of a way of constructing trial functions for the indirect Trefftz method as applied to 2-D creeping (Stokes) flow problems. The considered cases refer to the problems of flow around fixed and rotating circular cylinders, in corners with two walls fixed, or one wall moving, and flow possessing particular symmetry. The trial functions, proposed and systematically constructed fulfil exactly not only the governing equation, like T-complete Herrera functions, but also certain given boundary conditions and conditions resulting from assumed symmetry. A list of such trial functions, unavailable elsewhere, is presented. The derived functions can be treated as a subset of T-complete Herrera functions, which can be used for solving typical boundary-value problems.

1. Introduction

The concept of Trefftz method consists in the application of analytically derived trial functions, sometimes called T-functions, identically fulfilling a governing differential equation. In the traditional indirect formulation of the Trefftz method the solution of the boundary-volume problem is approximated by a linear combination of the T-complete functions and some coefficients. The unknown coefficients are then determined so as to make the boundary conditions satisfied approximately. The most popular trial functions are those known as Herrera functions, or T-complete Herrera sets of functions. These sets have been proposed for harmonic, biharmonic and Helmholtz equations [1,2]. The functions in Herrera sets satisfy a differential equation and do not result from boundary conditions. However, when solving some boundary-value problems better results my be obtained by using trial functions which satisfy exactly not only the governing equation, but also some of the boundary conditions on a part of the boundary, as well.

The purpose of the paper is to propose of a way of constructing trial functions for the indirect Trefftz method, as applied to 2-D creeping (Stokes) flow problems in presence fixed and rotating circular cylinders, corners with two walls fixed, corners with one wall moving, and possessing particular symmetry.

The trial functions, proposed and systematically constructed in this paper are related to biharmonic equation. These functions can be treated as a subset of T-complete Herrera functions, which can be used for solution of boundary value problems related with cylinders and corners. Such functions for harmonic problems were presented by one of the author at First International Workshop on Trefftz Method [3].

The derivation of the trial functions is based on the general solution of 2-D harmonic and biharmonic equations in polar co-ordinates system. The harmonic solution is used in the case when the boundary-value problem for creeping flow is formulated in terms of vorticity and stream function, whereas the biharmonic solution is used for the formulation of the boundary-value problem by stream function only. The separable solutions of harmonic and biharmonic equations in polar co-ordinates, in the most general form is a linear combination of the following expressions:

– for harmonic case

$$\left\{\begin{array}{l} (R^2\,,1)\,(\cos 2\theta\,,\sin 2\theta\,,1\,,\theta) \\ R(\cos\theta\,,\sin\theta) \\ (\sin\mu\theta\,,\cos\mu\theta)\,(R^{\mu}\,,R^{-\mu}) \\ (\sinh\nu\theta\,,\cosh\nu\theta)\,[\sin(\nu\ln R),\cos(\nu\ln R)] \\ (\cos\theta\,,\sin\theta)\,(R,R^{-1}) \\ (1\,,\theta)\,(1\,,\ln R) \end{array}\right\}, \tag{1}$$

– for biharmonic case

$$\left\{\begin{array}{l} (R^{1+\lambda}\,,R^{1-\lambda})\,[\cos(1+\lambda)\theta\,,\sin(1+\lambda)\theta\,,\cos(1-\lambda)\theta\,,\sin(1-\lambda)\theta] \\ {[R\sin(s\ln R)\,,R\cos(s\ln R)]\,[\cos\theta\cosh s\theta\,,\sin\theta\sinh s\theta\,,\cos\theta\sinh s\theta\,,\sin\theta\cosh s\theta]} \\ (R^2\,,1)\,(\cos 2\theta\,,\sin 2\theta\,,1\,,\theta) \\ (R\,,R\ln R)\,(\cos\theta\,,\sin\theta\,,\cos\theta\,,\sin\theta) \\ (\sin\mu\theta\,,\cos\mu\theta)\,(R^{\mu}\,,R^{-\mu}\,,R^{2+\mu}\,,R^{2-\mu}) \\ (\sinh\nu\theta\,,\cosh\nu\theta)\,[\sin(\nu\ln R)\,,\cos(\nu\ln R)\,,R^2\sin(\nu\ln R)\,,R^2\cos(\nu\ln R)] \\ (\cos\theta\,,\sin\theta)\,(R\,,R^{-1}\,,R\ln R\,,R^3) \\ (1\,,\theta)\,(1\,,\ln R\,,R^2\,,R^2\ln R) \end{array}\right\}, \tag{2}$$

where R, θ are polar co-ordinates, λ, s, μ, ν are constants. The results for seven cases of special T-functions are further considered. In two of these cases the detailed derivations are given (see Appendix A and B).

2. Trefftz method. Two indirect approximations

The boundary-value problems related to the two-dimensional creeping flow can be formulated in few different ways. The first one is based on the stream function and it is the following:

Biharmonic differential equation

$$\nabla^2\nabla^2\Psi = 0 \qquad \text{in } D \tag{3}$$

together with boundary conditions

$$\left.\begin{array}{l} \Psi = \bar{\Psi} \\ \dfrac{\partial\Psi}{\partial n} = \bar{q} \end{array}\right\} \qquad \text{on } \Gamma_1\,, \tag{4}$$

$$\left.\begin{array}{l} \Psi = \bar{\Psi} \\ \dfrac{\partial^2\Psi}{\partial n^2} = \bar{p} \end{array}\right\} \qquad \text{on } \Gamma_2\,, \tag{5}$$

where ∇^2 is the two-dimensional Laplace operator, Ψ is the unknown stream function (the velocity components are given by $V_X = \frac{\partial\Psi}{\partial Y}$, $V_Y = -\frac{\partial\Psi}{\partial X}$), $\frac{\partial}{\partial n}$ is the operator of normal derivative, $\bar{\Psi}$, $\bar{q}$, and $\bar{p}$ are given functions, and $\Gamma = \Gamma_1 + \Gamma_2$.

The second formulation is based on the stream function and vorticity. In such case the boundary-value problem is the following:

Two-dimensional Laplace equation for vorticity

$$\nabla^2\Omega = 0 \qquad \text{in } D, \tag{6}$$

two-dimensional Poisson equation for stream function:

$$\nabla^2\Psi = \Omega \qquad \text{in } D, \tag{7}$$

along with boundary conditions

$$\Omega = \bar{\Omega} \qquad \text{on } \Gamma_1, \tag{8}$$

$$\frac{\partial\Omega}{\partial n} = \bar{\omega} \qquad \text{on } \Gamma_2, \tag{9}$$

$$\Psi = \bar{\Psi} \qquad \text{on } \Gamma_1, \tag{10}$$

$$\frac{\partial\Psi}{\partial n} = \bar{q} \qquad \text{on } \Gamma_2, \tag{11}$$

or another combinations of the conditions.

The weak formulation of the boundary-value problem (3)–(5) can be expressed in weighted residual form as follows,

$$\int_D W\nabla^2\nabla^2\Psi \,\mathrm{d}D + \int_\Gamma W_1(\Psi - \bar{\Psi})\,\mathrm{d}\Gamma + \int_{\Gamma_1} W_2\left(\frac{\partial\Psi}{\partial n} - \bar{q}\right)\mathrm{d}\Gamma + \int_{\Gamma_2} W_3\left(\frac{\partial^2\Psi}{\partial n^2} - \bar{p}\right)\mathrm{d}\Gamma = 0, \tag{12}$$

where W, W_1, W_2, and W_3 are weighting functions.

Similarly, the weak formulation of the boundary-value problem (6)–(11) can be expressed as

$$\int_D \tilde{W}_1\nabla^2\Omega\,\mathrm{d}D + \int_D \tilde{W}_2\left(\nabla^2\Psi - \Omega\right)\mathrm{d}D + \int_{\Gamma_1}\tilde{W}_3(\Omega - \bar{\Omega})\,\mathrm{d}\Gamma$$

$$+ \int_{\Gamma_2}\tilde{W}_4\left(\frac{\partial\Omega}{\partial n} - \bar{\omega}\right)\mathrm{d}\Gamma + \int_{\Gamma_1}\tilde{W}_5(\Psi - \bar{\Psi})\,\mathrm{d}\Gamma + \int_{\Gamma_2}\tilde{W}_6\left(\frac{\partial\Psi}{\partial n} - \bar{q}\right)\mathrm{d}\Gamma = 0 \tag{13}$$

where $\tilde{W}_1$, $\tilde{W}_2$, $\tilde{W}_3$, $\tilde{W}_4$, $\tilde{W}_5$, and $\tilde{W}_6$ are weighting functions.

When using the Trefftz method, the solution of the boundary-value problem (3)–(5) is approximated by linear combination of a complete set of trial functions

$$\Psi = \sum_{k=1}^{2M} C_k U_k, \tag{14}$$

where C_k are the undetermined coefficients, and U_k are trial functions chosen in such way that they satisfy the equations

$$\nabla^2\nabla^2 U_k = 0. \tag{15}$$

In the same way, the solutions of the boundary-value problem (6)–(11) are approximated by:

$$\Omega = \sum_{k=1}^{2M} C_k T_k, \tag{16}$$

$$\Psi = \sum_{k=1}^{2M} C_k U_k, \tag{17}$$

where C_k are the undetermined coefficients, and T_k are trial functions which satisfy the equations

$$\nabla^2 T_k = 0, \tag{18}$$

$$\nabla^2 U_k = T_k\,. \tag{19}$$

Substituting (14) into (12) and (16)–(17) into (13), we get

$$\int_\Gamma W_1 \left(\sum_{k=1}^{2M} C_k U_k - \bar{\Psi} \right) \mathrm{d}\Gamma + \int_{\Gamma_1} W_2 \left(\sum_{k=1}^{2M} C_k \frac{\partial U_k}{\partial n} - \bar{q} \right) \mathrm{d}\Gamma + \int_{\Gamma_2} W_3 \left(\sum_{k=1}^{2M} C_k \frac{\partial^2 U_k}{\partial n^2} - \bar{p} \right) \mathrm{d}\Gamma = 0 \tag{20}$$

and

$$\int_{\Gamma_1} \tilde{W}_3 \left(\sum_{k=1}^{2M} C_k T_k - \bar{\Omega} \right) \mathrm{d}\Gamma + \int_{\Gamma_2} \tilde{W}_4 \left(\sum_{k=1}^{2M} C_k \frac{\partial T_k}{\partial n} - \bar{\omega} \right) \mathrm{d}\Gamma + \int_{\Gamma_1} \tilde{W}_5 \left(\sum_{k=1}^{2M} C_k U_k - \bar{\Psi} \right) \mathrm{d}\Gamma + \int_{\Gamma_2} \tilde{W}_6 \left(\sum_{k=1}^{2M} C_k \frac{\partial U_k}{\partial n} - \bar{q} \right) \mathrm{d}\Gamma = 0. \tag{21}$$

Depending on the selected weighting functions W_1, W_2, and W_3 or $\tilde{W}_3$, $\tilde{W}_4$, $\tilde{W}_5$, and $\tilde{W}_6$, one obtains different variants of the Trefftz method. In what follows the two variants, that can be identified as the boundary collocation method in the least square sense and the Galerkin method, are discussed.

