Computer Assisted Mechanics and Engineering Sciences, 8: 629–644, 2001.

Trefftz spectral method for elliptic equations of general type[1]

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(Received May 14, 2001)

A new numerical method for 2D linear elliptic partial differential equations in an arbitrary geometry is presented. The special feature of the method presented is that the trial functions, which are used to approximate a solution, satisfy the PDE only approximately. This reduction of the requirement to the trial functions extends the field of application of the Trefftz method. The method is tested on several one- and two-dimensional problems.

1. Introduction

The aim of the work is to describe a new numerical method for solving elliptic boundary value problems in irregular regions. The application of the method is illustrated for a general two-dimensional partial differential equation (PDE) of the type,

$$L(u) = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{i,j}(\mathbf{x})\frac{\partial u}{\partial x_j}\right) + \sum_{j=1}^{2} b_i(\mathbf{x})\frac{\partial u}{\partial x_i} + c(\mathbf{x})\,u = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega \subset \mathcal{R}^2, \tag{1}$$

where $\mathbf{x} \equiv (x_1, x_2) \equiv (x, y)$.

We assume that: 1) $a_{1,2}(\mathbf{x}) = a_{2,1}(\mathbf{x})$; 2) $a_{i,j}(\mathbf{x})$, $b_i(\mathbf{x})$, $c(\mathbf{x})$ and $f(\mathbf{x})$ are analytic functions in Ω; 3) Ω is a simply connected domain bounded by a simple closed curve $\partial\Omega$; 4) coefficients $a_{i,j}(x, y)$ satisfy, for some $p > 0$ and any ξ_1 and ξ_2, the condition,

$$\sum_{i,j=1}^{2} a_{i,j}(\mathbf{x})\,\xi_i\xi_j \geq p\sum_{i=1}^{2}\xi_i^2 \cdot \qquad \mathbf{x} \in \Omega. \tag{2}$$

The boundary condition is

$$l(u) = \alpha(\mathbf{x})u + \beta(\mathbf{x})\frac{\partial u}{\partial n} = g(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega, \tag{3}$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative and $\alpha(\mathbf{x})$, $\beta(\mathbf{x})$, $g(\mathbf{x})$ are prescribed functions of position.

[1] This paper was presented at the Second International Workshop on the Trefftz Method, Sintra, Portugal, in September 1999, and selected to be included in the special issue of *CAMES* (Vol. 8, Nos. 2/3). Following the Guest Editor's unfortunate ommision, this paper appears with delay in the current issue.
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The method presented falls into the group of embedding methods. The basic idea is to solve a given PDE in a simple cartesian domain Ω_0 in which the complex domain Ω is embedded. The initial PDE (1) is replaced by the following one,

$$L^{(0)}(u) = f^{(0)}(\mathbf{x}) + \sum_{k=1}^{K} q_k I(\mathbf{x} \mid \xi_k), \qquad \mathbf{x} \in \Omega_0, \qquad \xi_k \in \Omega_0 \backslash \Omega. \tag{4}$$

Here,

$$L^{(0)} = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{i,j}^{(0)}(\mathbf{x})\frac{\partial}{\partial x_j}\right) + \sum_{j=1}^{2} b_i^{(0)}(\mathbf{x})\frac{\partial}{\partial x_i} + c^{(0)}(\mathbf{x}). \tag{5}$$

$a_{i,j}^{(0)}$, $b_i^{(0)}$, $c^{(0)}$, $f^{(0)}$ are some extensions of $a_{i,j}$, b_i, c, f from Ω to Ω_0. This means that $a_{i,j}^{(0)}$, $b_i^{(0)}$, $c^{(0)}$, $f^{(0)}$ are analytic functions defined in Ω_0. They are approximations of the corresponding coefficients of the initial PDE in the sense of the $\mathcal{C}(\Omega)$-norm,

$$\|u\|_C = \max_{\Omega}\{|u(\mathbf{x})|\} \tag{6}$$

To obtain these extensions we use the so-called *C-expansions* technique developed by Smelov [11], which is briefly described in the next section (see [10] for more details). Note that all the terms on the right hand side of (4) and the solution are supposed to be analytic functions in Ω_0. They are approximated by truncated series using an orthogonal complete system in $\mathcal{L}^2(\Omega_0)$ of smooth global functions $\psi_n(\mathbf{x})$ defined on Ω_0, for example

$$u(\mathbf{x}) = \sum_{n=1}^{M} U_n \psi_n(\mathbf{x}). \tag{7}$$

So, from the point of view of the representation of solution, the method presented belongs to the group of spectral methods.