2.1. Boundary collocation method

Adopting the weighting functions in the form

$$W_1 = \delta(P_j)\,, \qquad P_j \in \Gamma\,, \qquad j = 1, \ldots, m_1\,, \tag{22}$$

$$W_2 = \delta(Q_j)\,, \qquad Q_j \in \Gamma_1\,, \qquad j = 1, \ldots, m_2\,, \tag{23}$$

$$W_3 = \delta(S_j)\,, \qquad S_j \in \Gamma_2\,, \qquad j = 1, \ldots, m_3\,, \tag{24}$$

where $\delta(P_j)$, $\delta(Q_j)$ and $\delta(S_j)$ are the Dirac delta functions, we fall into boundary collocation case. Putting these functions into (19), the system of linear equations for unknown coefficients C_k takes the form

$$\sum_{j=1}^{2M} C_k U_k(P_j) = \bar{\Psi}(P_j)\,, \qquad j = 1, \ldots, m, \tag{25}$$

$$\sum_{j=1}^{2M} C_k \frac{\partial U_k(P_j)}{\partial n} = \bar{q}(Q_j)\,, \qquad j = 1, \ldots, m_1\,, \tag{26}$$

$$\sum_{j=1}^{2M} C_k \frac{\partial^2 U_k(Q_j)}{\partial n^2} = \bar{p}(S_j)\,, \qquad j = 1, \ldots, m_2\,. \tag{27}$$

While the condition $m_1 + m_2 + m3 \geq 2M$ must be fulfilled and the system (25)–(27) can be solved in the least square sense.

2.2. Galerkin method

By taking

$$W_1 = \frac{\partial U_j}{\partial n}, \qquad W_2 = -U_j, \qquad W_3 = \frac{\partial^2 U_j}{\partial n^2}, \qquad j = 1, 2, \ldots, 2M, \tag{28}$$

and substituting these functions into Eq. (20) we get the Galerkin formulation

$$\int_\Gamma \frac{\partial U_j}{\partial n}\left(\sum_{k=1}^{2M} C_k U_k - \bar{\Psi}\right) \mathrm{d}\Gamma - \int_{\Gamma_1} U_j \left(\sum_{k=1}^{2M} C_k \frac{\partial U_k}{\partial n} - \bar{q}\right) \mathrm{d}\Gamma$$
$$+ \int_{\Gamma_2} \frac{\partial^2 U_j}{\partial n^2}\left(\sum_{k=1}^{2M} C_k \frac{\partial^2 U_k}{\partial n^2} - \bar{p}\right) \mathrm{d}\Gamma = 0, \qquad j = 1, 2, \ldots, 2M. \tag{29}$$

Matrix form of Eq. (29) is

$$\mathbf{KC} = \mathbf{f} \tag{30}$$

where

$$K_{ij} = \int_\Gamma \frac{\partial U_j}{\partial n} U_k \, \mathrm{d}\Gamma - \int_{\Gamma_1} U_j \frac{\partial U_k}{\partial n} \mathrm{d}\Gamma + \int_{\Gamma_2} \frac{\partial^2 U_j}{\partial n^2}\frac{\partial U_k}{\partial n} \mathrm{d}\Gamma, \tag{31}$$

$$f_j = \int_\Gamma \frac{\partial U_j}{\partial n} \bar{\Psi} \, \mathrm{d}\Gamma - \int_{\Gamma_1} U_j \bar{q} \, \mathrm{d}\Gamma + \int_{\Gamma_2} \frac{\partial^2 U_j}{\partial n^2} \bar{p} \, \mathrm{d}\Gamma. \tag{32}$$

3. Construction of trial functions

In the following seven paragraphs the trial functions corresponding to a chosen geometry of fluid flow problems are constructed. Detailed derivation for the first and fifth case are given in Appendixes A and B, respectively, while the derivations for the other cases are similar to these two cases.

3.1. Flow with two axes of symmetry around a circular cylinder

In many cases of creeping flow around a circular cylinder one can assume that flow is symmetric with respect to the axis perpendicular to mean flow direction (axes y on Fig. 1a). On the other hand, the axis of the cylinder parallels to mean flow direction can be assumed a line of constant stream function (axis x on Fig. 1a). Moreover, this axis can be treated as axes of symmetry of vorticity. In this way the flow around a cylinder can be represented by its properties in the fist quadrant (see Fig. 1b). The governing equations in this region and boundary conditions on some parts of the boundary are the following,

$$\frac{\partial^2 \Omega}{\partial R^2} + \frac{1}{R}\frac{\partial \Omega}{\partial R} + \frac{1}{R^2}\frac{\partial \Omega}{\partial \theta^2} = 0, \tag{33}$$

$$\frac{\partial^2 \Psi}{\partial R^2} + \frac{1}{R}\frac{\partial \Psi}{\partial R} + \frac{1}{R^2}\frac{\partial \Psi}{\partial \theta^2} = \Omega, \tag{34}$$

$$\Psi = 0 \qquad \text{for} \quad R = E, \tag{35}$$

$$\frac{\partial \Psi}{\partial R} = 0 \qquad \text{for} \quad R = E, \tag{36}$$

$$\Psi = 0 \qquad \text{for} \quad \theta = 0, \tag{37}$$

$$\Omega = 0 \qquad \text{for} \quad \theta = 0, \tag{38}$$

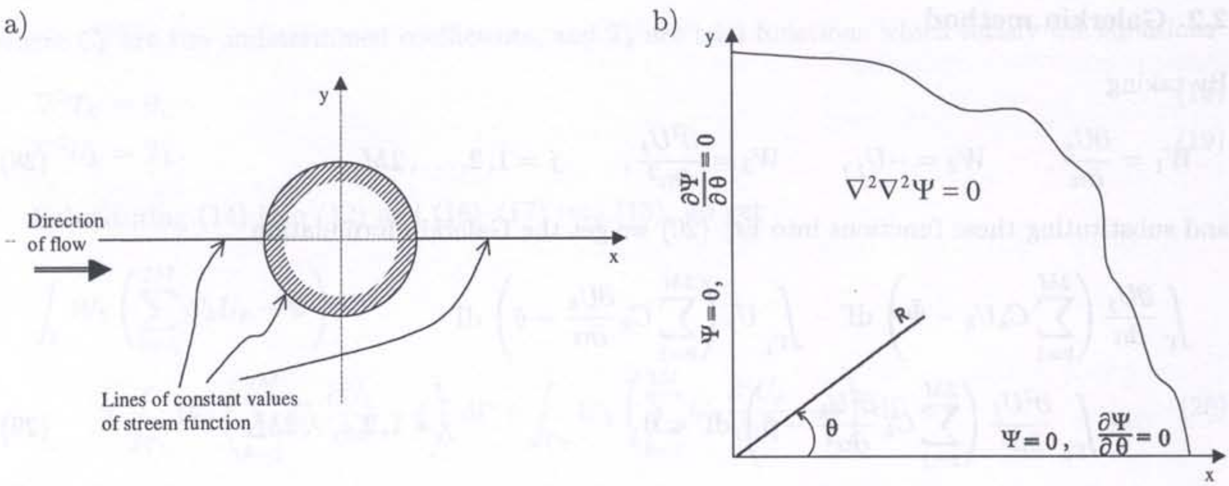

Fig. 1. Region of flow with two axes of symmetry around a circular cylinder

$$\frac{\partial \Psi}{\partial \theta} = 0 \qquad \text{for} \quad \theta = \frac{\pi}{2}, \tag{39}$$

$$\frac{\partial \Omega}{\partial \theta} = 0 \qquad \text{for} \quad \theta = \frac{\pi}{2}. \tag{40}$$

Taking into account a linear combination of functions given by (1) and (2) and performing the derivations presented in Appendix A, we propose the following solutions which satisfy Eqs. (33)–(34) and boundary conditions (35)–(40)

$$\Omega = \sum_{k=1}^{2M} C_k T_k(R,\theta), \tag{41}$$

$$\Psi = \sum_{k=1}^{2M} C_k U_k(R,\theta,E), \tag{42}$$

where

$$T_k(R,\theta) = R^{2k-1}\sin[(2k-1)\theta], \qquad k = 1,2,\ldots,M, \tag{43}$$

$$T_{M+k}(R,\theta) = R^{-(2k-1)}\sin[(2k-1)\theta], \qquad k = 1,2,\ldots,M, \tag{44}$$

$$U_1(R,\theta,E) = \frac{1}{8}(R^3 - 2E^2R + E^4R{-1})\sin\theta, \tag{45}$$

$$U_k(R,\theta,E) = \frac{(2k-1)R^{2k+1} - 2kE^2R^{2k-1} + E^{4k}R^{-(2k-1)}}{8k(2k-1)}\sin[(2k-1)\theta], \qquad k = 2,3,\ldots,M, \tag{46}$$

$$U_{M+1}(R,\theta,E) = \frac{1}{4}\left(E^2R^{-1} + 2R\ln\frac{R}{E} - R\right)\sin\theta, \tag{47}$$

$$U_{M+k}(R,\theta,E) = \frac{E^{-4(k-1)}R^{2k+1} - (2k-1)R^{-2k+3} + 2(k-1)E^2R^{-(2k-1)}}{8(2k-1)(k-1)}\sin[(2k-1)\theta], \qquad k = 2,3,\ldots,M. \tag{48}$$

The constant coefficients C_k must be determined from another boundary conditions. An example of boundary value problem for which trial functions given by (41)–(48) are useful is presented in Section 4 when creeping flow through periodic array of cylinders is considered.

3.2. Flow around a circular cylinder symmetrical with respect to the direction of main flow

Consider the region in the surrounding a circular cylinder of radius E, in the case when the vorticity field is symmetrical with respect to the direction of the main flow. In this case the pattern of the flow may not be the same before and after the cylinder, as shown on Fig. 2a. In order to study the flow around the cylinder we can restrict ourselves to quadrants I and IV (Fig. 2b). Now, the governing equations in the considered region are the same as in the previous case, i.e. (33)–(34), and the boundary conditions on some parts of its boundary the are following,

$$\Psi = 0 \qquad \text{for} \quad R = E, \tag{49}$$

$$\frac{\partial \Psi}{\partial R} = 0 \qquad \text{for} \quad R = E, \tag{50}$$

$$\Psi = 0 \qquad \text{for} \quad \theta = 0, \tag{51}$$

$$\Omega = 0 \qquad \text{for} \quad \theta = 0, \tag{52}$$

$$\frac{\partial \Psi}{\partial \theta} = 0 \qquad \text{for} \quad \theta = \pi, \tag{53}$$

$$\frac{\partial \Omega}{\partial \theta} = 0 \qquad \text{for} \quad \theta = \pi. \tag{54}$$

Using a derivation similar to that given in Appendix A one can find a solution, which fulfils Eqs. (33)–(34) and boundary conditions (49)–(54)

$$\Omega = \sum_{k=1}^{2M} C_k T_k(R, \theta), \tag{55}$$

$$\Psi = \sum_{k=1}^{2M} C_k U_k(R, \theta, E), \tag{56}$$

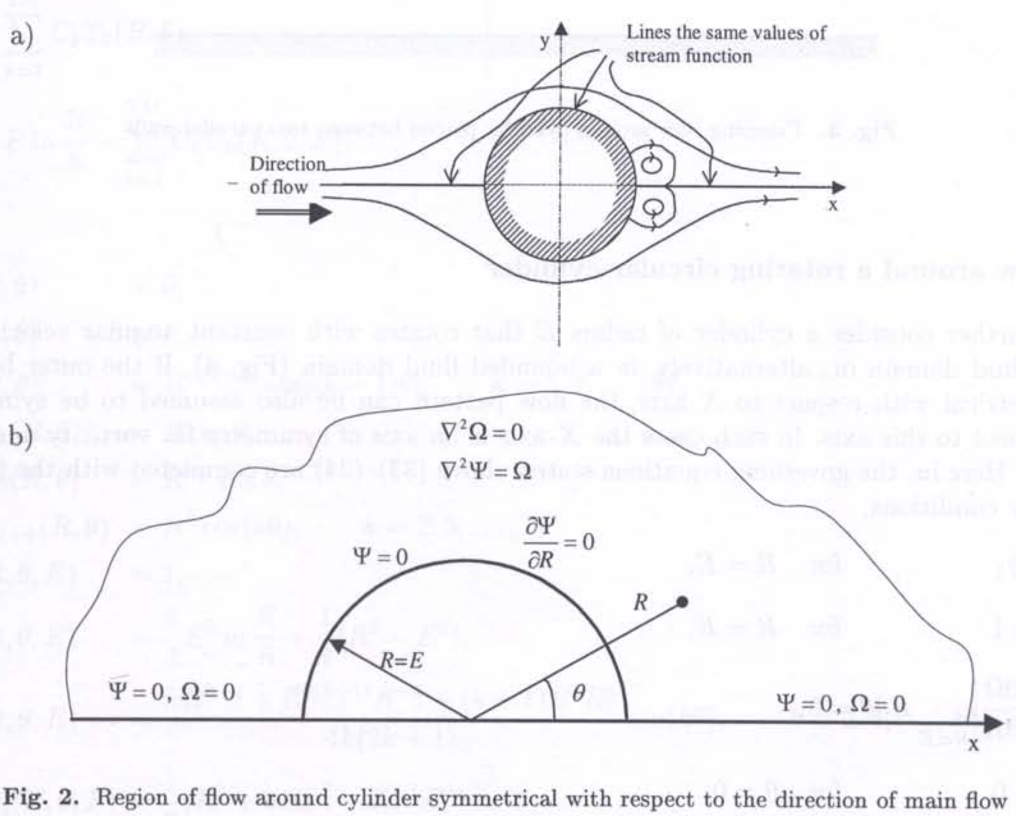

Fig. 2. Region of flow around cylinder symmetrical with respect to the direction of main flow

where

$$T_k(R,\theta) = R^k \sin(k\theta), \qquad k = 1,2,\ldots,M, \tag{57}$$

$$T_{M+k}(R,\theta) = R^{-k} \sin(k\theta), \qquad k = 1,2,\ldots,M, \tag{58}$$

$$U_1(R,\theta,E) = \frac{1}{8}(R^3 - 2E^2R + E^4R^{-1}) \sin\theta, \tag{59}$$

$$U_k(R,\theta,E) = \frac{kR^{k+2} + E^{2(k+1)}R^{-k} + (k+1)E^2R^k}{4k(2k+1)} \sin(k\theta), \qquad k = 2,3,\ldots,M, \tag{60}$$

$$U_{M+1}(R,\theta,E) = \frac{1}{4}\left(E^2R^{-1} + 2R\ln\frac{R}{E} - R\right)\sin\theta, \tag{61}$$

$$U_{M+k}(R,\theta,E) = \frac{E^{-2(k-1)}R^k - kR^{-k+2} + (k-1)E^2R^{-k}}{4k(k-1)} \sin(k\theta), \qquad k = 2,3,\ldots,M. \tag{62}$$

These functions can by used, for instance, in solution of the boundary-value problem connected with creeping flow around a circular cylinder placed between two parallel walls, as shown in Fig. 3. In such case the constants C_k can be determined from approximate satisfaction of boundary conditions at the boundary AGFE.

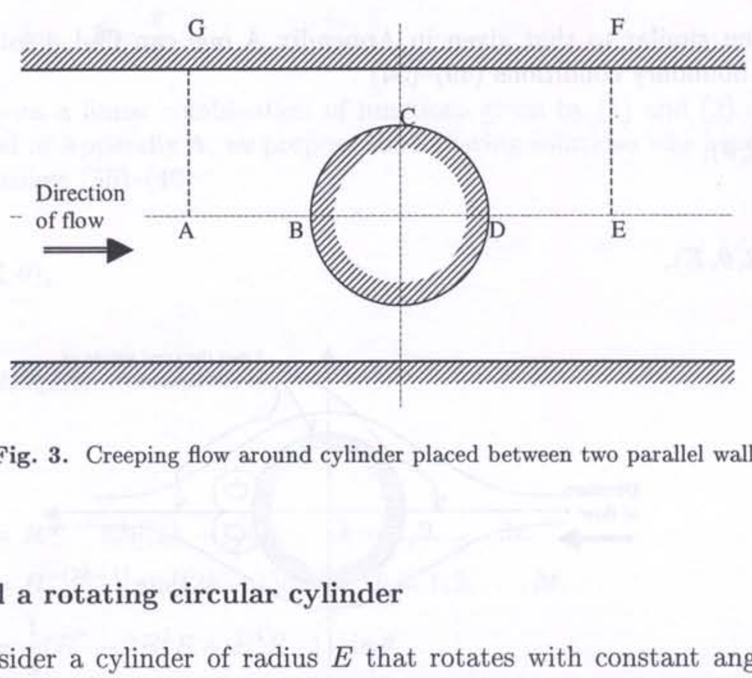

Fig. 3. Creeping flow around cylinder placed between two parallel walls

3.3. Flow around a rotating circular cylinder

Let us further consider a cylinder of radius E that rotates with constant angular velocity in an infinite fluid domain or, alternatively, in a bounded fluid domain (Fig. 4). If the outer boundary is symmetrical with respect to X-axis, the flow pattern can be also assumed to be symmetrical with respect to this axis. In such cases the X-axis is an axis of symmetry for vorticity and stream function. Here in, the governing equations stated above (33)–(34) are completed with the following boundary conditions,

$$\Psi = \Psi_1 \qquad \text{for} \quad R = E, \tag{63}$$

$$\frac{\partial\Psi}{\partial R} = 1 \qquad \text{for} \quad R = E, \tag{64}$$

$$\int_0^{2\pi} \frac{\partial\Omega}{\partial R}\bigg|_{R=E} E d\theta = 0, \tag{65}$$

$$\frac{\partial\Omega}{\partial\theta} = 0 \qquad \text{for} \quad \theta = 0, \tag{66}$$

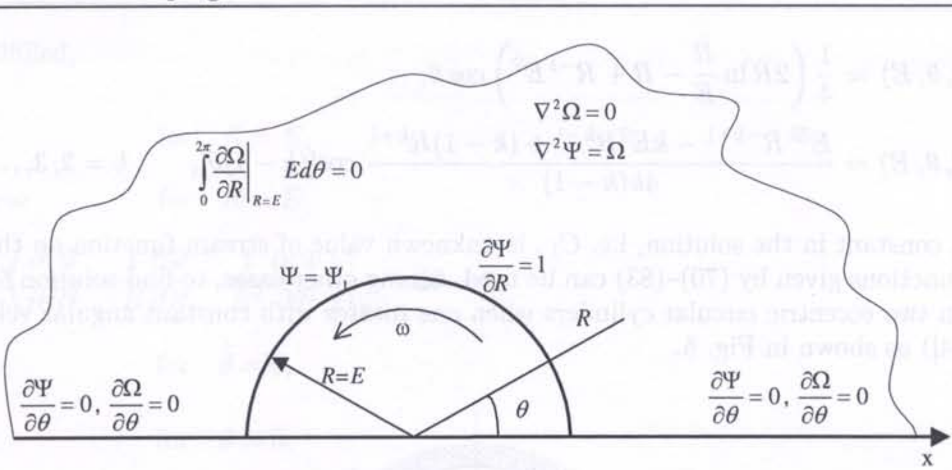

Fig. 4. Region of flow around rotating circular cylinder

$$\frac{\partial\Psi}{\partial\theta} = 0 \qquad \text{for} \quad \theta = 0, \tag{67}$$

$$\frac{\partial\Psi}{\partial\theta} = 0 \qquad \text{for} \quad \theta = \pi, \tag{68}$$

$$\frac{\partial\Omega}{\partial\theta} = 0 \qquad \text{for} \quad \theta = \pi. \tag{69}$$

Boundary conditions (63)–(64) express the non-slip conditions on the cylinder surfaces. The condition (65) reflects the fact that pressure is periodic around the inner cylinder. The remaining conditions result from symmetry of problem. Using a procedure similar to that presented in Appendix A one can get the following solution, which fulfils equations (33)–(34) and boundary conditions (64)–(69),

$$\Omega = \sum_{k=1}^{2M} C_k T_k(R,\theta), \tag{70}$$

$$\Psi = E \ln\frac{R}{E} + \sum_{k=1}^{2M} C_k U_k(R,\theta,E), \tag{71}$$

where

$$T_1(R,\theta) = 0, \tag{72}$$

$$T_2(R,\theta) = 1, \tag{73}$$

$$T_k(R,\theta) = R^{-(k-1)} \cos[(k-1)\theta], \qquad k = 3,4,\ldots,M, \tag{74}$$

$$T_{M+1}(R,\theta) = R\cos\theta, \tag{75}$$

$$T_{M+2}(R,\theta) = R^{-1}\cos\theta, \tag{76}$$

$$T_{M+1+k}(R,\theta) = R^k \cos(k\theta), \qquad k = 2,3,\ldots,M, \tag{77}$$

$$U_1(R,\theta,E) = 1, \tag{78}$$

$$U_2(R,\theta,E) = \frac{1}{2}E^2 \ln\frac{E}{R} + \frac{1}{4}(R^2 - E^2), \tag{79}$$

$$U_k(R,\theta,E) = \frac{kR^{k+2} + E^{2(k+1)}R^{-k} + (k+1)E^2R^k}{4k(2k+1)} \sin(k\theta), \qquad k = 2,3,\ldots,M, \tag{80}$$

$$U_{M+1}(R,\theta,E) = \frac{1}{8}(R^3 + 2RE^2 - R^{-1}E^4)\cos\theta, \tag{81}$$

$$U_{M+2}(R,\theta,E) = \frac{1}{4}\left(2R\ln\frac{R}{E} - R + R^{-1}E^2\right)\cos\theta, \tag{82}$$

$$U_{M+k}(R,\theta,E) = \frac{E^{2k}R^{-k+1} - kE^2R^{k-1} + (k-1)R^{k+1}}{4k(k-1)}\cos[(k-1)\theta], \qquad k = 2,3,\dots,M. \tag{83}$$

The first constant in the solution, i.e. C_1, is unknown value of stream function on the cylinder, $C_1 = \Psi_1$. Functions given by (70)–(83) can be used, among other cases, to find solution for creeping flow between two eccentric circular cylinders when one rotates with constant angular velocity (flow in bearing [4]) as shown in Fig. 5.