The additional term on the right hand side of (4) contains the δ-shaped source functions $I(\mathbf{x}|\xi)$ which essentially differ from zero only inside some neighborhood of the source point ξ. The general method of constructing such functions in the form of expansion over a broad class of complete orthogonal systems in $\mathcal{L}^2(\Omega_0)$ is described in [10]. Some examples of such functions are presented in Section 2.

The coefficients q_k are the free parameters of the algorithm. They should be determined from the boundary conditions (3). As it follows from (4) an approximate solution can be written in the form of a linear combination,

$$u(\mathbf{x} \mid q_1, \ldots, q_K) = v(\mathbf{x}) + \sum_{k=1}^{K} q_k \Psi_k(\mathbf{x}), \tag{8}$$

where $v(\mathbf{x})$ is the particular integral and the trial functions $\Psi_k(\mathbf{x})$ satisfy the equation,

$$L^{(0)}(\Psi_k) = I(\mathbf{x} \mid \xi_k), \qquad \mathbf{x} \in \Omega_0, \tag{9}$$

or, if it is regarded on Ω,

$$L(\Psi_k) = \left(L - L^{(0)}\right)(\Psi_k) + I(\mathbf{x} \mid \xi_k) \overset{\text{def}}{=} \epsilon(\mathbf{x}), \qquad \mathbf{x} \in \Omega. \tag{10}$$

Note that if $L^{(0)}$ approximates L well and the source point ξ_k is removed from Ω, then $\|\epsilon\|_C$ is a small value.

We obtain parameters q_k as a solution of the minimization problem,

$$\min_{q_k}\left\{\sum_{i=1}^{N_c}\left[l(u(\mathbf{x}_i \mid q_1, \ldots, q_K)) - g(\mathbf{x}_i)\right]^2\right\}, \tag{11}$$

with $N_c > K$ collocation points $\mathbf{x}_i$ distributed uniformly on the boundary $\partial\Omega$.

It follows from (8)–(11) that the method presented can also be regarded as a generalization of the Trefftz methods [2, 13, 14, 15]. It should be recalled that the main idea of Trefftz-type methods consists in looking for an approximate solution of the boundary value problem,

$$L(u) = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{12}$$

$$l(u) = g(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega, \tag{13}$$

in the form of a linear combination,

$$u(\mathbf{x}) = v(\mathbf{x}) + \sum_{k=1}^{K} q_k \Phi_k(\mathbf{x}). \tag{14}$$

Here the trial functions $\Phi_k(\mathbf{x})$ satisfy exactly the homogeneous PDE $L(\Phi_k) = 0$ but do not necessarily satisfy the boundary condition (13). This condition is used to determine the unknowns q_k.

Comparing (8)–(11) and (12)–(14) we can conclude that the method presented follows the basic scheme of the Trefftz methods. However, it uses trial functions $\Psi_k(\mathbf{x})$ which satisfy the initial PDE only approximately. It becomes a Trefftz method when $\epsilon(\mathbf{x}) = 0$, i.e. when $L^{(0)} \equiv L$ and $I(\mathbf{x} \mid \xi_k) = 0$ for $\mathbf{x} \in \Omega$.

This reduction of the Trefftz requirement on the trial functions extends the field of application of the method because such trial functions can be found for a broad class of differential operators (e.g. see [10]).

This method was first suggested for studying magnetohydrodynamic flows in a complex geometry [9]. In the first publication it was called the Method of Equivalent Charges, and the designation of QTSM (Quasi Trefftz Spectral Method) was adopted from then onwards. QTSM has been used successfully in the solution of certain elliptic boundary value problems [5], initial value problems [6], problems with moving boundaries [7] and the Stokes problem [8].

A brief outline of the paper is as follows. In Section 2 the *C-expansions* and the source functions are described. The main algorithm and numerical results are considered in Section 3. Important particular cases when the basic algorithm can be simplified considerably are discussed briefly in Section 4. A conclusion is given in Section 5.

2. *C-EXPANSIONS* AND δ-SHAPED FUNCTIONS

The first step in applying the method is to extend the coefficients $a_{i,j}(\mathbf{x})$, $b_i(\mathbf{x})$, $c(\mathbf{x})$ and the right hand side $f(\mathbf{x})$ from Ω to Ω_0. We use a *C-expansions* procedure developed by Smelov [11] to obtain these extensions.