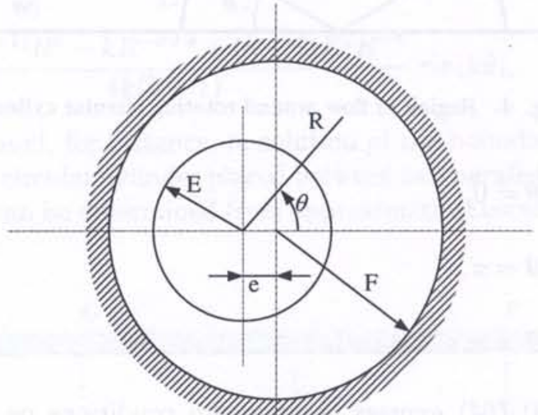

Fig. 5. Bearing geometry

3.4. Flow around a freely rotating circular cylinder

We now consider a cylinder of radius E that may rotate freely under the action of the flow (see Fig. 6). The angular velocity ω is unknown and must be determined from the solution of the appropriate boundary-value problem. Moreover, we assume that the solution is symmetrical with respect to X and Y axes. In the considered case, together with Eqs. (33)–(34), the following boundary conditions

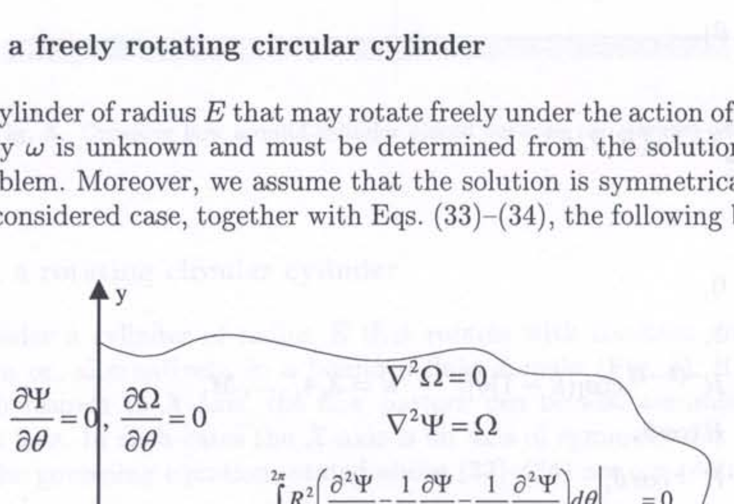

Fig. 6. Region of flow around freely rotating circular cylinder

must be fulfilled,

$$\Psi = 0 \qquad \text{for} \quad R = E, \tag{84}$$

$$\frac{\partial \Psi}{\partial R} = -\omega \qquad \text{for} \quad R = E, \tag{85}$$

$$\int_0^{2\pi} R^2 \left(\frac{\partial^2 \Psi}{\partial R^2} - \frac{1}{R}\frac{\partial \Psi}{\partial R} - \frac{1}{R^2}\frac{\partial^2 \Psi}{\partial \theta^2} \right)\Bigg|_{R=E} \mathrm{d}\theta = 0, \tag{86}$$

$$\frac{\partial \Omega}{\partial \theta} = 0 \qquad \text{for} \quad \theta = 0, \tag{87}$$

$$\frac{\partial \Psi}{\partial \theta} = 0 \qquad \text{for} \quad \theta = 0, \tag{88}$$

$$\frac{\partial \Psi}{\partial \theta} = 0 \qquad \text{for} \quad \theta = \frac{\pi}{2}, \tag{89}$$

$$\frac{\partial \Omega}{\partial \theta} = 0 \qquad \text{for} \quad \theta = \frac{\pi}{2}. \tag{90}$$

Boundary conditions (84)–(85) express the non-slip condition on the cylinder surface. The condition (86) states zero hydrodynamic moment acting on the cylinder, as it is free to rotate. Using a derivation similar to that of Appendix A, one can get a solution, which fulfils Eqs. (33)–(34) and boundary conditions (84)–(90),

$$\Omega = \sum_{k=1}^{2M+1} C_k T_k(R,\theta), \tag{91}$$

$$\Psi = E \ln \frac{R}{E} + \sum_{k=1}^{2M} C_k U_k(R,\theta,E), \tag{92}$$

where

$$T_1(R,\theta) = -\frac{2}{E}, \tag{93}$$

$$T_k(R,\theta) = R^{-2(k-1)} \cos[2(k-1)\theta], \qquad k = 2,3,\ldots,M, \tag{94}$$

$$T_{M+1+k}(R,\theta) = R^{2k} \cos(2k\theta), \qquad k = 1,2,\ldots,M, \tag{95}$$

$$U_1(R,\theta,E) = \frac{1}{2}\left(E - \frac{R^2}{E} \right), \tag{96}$$

$$U_k(R,\theta,E) = \frac{R^{k-1}E^{-2(k-2)} + (k-2)E^2 R^{-k+1} - (k-1)R^{-k+3}}{4(k-1)(k-2)} \cos[(k-1)\theta], \qquad k = 2,3,\ldots,M+1, \tag{97}$$

$$U_{M+k}(R,\theta,E) = \frac{E^{2k}R^{-k+1} - kE^2R^{k-1} + (k-1)R^{k+1}}{4k(k-1)} \cos[(k-1)\theta], \qquad k = 2,3,\ldots,M. \tag{98}$$

The first constant in the solution is equal to the unknown values of angular velocity of cylinder, $C_1 = \omega$. Functions given by (91)–(98) can be used, for instance, for the solution shear flow induced by motion of two walls in opposite directions, as shown in Fig. 7.

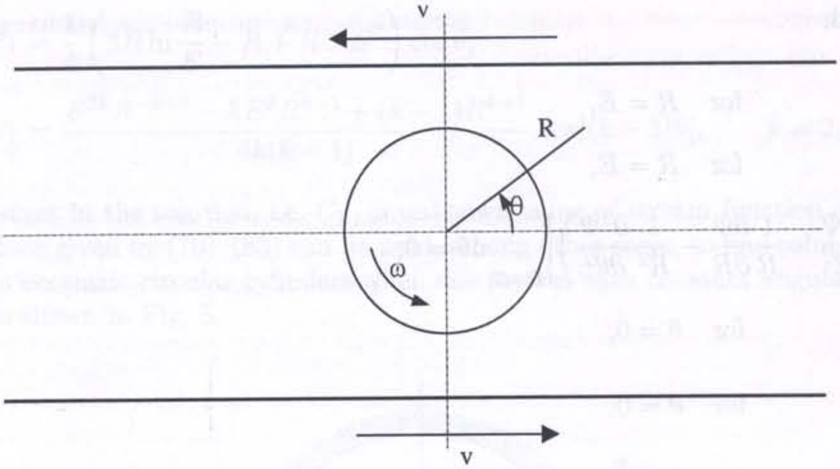

Fig. 7. Circular cylinder in shear flow

3.5. Flow in rigid corner

Consider the region surrounded by a rigid corner of $\frac{\pi}{2}$ angle as shown in Fig. 8. The governing equation in the region and boundary conditions on a part of the boundary of region are the following,

$$\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \Psi(R,\theta) = 0, \tag{99}$$

$$\Psi = 0 \qquad \text{for} \quad \theta = 0, \tag{100}$$

$$\frac{\partial \Psi}{\partial \theta} = 0 \qquad \text{for} \quad \theta = 0, \tag{101}$$

$$\Psi = 0 \qquad \text{for} \quad \theta = \frac{\pi}{2}, \tag{102}$$

$$\frac{\partial \Psi}{\partial \theta} = 0 \qquad \text{for} \quad \theta = \frac{\pi}{2}. \tag{103}$$

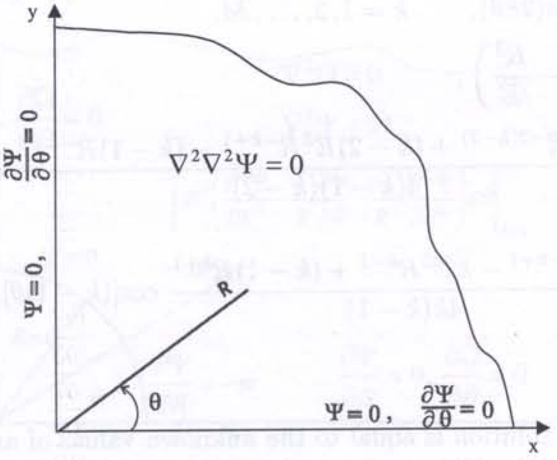

Fig. 8. Flow in rigid corner

If we now use the derivation presented in Appendix B, we may propose the following solution, which satisfy Eq. (99) and boundary conditions (100)–(103),

$$\Psi = \sum_{k=1}^{2M} C_k U_k(R,\theta), \tag{104}$$

where

$$U_k(R,\theta) = \mathrm{Re}\left\{ R^{\lambda_k+1}[\cos(\lambda_k+1)\theta - \cos(\lambda_k-1)\theta] + \frac{\lambda_k \cot\frac{\lambda_k \pi}{2}}{(\lambda_k+1)}\left[\sin(\lambda_k+1)\theta - \frac{\lambda_k+1}{\lambda_k-1}\sin(\lambda_k-1)\theta\right]\right\}, \qquad k=1,2,\ldots,M, \tag{105}$$

and $\lambda_k = a_k + ib_k$ are complex numbers, which are solutions of the following transcendental equation,

$$\sin\frac{\pi\lambda}{2} = \pm\lambda. \tag{106}$$

Solutions of equation (106) are given in Table 1.

The functions given by (104)–(105) can be used for solving of creeping flow in a cavity, as shown in Fig. 9. In such case the flow region can be divided into four large elements. For two of them (elements III and IV on Fig. 9), solution (104)–(105) can be useful.

Table 1. Subsequent roots of Eq. (106)

k	a_k	b_k	k	a_k	b_k
1	0.0	0.0	27	50.96324700	2.944890604
2	1.0	0.0	28	52.96434003	2.969351749
3	2.739593356	1.119024534	29	54.96536403	2.992907843
4	4.808250761	1.463928121	30	56.96632555	3.015623484
5	6.845135158	1.681634696	31	58.96723031	3.037556589
6	8.868825976	1.842383989	32	60.96808336	3.058759286
7	10.88555235	1.970199497	33	62.96888915	3.079278866
8	12.89809091	2.076415813	34	64.96965162	3.099157389
9	14.90789080	2.167332596	35	66.97037429	3.118434266
10	16.91579046	2.246828034	36	68.97106029	3.137144667
11	18.92231199	2.317464558	37	70.97171243	3.155320937
12	20.92779903	2.381025854	38	72.97233324	3.172992728
13	22.93248780	2.438804427	39	74.97292499	3.190187287
14	24.93654647	2.491767285	40	76.97348974	3.206929715
15	26.94009826	2.540657034	41	78.97402935	3.223243179
16	28.94323570	2.586056465	42	80.97454552	3.239149113
17	30.94602969	2.628431430	43	82.97503978	3.254667381
18	32.94853556	2.668160213	44	84.97551356	3.269816430
19	34.95079716	2.705554217	45	86.97596813	3.284613422
20	36.95284973	2.740872866	46	88.97640467	3.299074348
21	38.95472190	2.774334582	47	90.97682427	3.313214135
22	40.95643725	2.806124999	48	92.97722793	3.327046735
23	42.95801532	2.836403219	49	94.97761657	3.340585212
24	44.95947250	2.865306641	50	96.97799103	3.353841813
25	46.96082261	2.892954743	51	98.97835210	3.366828036
26	48.96207742	2.919452064	52	100.9787005	3.379554689

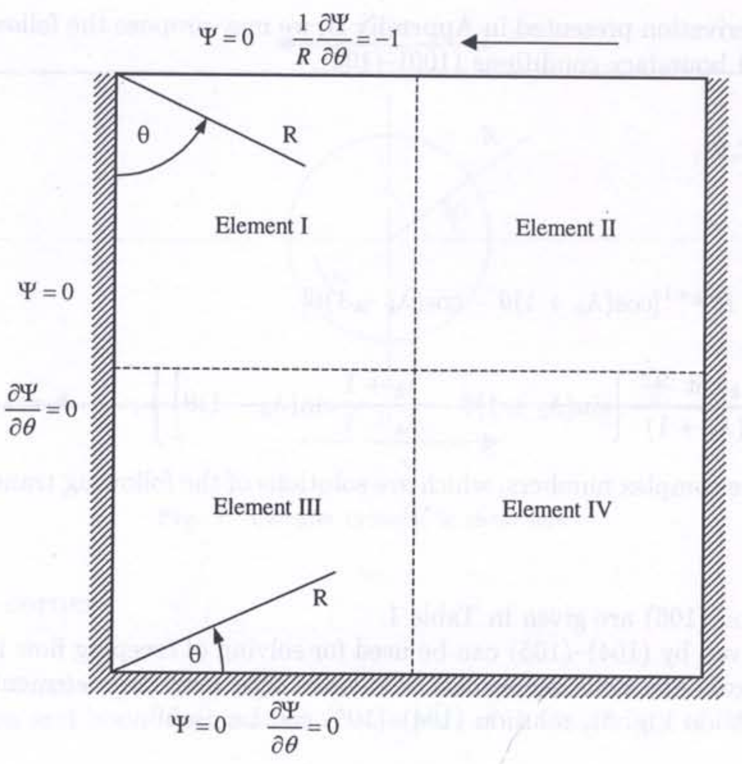

Fig. 9. Flow in square cavity

3.6. Stick-slip problem

Let us consider the region surrounding the edge of a semi-infinite plate (see Fig. 10). The main direction of the flow, far from plate, is parallel to the plate. The governing equation is the same as in the previous case, i.e. Eq. (99), whereas the boundary conditions on a part of the boundary are the following,

$$\Psi = 0 \qquad \text{for} \quad \theta = 0, \tag{107}$$

$$\frac{\partial \Psi}{\partial \theta} = 0 \qquad \text{for} \quad \theta = 0, \tag{108}$$

$$\Psi = 0 \qquad \text{for} \quad \theta = \pi, \tag{109}$$

$$\frac{\partial^2 \Psi}{\partial \theta^2} = 0 \qquad \text{for} \quad \theta = \pi. \tag{110}$$

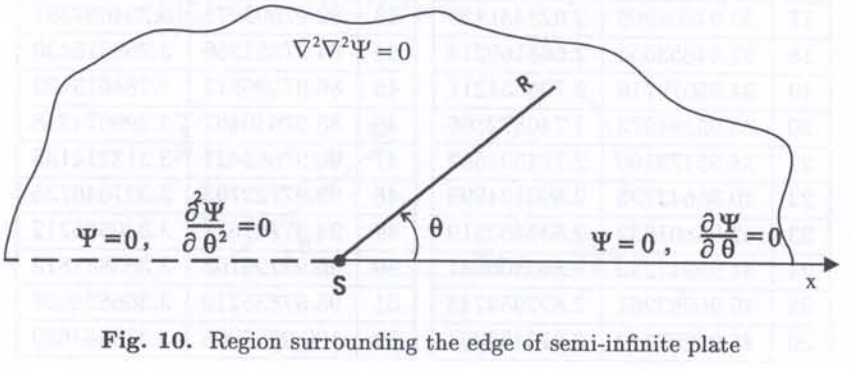

Fig. 10. Region surrounding the edge of semi-infinite plate

Following a derivation similar to the one presented in Appendix B, we propose the following solution which satisfy Eq. (99) and boundary conditions (107)–(110),

$$\Psi = \sum_{k=1}^{2M} C_k U_k(R,\theta), \tag{111}$$

where

$$U_k(R,\theta) = R^{\lambda_k+1}[\cos(\lambda_k+1)\theta - \cos(\lambda_k-1)\theta], \tag{112}$$

$$U_{M+k}(R,\theta) = R^{\lambda_k+1}\left[\sin(\lambda_k+1)\theta - \frac{\lambda_k+1}{\lambda_k-1}\sin(\lambda_k-1)\theta\right], \qquad k=1,2,\ldots,M, \tag{113}$$

with

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{3}{2}, \quad \lambda_3 = 2, \quad \lambda_4 = \frac{5}{2}, \quad \lambda_5 = 3, \quad \lambda_6 = \frac{7}{2}, \quad \lambda_7 = 4, \quad \ldots \tag{114}$$

Functions given by (111)–(114) can be used, for instance, in the analysis of stick-slip problem [6] shown in Fig. 11.

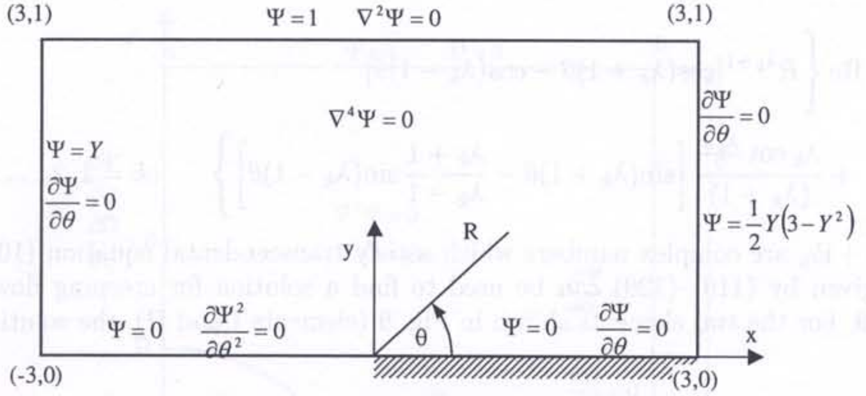

Fig. 11. Stick-slip flow

3.7. Flow in corner $\frac{\pi}{2}$ when one wall moves with constant velocity

Consider the region in the vicinity of the corner $\frac{\pi}{2}$ angle, when one wall moves with a constant velocity, as shown on Fig. 12. The governing equation in the region is the same as for the rigid corner, whereas the boundary conditions on some parts of the boundary of region are as follows,

$$\Psi = 0 \qquad \text{for} \quad \theta = 0, \tag{115}$$

$$\frac{1}{R}\frac{\partial\Psi}{\partial\theta} = -1 \qquad \text{for} \quad \theta = 0, \tag{116}$$

$$\Psi = 0 \qquad \text{for} \quad \theta = \frac{-\pi}{2}, \tag{117}$$

$$\frac{\partial\Psi}{\partial\theta} = 0 \qquad \text{for} \quad \theta = \frac{-\pi}{2}. \tag{118}$$

Using a derivation similar to the one presented in Appendix B, we propose the following solutions which satisfy Eq. (99) and boundary conditions (115)–(118),

$$\Psi = R\frac{1}{1-\left(\frac{\pi}{2}\right)^2}\left[\left(\frac{\pi}{2}\right)^2\sin\theta - \theta\cos\theta + \frac{\pi}{2}\theta\sin\theta\right] + \sum_{k=1}^{2M} C_k U_k(R,\theta), \tag{119}$$

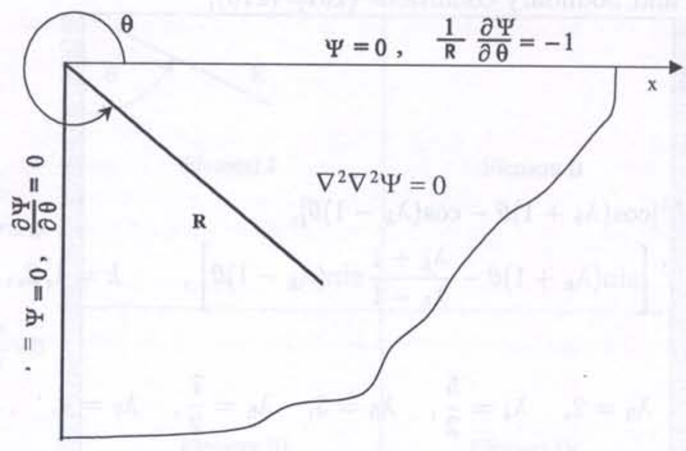

Fig. 12. Flow in corner when one wall moves with constant velocity

where

$$U_k(R,\theta) = \mathrm{Re}\left\{ R^{\lambda_k+1}[\cos(\lambda_k+1)\theta - \cos(\lambda_k-1)\theta] + \frac{\lambda_k \cot\frac{\lambda_k\pi}{2}}{(\lambda_k+1)}\left[\sin(\lambda_k+1)\theta - \frac{\lambda_k+1}{\lambda_k-1}\sin(\lambda_k-1)\theta\right]\right\} \qquad k=1,2,\ldots,M, \tag{120}$$

where $\lambda_k = a_k + ib_k$ are complex numbers which satisfy transcendental equation (106).

Functions given by (119)–(120) can be used to find a solution for creeping flow in the cavity shown in Fig. 9. For the two elements shown in Fig. 9 (elements I and II), the solution (119)–(120) can be used.