According to the *C-expansions* algorithm, the eigenfunctions $\varphi_n(x)$ of a one-dimensional *Sturm–Liouville* problem,

$$\begin{cases} \mathrm{d}_x(p(x)\mathrm{d}_x\varphi) - q(x)\varphi = -\lambda r(x)\varphi, & x \in [A, B], \\ \alpha_1\,\varphi(A) + \beta_1\,\mathrm{d}_x\varphi(A) = 0, & \alpha_2\,\varphi(B) + \beta_2\,\mathrm{d}_x\varphi(B) = 0, \end{cases} \tag{15}$$

defined in an interval $[A, B]$ are used to approximate a smooth enough function $f(x)$ defined in a smaller interval $[\alpha, \beta] \subset [A, B]$.

It assumes a representation of the approximated functions in a form similar to Fourier series,

$$f(x) = \sum_{n=1}^{\infty} F_n \varphi_n(x). \tag{16}$$

The coefficients F_n are obtained as a solution of the minimization problem

$$\min_{F_n} \left\| f(x) - \sum_{n=1}^{\infty} F_n \varphi_n(x) \right\|_{[\alpha,\beta]}^2 \tag{17}$$

because the functions $\varphi_n(x)$ are nonorthogonal on $[\alpha, \beta]$. Here $\| \dots \|_{[\alpha,\beta]}$ denotes some norm defined on $[\alpha, \beta]$.

As it was demonstrated in [11] and [10] a smooth enough function $f(x)$, together with its derivatives defined on $[\alpha, \beta]$, can be well approximated by the following finite sum, using the algorithm of *C-expansions* with a small number of terms,

$$f(x) \simeq \tilde{f}(x \mid F_1, \dots, F_N) = \sum_{n=1}^{N} F_n \varphi_n(x). \tag{18}$$

In practical calculations, to obtain F_n we write conditions

$$\tilde{f}(x_i \mid F_1, \dots F_N) = \sum_{n=1}^{N} F_n \varphi_n(x_i) = f(x_i) \tag{19}$$

at the collocation points x_i, $i = 1, \dots, N_1 > N$ which are distributed uniformly inside $[\alpha, \beta]$. As a result we get an overdetermined linear system,

$$\hat{\mathbf{A}}\mathbf{F} = \mathbf{f}, \quad \hat{\mathbf{A}} = \begin{pmatrix} \varphi_1(x_1) & \dots & \varphi_N(x_1) \\ \vdots & & \vdots \\ \varphi_1(x_{N_1}) & \dots & \varphi_N(x_{N_1}) \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F_1 \\ \vdots \\ F_N \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_{N_1}) \end{pmatrix}, \tag{20}$$

which is solved in the least squares sense,

$$\mathbf{F} : \quad \min \| \hat{\mathbf{A}}\mathbf{F} - \mathbf{f} \|. \tag{21}$$

Here $\|\mathbf{z}\|$ denotes the euclidean norm of $\mathbf{z}$.

In this work we use the following two systems of orthogonal functions,

$$\varphi_n^{(1)}(x) = \sin(n\pi x) \quad \text{and} \quad \varphi_n^{(2)}(x) = \cos((n-1)\pi x), \quad n = 1, \dots, +\infty, \tag{22}$$

which are solutions of the *Sturm-Liouville* problems (15) with boundary conditions $\varphi_n^{(1)}(0) = \varphi_n^{(1)}(1) = 0$ and $\mathrm{d}_x \varphi_n^{(2)}(0) = \mathrm{d}_x \varphi_n^{(2)}(1) = 0$.

As an example of such technique we present the results obtained by applying the *C-expansions* procedure to the following three functions defined on $[\alpha, \beta] = [0.3, 0.7]$,

$$a(x) = 1 + x^2, \qquad b(x) = \sin^2(x), \qquad c(x) = e^x. \tag{23}$$

To estimate the accuracy we use the maximal absolute error defined as

$$e_{\mathrm{a}} = \max_{j=1,\dots,N_2} \left| f(x_j) - \tilde{f}(x_j \mid F_1, \dots, F_N) \right|. \tag{24}$$

Here x_j, $j = 1, \dots, N_2$, are the checking points distributed uniformly inside $[\alpha, \beta]$. For this we decompose $[\alpha, \beta]$ into N_2 subintervals and place x_j in the middle of each interval.