4. Numerical example of application of the proposed T-functions: Creeping flow through a periodic array of cylinders

Consider the steady-state flow of an incompressible fluid thorough a periodic infinite square array of cylinders, each of radius $2a$, with $2b$ being the center-to-center distance between two adjanced cylinders (see Fig. 13). The mean flow velocity of magnitude U is in the X-direction. Due to the symmetry, the creeping flow can be considered in repeated element $OABC$, shown in Fig. 14.

The boundary-value problem for this element is the following,

governing equations,

$$\nabla^2\omega = 0, \tag{121}$$

$$\nabla^2\psi = \omega, \tag{122}$$

and boundary conditions

$$\psi = 0 \qquad \text{for} \quad r = a, \tag{123}$$

$$\frac{\partial\psi}{\partial r} = 0 \qquad \text{for} \quad r = a, \tag{124}$$

$$\psi = 0 \qquad \text{for} \quad \theta = 0, \tag{125}$$

$$\omega = 0 \qquad \text{for} \quad \theta = 0, \tag{126}$$

$$\frac{\partial\psi}{\partial\theta} = 0 \qquad \text{for} \quad \theta = \frac{\pi}{2}, \tag{127}$$

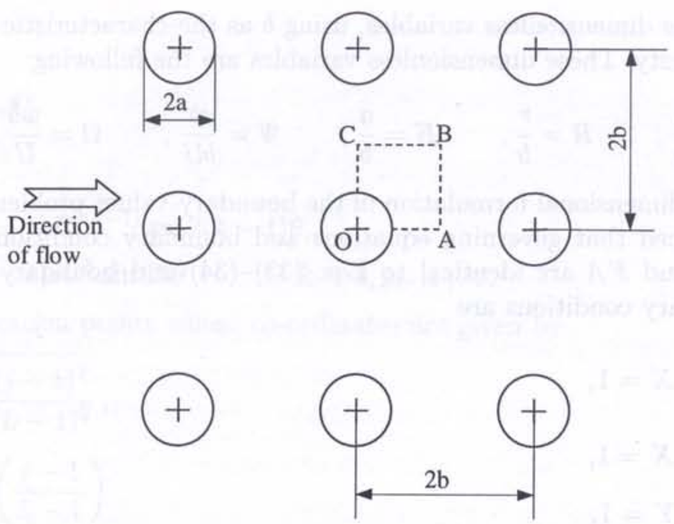

Fig. 13. Flow through a periodic array of cylinders

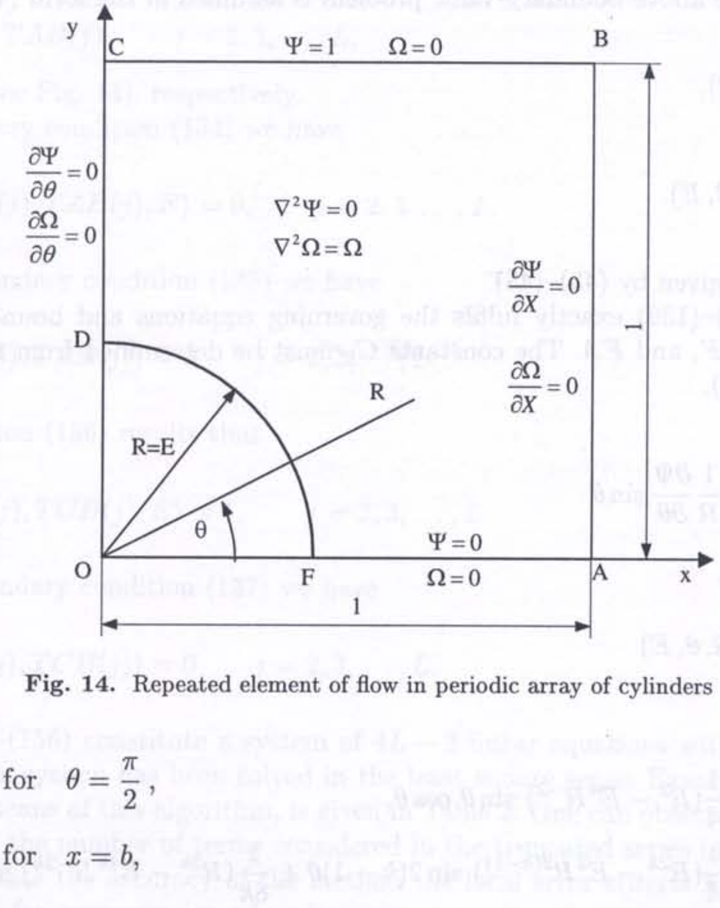

Fig. 14. Repeated element of flow in periodic array of cylinders

$$\frac{\partial \omega}{\partial \theta} = 0 \quad \text{for} \quad \theta = \frac{\pi}{2}, \tag{128}$$

$$\frac{\partial \psi}{\partial x} = 0 \quad \text{for} \quad x = b, \tag{129}$$

$$\frac{\partial \omega}{\partial x} = 0 \quad \text{for} \quad x = b, \tag{130}$$

$$\psi = Ub \quad \text{for} \quad y = b, \tag{131}$$

$$\omega = 0 \quad \text{for} \quad y = b, \tag{132}$$

where ω is vorticity, ψ is the stream function, U is the velocity of mean flow (average velocity in x direction).

Let us introduce the dimensionless variables, using b as the characteristic length scale and U as the characteristic velocity. These dimensionless variables are the following:

$$X = \frac{x}{b}, \qquad Y = \frac{y}{b}, \qquad R = \frac{r}{b}, \qquad E = \frac{a}{b}, \qquad \Psi = \frac{\psi}{bU}, \qquad \Omega = \frac{\omega b}{U}. \tag{133}$$

This leads to the non-dimensional formulation of the boundary-values problem presented in Fig. 14. Here, it must be noticed that governing equations and boundary conditions on that part of the boundary CD, DF, and FA are identical to Eqs. (33)–(34) and boundary conditions (35)–(40). The additional boundary conditions are

$$\frac{\partial \Psi}{\partial X} = 0 \qquad \text{for} \quad X = 1, \tag{134}$$

$$\frac{\partial \Omega}{\partial X} = 0 \qquad \text{for} \quad X = 1, \tag{135}$$

$$\Psi = 1 \qquad \text{for} \quad Y = 1, \tag{136}$$

$$\Omega = 0 \qquad \text{for} \quad Y = 1. \tag{137}$$

The solution of the above boundary-value problem is assumed in the form (41)–(42), namely

$$\Omega = \sum_{k=1}^{2M} C_k T_k(R, \theta), \tag{138}$$

$$\Psi = \sum_{k=1}^{2M} C_k U_k(R, \theta, E). \tag{139}$$

where T_k and U_k are given by (43)–(48).

The solution (138)–(139) exactly fulfils the governing equations and boundary conditions on the boundary CD, DF, and FA. The constants C_k must be determined from the other boundary conditions (134)–(137).

Using the formula

$$\frac{\partial \Psi}{\partial X} = \frac{\partial \Psi}{\partial R} \cos\theta - \frac{1}{R} \frac{\partial \Psi}{\partial \theta} \sin\theta \tag{140}$$

one can obtain

$$\frac{\partial \Psi}{\partial X} = \sum_{k=1}^{2M} C_k W_k(R, \theta, E) \tag{141}$$

where

$$W_1(R, \theta, E) = \frac{1}{4}(R^2 - E^4 R^{-2}) \sin\theta \cos\theta, \tag{142}$$

$$W_k(R, \theta, E) = \frac{1}{4}(R^{2k} - E^2 R^{2(k-1)}) \sin 2(k-1)\theta + \frac{1}{8k}(R^{2k} - E^{4k} R^{-2k}) \sin 2k\theta, \quad k = 2, 3, \ldots, M, \tag{143}$$

$$W_{M+1}(R, \theta, E) = \frac{1}{2}(1 - E^2 R^{-2}) \sin\theta \cos\theta, \tag{144}$$

$$W_{M+k}(R, \theta, E) = \frac{E^{-4(k-1)} R^{2(k-1)}}{8(k-1)} \sin(2(k-1)\theta + \left[\frac{k}{4(k-1)} R^{-2(k-1)} - \frac{E^2 R^{-2k}}{4}\right] \sin 2k\theta - \frac{R^{-2(k-1)}}{8(k-1)} [3 \sin(2k-1)\theta \cos\theta + \cos(2k-1)\theta \sin\theta]. \tag{145}$$

In a similar way

$$\frac{\partial \Omega}{\partial X} = \sum_{k=1}^{2M} C_k V_k(R,\theta) \tag{146}$$

where

$$V_k(R,\theta) \quad = (2k-1)R^{2(k-1)} \sin 2(k-1)\theta, \tag{147}$$

$$V_{M+k}(R,\theta) = (2k-1)R^{2k} \sin 2k\theta \qquad k = 1,2,\ldots,M. \tag{148}$$

We chose the collocation points whose co-ordinates are given by

$$RAB(j) = \sqrt{1 + \frac{(j-1)^2}{(L-1)^2}}, \tag{149}$$

$$TAB(j) = \arctan\left(\frac{j-1}{L-1}\right), \tag{150}$$

on boundary AB (see Fig. 14) and

$$RCB(j) = RAB(j), \tag{151}$$

$$TCB(j) = \frac{\pi}{2} - TAB(j), \qquad j = 2,3,\ldots,L, \tag{152}$$

on boundary CB (see Fig. 14), respectively.

By using boundary condition (134) we have

$$\sum_{k=1}^{2M} C_k W_k(RAB(j), TAB(j), E) = 0, \qquad j = 2,3,\ldots,L. \tag{153}$$

Similarly using boundary condition (135) we have

$$\sum_{k=1}^{2M} C_k V_k(RAB(j), TAB(j)) = 0 \qquad j = 2,3,\ldots,L. \tag{154}$$

From the condition (136) results that

$$\sum_{k=1}^{2M} C_k U_k(RCB(j), TCB(j), E) = 1, \qquad j = 2,3,\ldots,L. \tag{155}$$

Similarly using boundary condition (137) we have

$$\sum_{k=1}^{2M} C_k T_k(RCB(j), TCB(j)) = 0 \qquad j = 2,3,\ldots,L. \tag{156}$$

Equations (153)–(156) constitute a system of $4L-2$ linear equations with $N = 2M$ unknown coefficients C_k. This system has been solved in the least square sense. Example of the coefficients C_k, calculated by means of this algorithm, is given in Table 2. One can observe that the coefficients quickly decrease, as the number of terms considered in the truncated series increases.

In order to estimate the accuracy of the method the local error criteria has been applied. The following definitions for errors are introduced,

$$ER1 = \max|\Psi(X,1) - 1|, \tag{157}$$

$$ER2 = \max\left|\frac{\partial \Psi(1,Y)}{\partial X}\right|, \tag{158}$$

$$ER3 = \max|\Omega(X,1)|, \tag{159}$$

$$ER4 = \max\left|\frac{\partial \Omega(1,Y)}{\partial X}\right|, \tag{160}$$

Table 2. Coefficients of expansion in solution (138)–(139). Number of collocation points per unit length of boundary $L = 25$. The ratio of diameter of cylinders to distance between neighbouring cylinders $E = 0.1$

$C(1) = -0.884966408993115183D{+}00$	$C(8) = 0.112861886362430064D{+}01$
$C(2) = 0.222942184902081481D{+}00$	$C(9) = -0.300868155068206953D{-}03$
$C(3) = 0.316949653135075663D{-}03$	$C(10) = -0.302089135558947975D{-}07$
$C(4) = 0.180602277863933305D{-}01$	$C(11) = -0.180701348649491536D{-}11$
$C(5) = 0.540035521614847577D{-}03$	$C(12) = -0.176197412254323144D{-}16$
$C(6) = 0.673371493354316683D{-}03$	$C(13) = -0.955714134181379626D{-}20$
$C(7) = 0.101756084072560863D{-}03$	$C(14) = 0.266208914274642255D{-}24$

Table 3. Values of maximum local errors versus number of terms series $N = 2M$ in solution (138)–(139). The ratio of diameter of cylinders to distance between neighbouring cylinders $E = 0.1$. Number of collocation points per unit length of boundary $L = 25$

N	ER1	ER2	ER3	ER4
6	$0.14486E{+}00$	$0.19778E{-}01$	$0.18930E{+}00$	$0.22354E{+}00$
8	$0.40995E{-}01$	$0.17262E{-}01$	$0.28953E{-}01$	$0.50270E{-}01$
10	$0.48470E{-}01$	$0.14454E{-}01$	$0.30871E{-}01$	$0.24145E{-}01$
12	$0.46973E{-}01$	$0.14051E{-}01$	$0.52413E{-}02$	$0.88743E{-}02$
14	$0.70820E{-}01$	$0.14185E{-}01$	$0.46934E{-}02$	$0.46594E{-}02$
16	$0.69712E{-}01$	$0.14190E{-}01$	$0.14459E{-}01$	$0.52028E{-}02$
18	$0.29472E{-}01$	$0.14387E{-}01$	$0.12810E{-}01$	$0.59318E{-}02$
20	$0.32266E{-}01$	$0.13758E{-}01$	$0.59177E{-}02$	$0.46124E{-}02$
22	$0.49571E{-}01$	$0.13618E{-}01$	$0.55568E{-}02$	$0.33682E{-}02$
24	$0.64872E{-}01$	$0.13804E{-}01$	$0.12451E{-}01$	$0.39154E{-}02$
26	$0.28288E{-}01$	$0.14283E{-}01$	$0.11530E{-}01$	$0.47579E{-}02$
28	$0.23598E{-}01$	$0.13648E{-}01$	$0.76384E{-}02$	$0.44121E{-}02$
30	$0.34713E{-}01$	$0.13280E{-}01$	$0.55524E{-}02$	$0.35546E{-}02$

where functions $\Omega(X, Y)$ and $\Psi(X, Y)$ are given by (138)–(139). These errors are measures of satisfaction of boundary conditions (134), (135), (136), and (137), respectively. Examples of values of errors are presented in Table 3. One may notice that the method applied above is relatively accurate.

5. Conclusions

A new way of constructing trial functions for the indirect Tefftz method as applied to 2-D creeping flow problems has been proposed. The functions fulfil exactly not only the governing equations, as in the traditional Trefftz formulation, but also some additional boundary conditions of the problem. The boundary conditions considered throughout the paper concern particular geometries that are difficult to tackle numerically, like corners. Moreover, the method is especially efficient in treating the problems of creeping flow around cylinders in bounded or unbounded domains. In the case study that was worked out in the paper numerical results are presented for values of maximum local errors, as a function of the number of trial functions. The local errors observed on the stream function at the boundary, where the boundary conditions are fulfilled approximately, are below 4% for 30 trial functions. The precision of the method can still be improved by an adaptation procedure, like the one presented in [4]. The practical applications of the proposed algorithms are related to the steady-state flow of a viscous fluid in bearings [4], stick-slip flow [6], and flow in fibrous porous media (flow in infinite array of cylinders), among others.

ACKNOWLEDGEMENTS

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APPENDIX

A. Derivation of special purpose T-functions for flow with two axes of symmetry around a circular cylinder

A general solution of Eq. (33) in polar co-ordinate system is:

$$\Omega = A_1 + A_2 \ln R + A_3\theta + A_4\theta \ln R + \sum_{k=1}^{\infty}\Big[(B_k R^k + C_k R^{-k})\cos k\theta + (D_k R^k + E_k R^{-k})\sin k\theta\Big] \tag{A1}$$

where A_1, A_2, A_3, A_4, B_k, C_k, D_k, E_k, are integral constants.

The boundary condition (38) will be fulfilled if

$$A_1 = A_2 = A_4 = B_k = C_k = 0. \tag{A2}$$

Hence

$$\Omega = A_3\theta + \sum_{k=1}^{\infty}(D_k R^k + E_k R^{-k})\sin k\theta. \tag{A3}$$

The derivative of (A3) with respect to θ is equal

$$\frac{\partial\Omega}{\partial\theta} = A_3 + \sum_{k=1}^{\infty} k(D_k R^k + E_k R^{-k})\cos k\theta. \tag{A4}$$

Taking $A_3 = 0$ and putting $\theta = \frac{\pi}{2}$ as in the boundary condition (40) we have

$$\sum_{k=1}^{\infty} k(D_k R^k + E_k R^{-k})\cos k\frac{\pi}{2} = 0 \tag{A5}$$

which is satisfied if

$$k = 1, 3, 5, 7, \ldots \tag{A6}$$

Thus without lose of generality one can write

$$\Omega = \sum_{k=1}^{\infty}(D_k R^{2k-1} + E_k R^{-(2k-1)})\sin[(2k-1)\theta]. \tag{A7}$$

The general solution of Eq. (34) can be written in the form

$$\Psi = \Psi_h + \Psi_{ps} \tag{A8}$$

where

$$\Psi = F_1 + F_2 \ln R + F_3\theta + F_4\theta \ln R + \sum_{k=1}^{\infty}\Big[(G_k R^k + H_k R^{-k})\sin k\theta + (P_k R^k + Q_k R^{-k})\cos k\theta\Big] \tag{A9}$$

and F_1, F_2, F_3, F_4, G_k, H_k, P_k, Q_k are constants, and Ψ_{ps} is particular solution of the nonhomogeneous Eq. (34), i.e.

$$\frac{\partial^2\Psi_{ps}}{\partial R^2}+\frac{1}{R}\frac{\partial\Psi_{ps}}{\partial R}+\frac{1}{R^2}\frac{\partial^2\Psi_{ps}}{\partial\theta^2}=\sum_{k=1}^{\infty}(D_kR^{2k-1}-E_kR)\sin[(2k-1)\theta]. \tag{A10}$$

It can be easily checked that the above equation is satisfied by the function

$$\Psi_{ps}=\left(\frac{1}{8}R^3D_1+\frac{1}{2}E_1R\ln R\right)\sin\theta+\sum_{k=2}^{\infty}\left[\frac{1}{8k}D_kR^{2k+1}-\frac{1}{8(k-1)}E_kR^{-2k+3}\right]\sin[(2k-1)\theta]. \tag{A11}$$

Hence, the general solution of Eq. (34) has the form

$$\begin{aligned}\Psi &= F_1+F_2\ln R+F_3\theta+F_4\theta\ln R\\ &+\sum_{k=1}^{\infty}\left[(G_kR^k+H_kR^{-k})\sin k\theta+(P_kR^k+Q_kR^{-k})\cos k\theta\right]\\ &+\left(\frac{1}{8}R^3D_1+\frac{1}{2}E_1R\ln R\right)\sin\theta\\ &+\sum_{k=2}^{\infty}\left[\frac{1}{8k}D_kR^{2k+1}-\frac{1}{8(k-1)}E_kR^{-2k+3}\right]\sin[(2k-1)\theta].\end{aligned} \tag{A12}$$

Taking into account the boundary conditions (37) and (39) we have

$$F_1=F_2=F_3=F_4=P_k=Q_k=0. \tag{A13}$$

Then, the general solution (A12) is reduced to the form

$$\begin{aligned}\Psi &= \sum_{k=1}^{\infty}\left[(G_kR^k+H_kR^{-k})\sin k\theta+\right]+\left(\frac{1}{8}R^3D_1+\frac{1}{2}E_1R\ln R\right)\sin\theta\\ &+\sum_{k=2}^{\infty}\left[\frac{1}{8k}D_kR^{2k+1}-\frac{1}{8(k-1)}E_kR^{-2k+3}\right]\sin[(2k-1)\theta].\end{aligned} \tag{A14}$$

Taking into account condition (35) we have

$$\begin{aligned}0 &= \sum_{k=1}^{\infty}\left[(G_kE^k+H_kE^{-k})\sin k\theta+(P_kE^k+Q_kE^{-k})\cos k\theta\right]+\left(\frac{1}{8}E^3D_1+\frac{1}{2}E_1E\ln E\right)\sin\theta\\ &+\sum_{k=2}^{\infty}\left[\frac{1}{8k}D_kE^{2k+1}-\frac{1}{8(k-1)}E_kE^{-2k+3}\right]\sin[(2k-1)\theta].\end{aligned} \tag{A15}$$

This will be fulfilled if

$$\frac{1}{8}E^3D_1+\frac{1}{2}E_1E\ln E+G_1E+H_1E^{-1}=0, \tag{A16}$$

$$G_kE^{2k-1}+H_kE^{-(2k-1)}+\frac{1}{8k}D_kE^{2k+1}-\frac{1}{8(k-1)}E_kE^{-2k+3}=0,\qquad k=2,3,\ldots \tag{A17}$$

Taking into account condition (36) we must calculate derivative of (A14) with respect to R,

$$\begin{aligned}\frac{\partial\Psi}{\partial R} &= \left[\frac{3}{8}R^2D_1+\frac{1}{2}E_1(\ln R+1)+G_1-H_1R^{-2}\right]\sin\theta\\ &+\sum_{k=2}^{\infty}\Big[(2k-1)G_kR^{2k-2}-(2k-1)H_kR^{-2k}\\ &\quad+\frac{2k+1}{8k}D_kR^{2k}+\frac{2k-3}{8(k-1)}E_kR^{-2k+2}\Big]\sin(2k-1)\theta.\end{aligned}$$

From condition (36) we have

$$\frac{3}{8}E^2D_1+\frac{1}{2}E_1(\ln E+1)+G_1-H_1E^{-2}=0 \tag{A18}$$

or

$$\frac{3}{8}E^3D_1+\frac{1}{2}E_1E(\ln E+1)+G_1E-H_1E^{-1}=0 \tag{A19}$$

and

$$(2k-1)G_kE^{2k-2}-(2k-1)H_kE^{-2k}+\frac{2k+1}{8k}D_kE^{2k}+\frac{2k-3}{8(k-1)}E_kE^{2k+2}=0 \tag{A20}$$

or

$$(2k-1)G_kE^{2k-1}-(2k-1)H_kE^{-2k+1}+\frac{2k+1}{8k}D_kE^{2k+1}+\frac{2k-3}{8(k-1)}E_kE^{2k+3}=0. \tag{A21}$$