Table 1. Maximal absolute error e_a

N	$\varphi_n^{(1)}(x)$			$\varphi_n^{(2)}(x)$		
	$a(x)$	$b(x)$	$c(x)$	$a(x)$	$b(x)$	$c(x)$
5	$4.8 \cdot 10^{-3}$	$2.4 \cdot 10^{-3}$	$7.2 \cdot 10^{-3}$	$4.0 \cdot 10^{-4}$	$1.9 \cdot 10^{-4}$	$6.6 \cdot 10^{-4}$
10	$3.0 \cdot 10^{-5}$	$8.5 \cdot 10^{-6}$	$3.9 \cdot 10^{-5}$	$7.2 \cdot 10^{-7}$	$3.3 \cdot 10^{-7}$	$7.8 \cdot 10^{-7}$
15	$7.5 \cdot 10^{-8}$	$3.6 \cdot 10^{-8}$	$1.1 \cdot 10^{-7}$	$2.0 \cdot 10^{-9}$	$9.1 \cdot 10^{-10}$	$3.2 \cdot 10^{-9}$
20	$6.3 \cdot 10^{-10}$	$1.8 \cdot 10^{-10}$	$8.2 \cdot 10^{-10}$	$7.7 \cdot 10^{-12}$	$3.5 \cdot 10^{-12}$	$8.6 \cdot 10^{-12}$

In all the computations, presented in Table 1, we take $N_1 = 100$ collocation points in (19) and $N_2 = 1000$ checking points in the calculation of e_a.

It should be underlined that the *C-expansions* procedure provides a high precision inside the smaller interval $[\alpha, \beta]$ only. For example, all the *C-expansions* which use $\sin(n\pi x)$ as a basis system become equal to zero at the endpoints of the interval $[0,1]$ independently of the functions being approximated.

The same algorithm can also be applied in the two-dimensional case. For more detailed information and numerical examples see [10].

Now let us consider δ–shaped source functions on the right hand side of (4). A general procedure of constructing such functions in the form of truncated series over a broad class of eigenfunctions is described in [10]. In particular, it is shown there that, if $\{\varphi_n(x), \lambda_n\}$ is a solution of a *Sturm-Liouville* problem (15), then a δ–shaped source function can be represented in the form,

$$I(x \mid \xi) = \beta_0(\xi) + \sum_{n=1}^{M} \left[\frac{\sin \nu(n,M)}{\nu(n,M)}\right]^l \beta_n(\xi)\,\varphi_n(x), \tag{25}$$

$$\beta_n(\xi) = r(\xi)\varphi_n(\xi)/g_n\,, \qquad g_n = \int_A^B r(x)\varphi_n^2(x)\,\mathrm{d}x,$$

$$\nu(n,M) = \pi\frac{\sqrt{\lambda_n}}{n\gamma(M+1)}, \qquad \gamma = \lim_{n\to\infty}\left(\sqrt{\lambda_n}/n\right).$$

The first term in (25) is absent when there is no zero eigenvalue.

It is shown in [10] that (25) can be regarded as the result of applying the Riemann (R,l)-method of summation to the divergent series, see [1],

$$\delta(x-\xi) = \sum_{n=0}^{\infty} \beta_n(\xi)\varphi_n(x). \tag{26}$$

In this paper we use the orthogonal system $\varphi_n^{(1)}(x) = \sin(n\pi x)$ to obtain the source functions. In this particular case (25) is replaced by Lanczos σ-factors method,

$$I(x \mid \xi) \;=\; \sum_{n=1}^{M} c_n(\xi)\sin \pi n x, \tag{27}$$

$$c_n(\xi) \;=\; \frac{1}{2} r_n(l,M)\sin \pi n\xi, \tag{28}$$

$$r_n(l,M) = (\sigma_n(M))^l, \qquad \sigma_n(M) = \sin\frac{n\pi}{(M+1)} \Big/ \frac{n\pi}{(M+1)}\,, \tag{29}$$

where $\sigma_n(M)$ are Lanczos σ-factors, see [4].

The normalized value

$$\hat{I}(x \mid \xi) = \left|\frac{I(x \mid \xi)}{I(\xi \mid \xi)}\right| \tag{30}$$

Table 2. Normalized δ–shaped source function $\widetilde{I}(x \mid \xi)$ in the form of expansion over $\varphi_n^{(1)}(x)$, with source centered at point $\xi = 0.5$