Using Eqs. (A16) and (A19) we can express G_1 and H_1 in terms of D_1 and E_1 , namely

$$G_1=-\frac{1}{4}E^2D_1-\frac{1}{2}E_1\left(\ln E+\frac{1}{2}\right), \tag{A22}$$

$$H_1=\frac{1}{8}E^4D_1+\frac{1}{4}E^2E_1. \tag{A23}$$

Using Eqs. (A17) and (A21) we can express G_k and H_k in terms of D_k and E_k , namely

$$G_k=\frac{E^{-4(k-1)}}{8(k-1)(2k-1)}E_k-\frac{E^2}{4(2k-1)}D_k, \tag{A24}$$

$$H_k=\frac{E^{4k}}{8k(2k-1)}D_k+\frac{E^2}{4(2k-1)}E_k. \tag{A25}$$

Putting (A22), (A23), (A24), and (A25) into (A14), we can express stream function in terms D_1 , E_1 , D_k , E_k , only

$$\begin{aligned}\Psi=&\left[\frac{1}{8}(R^3-2E^2R+E^4R^{-1})\sin\theta\right]D_1+\left[\frac{1}{4}\left(E^2R^{-1}+2R\ln\frac{R}{E}-R\right)\sin\theta\right]E_1\\&+\sum_{k=2}^{\infty}D_k\frac{(2k-1)R^{2k+1}-2kE^2R^{2k+1}+E^{4k}R^{-(2k-1)}}{8k(2k-1)}\sin[(2k-1)\theta]\,D_k\\&+\sum_{k=2}^{\infty}E_k\frac{E^{-4(k-1)}R^{2k+1}+2(k-1)E^2R^{-2k+1}-(2k-1)R^{-2k+3}}{8(k-1)(2k-1)}\sin[(2k-1)\theta]\,E_k.\end{aligned} \tag{A26}$$

Now we introduce the following nomenclature,

$$\begin{aligned}&C_1=D_1,\quad C_2=D_2,\quad\ldots,\quad C_k=D_k &&(k=2,3,\ldots,M),\\&C_{M+1}=E_1,\quad C_{M+2}=E_2,\quad\ldots,\quad C_{M+k}=E_k &&(k=2,3,\ldots,M),\end{aligned} \tag{A27}$$

and truncating infinite series to finite number M terms we can write

$$\Omega=\sum_{k=1}^{2M}C_kT_k(R,\theta), \tag{A28}$$

$$\Psi=\sum_{k=1}^{2M}C_kU_k(R,\theta,E), \tag{A29}$$

where $T_k(R,\theta)$ and $U_k(R,\theta,E)$ are functions given by (43)–(48).

B. Derivation of special T-functions for flow in $\frac{\pi}{2}$ rigid corner

Following Moffat [5] we introduce the separated solution $\Psi(R,\theta)$ of Eq. (99) in plane polar coordinate system appropriate for corner flow. Moffat has used the following solution,

$$\Psi = R^{\lambda_k+1} f_k(\theta), \tag{B1}$$

where λ_k is real or complex constant and called the exponent of the solution. For $k=0$ and $k=1$ the function $f_k(\theta)$ take special forms. In considered case Moffat's solution of Eq. (99) is assumed in the form

$$\Psi(R,\theta) = \beta_0 R f_0(\theta) + \beta_1 R^2 f_1(\theta) + \sum_{k=2}^{\infty} \beta_k R^{\lambda_k+1} f_k(\theta) \tag{B2}$$

where β_0, β_1, β_2 are constants and

$$\begin{aligned} f_0(\theta) &= A_0\cos\theta + B_0\sin\theta + C_0\theta\cos\theta + D_0\theta\sin\theta, & \text{(B3)}\\ f_1(\theta) &= A_1\cos 2\theta + B_1\sin 2\theta + C_1\theta + D_1, & \text{(B4)}\\ f_k(\theta) &= A_k\cos(\lambda_k+1)\theta + B_k\sin(\lambda_k+1)\theta + C_k\cos(\lambda_k-1)\theta + D_k\sin(\lambda_k-1)\theta, & \text{(B5)}\\ & \qquad k = 2,,3,\ldots \end{aligned}$$

The functions $f_k(\theta)$ involve arbitrary constants A_0, B_0, C_0, D_0, A_1, B_1, C_1, D_1, A_k, B_k, C_k, and D_k, which must be determined from boundary conditions.

The boundary condition (100) will be satisfied if

$$\begin{aligned} A_0 &= 0, & \text{(B6)}\\ A_1 + D_1 &= 0, & \text{(B7)}\\ A_\lambda + D_\lambda &= 0. & \text{(B8)} \end{aligned}$$

The derivatives of functions $f_k(\theta)$ with respect to θ are following,

$$\begin{aligned} f_0' &= -A_0\sin\theta + B_0\cos\theta C_0(\cos\theta - \theta\sin\theta) + D_0(\sin\theta + \theta\cos\theta), & \text{(B9)}\\ f_1' &= -2A_1\sin 2\theta + 2B_1\cos 2\theta + C_1, & \text{(B10)}\\ f_k' &= -A_k(\lambda_k+1)\sin(\lambda_k+1)\theta + B_k(\lambda_k+1)\cos(\lambda_k+1)\theta \\ &\quad - C_k(\lambda_k-1)\sin(\lambda_k-1)\theta + D_k(\lambda_k-1)\cos(\lambda_k-1)\theta. & \text{(B11)} \end{aligned}$$

After using (B9)–(B11) from the boundary condition (101) results that

$$\begin{aligned} & B_0 + C_0 = 0, & \text{(B12)}\\ & 2B_1 + C_1 = 0, & \text{(B13)}\\ & B_k(\lambda_k+1) + D_k(\lambda_k-1) = 0. & \text{(B14)} \end{aligned}$$

The boundary condition (102) will be satisfied if

$$B_0 + D_0\frac{\pi}{2} = 0, \tag{B15}$$

$$-A_1 + C_1\frac{\pi}{2} + D_1 = 0, \tag{B16}$$

$$A_k\cos(\lambda_k+1)\frac{\pi}{2} + B_k\sin(\lambda_k+1)\frac{\pi}{2} + C_k\cos(\lambda_k-1)\frac{\pi}{2} + D_k\sin(\lambda_k-1)\frac{\pi}{2} = 0. \tag{B17}$$

The condition (103) leads to

$$-A_0 - C_0\frac{\pi}{2} + D_0 = 0, \tag{B18}$$

$$-2B_1 + C_1 = 0, \tag{B19}$$

$$A_k(\lambda_k+1)\sin(\lambda_k+1)\frac{\pi}{2} + B_k(\lambda_k+1)\cos(\lambda_k+1)\frac{\pi}{2} - C_k(\lambda_k-1)\sin(\lambda_k-1)\frac{\pi}{2} + D_k(\lambda_k-1)\sin(\lambda_k-1)\frac{\pi}{2} = 0. \tag{B20}$$

From (B6), (B12), (B15), and (B18) results that

$$A_0 = B_0 = C_0 = D_0\,. \tag{B21}$$

Similarly, from (B7), (B13), (B16), and (B19) results that

$$A_1 = B_1 = C_1 = D_1\,. \tag{B22}$$

Equations (B8) and (B14) permit to express C_k and D_k by A_k and B_k, namely

$$C_k = -A_k\,, \tag{B23}$$

$$D_k = -\frac{\lambda_k+1}{\lambda_k-1}B_k\,. \tag{B24}$$

Now, Eqs. (B17) and (B20) can be written as

$$A_k\left[\cos(\lambda_k+1)\frac{\pi}{2} - \cos(\lambda_k-1)\frac{\pi}{2}\right] + B_k\left[\sin(\lambda_k+1)\frac{\pi}{2} - \frac{\lambda_k+1}{\lambda_k-1}\sin(\lambda_k-1)\frac{\pi}{2}\right] = 0, \tag{B25}$$

$$-A_k\left[(\lambda_k+1)\sin(\lambda_k+1)\frac{\pi}{2} - (\lambda_k-1)\sin(\lambda_k-1)\frac{\pi}{2}\right] + B_k\left[(\lambda_k+1)\cos(\lambda_k+1)\frac{\pi}{2} - (\lambda_k-1)\cos(\lambda_k-1)\frac{\pi}{2}\right] = 0. \tag{B26}$$

Because

$$\cos(\lambda+1)\frac{\pi}{2} = \cos\frac{\lambda\pi}{2}\,\cos\frac{\pi}{2} - \sin\frac{\lambda\pi}{2}\,\sin\frac{\pi}{2} = -\sin\frac{\lambda\pi}{2}\,, \tag{B27}$$

$$\cos(\lambda-1)\frac{\pi}{2} = \cos\frac{\lambda\pi}{2}\,\cos\frac{\pi}{2} + \sin\frac{\lambda\pi}{2}\,\sin\frac{\pi}{2} = \sin\frac{\lambda\pi}{2}\,, \tag{B28}$$

$$\sin(\lambda+1)\frac{\pi}{2} = \sin\frac{\lambda\pi}{2}\,\cos\frac{\pi}{2} + \cos\frac{\lambda\pi}{2}\,\sin\frac{\pi}{2} = \cos\frac{\lambda\pi}{2}\,, \tag{B29}$$

$$\sin(\lambda-1)\frac{\pi}{2} = \sin\frac{\lambda\pi}{2}\,\cos\frac{\pi}{2} - \cos\frac{\lambda\pi}{2}\,\sin\frac{\pi}{2} = -\cos\frac{\lambda\pi}{2}\,, \tag{B30}$$

Eqs. (B25) and (B26) can be written as

$$A_k(\lambda_k-1)\sin\frac{\lambda_k\pi}{2} = B_k\lambda_k\cos\frac{\lambda_k\pi}{2}\,, \tag{B31}$$

$$A_k\lambda_k\cos\frac{\lambda_k\pi}{2} = -B_k(\lambda_k+1)\sin\frac{\lambda_k\pi}{2}\,. \tag{B32}$$

Dividing by sides the last equations we obtain

$$\lambda_k^2\cos^2\frac{\lambda_k\pi}{2} = -(\lambda_k^2-1)\sin^2\frac{\lambda_k\pi}{2}\,, \tag{B33}$$

which gives the transcendental equation (Eq. (106)) in the form

$$\sin^2\frac{\lambda_k\pi}{2} = \pm\lambda_k\,. \tag{B34}$$

Solutions of this equation are given in Table 1.

Due to Eq. (B32) we have

$$B_k = -A_k \frac{\lambda_k \cot \frac{\lambda_k \pi}{2}}{\lambda_k - 1} . \tag{B35}$$

Taking into account the above results in solution (B2) we have

$$\Psi = \sum_{k=1}^{2M} C_k U_k(R, \theta) \tag{B36}$$

where $U_k(R, \theta)$ are functions given by (105).

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