x	Parameters of expansion				
	$M = 10, l = 4$	$M = 15, l = 6$	$M = 20, l = 6$	$M = 50, l = 8$	$m = 100, l = 8$
0.5	$1.0 \cdot 10^{0}$	$1.0 \cdot 10^{0}$	$1.0 \cdot 10^{0}$	$1.0 \cdot 10^{0}$	$1.0 \cdot 10^{0}$
0.55	$9.1 \cdot 10^{-1}$	$8.6 \cdot 10^{-1}$	$8.2 \cdot 10^{-1}$	$3.1 \cdot 10^{-1}$	$6.3 \cdot 10^{-3}$
0.60	$6.7 \cdot 10^{-1}$	$5.5 \cdot 10^{-1}$	$4.6 \cdot 10^{-1}$	$5.6 \cdot 10^{-3}$	$6.6 \cdot 10^{-9}$
0.65	$4.0 \cdot 10^{-1}$	$2.6 \cdot 10^{-1}$	$1.6 \cdot 10^{-1}$	$1.2 \cdot 10^{-7}$	$1.6 \cdot 10^{-10}$
0.70	$1.8 \cdot 10^{-1}$	$8.1 \cdot 10^{-2}$	$3.5 \cdot 10^{-2}$	$7.5 \cdot 10^{-9}$	$6.0 \cdot 10^{-12}$
0.75	$6.0 \cdot 10^{-2}$	$1.5 \cdot 10^{-2}$	$3.8 \cdot 10^{-3}$	$3.4 \cdot 10^{-10}$	$2.1 \cdot 10^{-13}$
0.80	$1.0 \cdot 10^{-2}$	$1.2 \cdot 10^{-3}$	$1.3 \cdot 10^{-4}$	$9.4 \cdot 10^{-11}$	$4.8 \cdot 10^{-14}$
0.90	$1.8 \cdot 10^{-4}$	$4.0 \cdot 10^{-6}$	$4.6 \cdot 10^{-8}$	$7.9 \cdot 10^{-13}$	$1.2 \cdot 10^{-14}$

is presented in Table 2 for different parameters M and l. The source point is $\xi = 0$. One can see that $I(x \mid \xi)$ has a δ-shaped form.

Using (27) the two-dimensional source function in (4) can be written as a product

$$I(x, y \mid \xi, \eta) = I(x \mid \xi)\, I(y \mid \eta) = \sum_{n,m=1}^{M} c_{nm}(\xi, \eta) \sin(n\pi x) \sin(m\pi y), \tag{31}$$

$$c_{nm}(\xi, \eta) \quad = c_n(\xi)\, c_m(\eta). \tag{32}$$

A general method of constructing such source functions is presented in [10].

3. Main algorithm

The material of this section is divided into four parts, addressing the formulation and the solution of one- and two-dimensional eliptic PDE.

3.1. One–dimensional problems

It is reasonable, for the sake of simplicity, to consider first the following two–point value problem, where $a(x)$, $b(x)$, $c(x)$, $f(x)$ are analytic functions on $[\alpha, \beta]$,

$$a(x) \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} + b(x) \frac{\mathrm{d}u(x)}{\mathrm{d}x} + c(x) u(x) = f(x), \qquad 0 < \alpha < x < \beta < 1. \tag{33}$$

The boundary conditions are

$$\beta_1 \frac{\mathrm{d}u(\alpha)}{\mathrm{d}x} + \alpha_1 u(\alpha) = g_1, \qquad \beta_2 \frac{\mathrm{d}u(\beta)}{\mathrm{d}x} + \alpha_2 u(\beta) = g_2, \qquad \alpha_i^2 + \beta_i^2 \neq 0, \quad i = 1, 2. \tag{34}$$

According to the general scheme described above we replace (33) by,

$$a^{(0)}(x) \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} + b^{(0)}(x) \frac{\mathrm{d}u(x)}{\mathrm{d}x} + c^{(0)}(x)\, u(x) = f^{(0)}(x) + q_1 I(x|\xi_1) + q_2 I(x|\xi_2), \qquad 0 \leq x \leq 1, \tag{35}$$

where the one-dimensional source function $I(x \mid \xi)$ is given by (27). The source points ξ_1 and ξ_2 are constrained to: $0 < \xi_1 < \alpha$, $\beta < \xi_2 < 1$.

Functions $a^{(0)}(x)$, $b^{(0)}(x)$, $c^{(0)}(x)$, $f^{(0)}(x)$ result from applying the *C-expansion* procedure to the coefficients of (33), where $\varphi_n^{(1)}$, $\varphi_n^{(2)}$ are given by (22):

$$a^{(0)}(x) = \sum_{k=1}^{N} A_k \varphi_n^{(2)}(x), \tag{36}$$

$$b^{(0)}(x) = \sum_{k=1}^{N} B_k \varphi_n^{(1)}(x), \tag{37}$$

$$c^{(0)}(x) = \sum_{k=1}^{N} C_k \varphi_n^{(2)}(x), \tag{38}$$

$$f^{(0)}(x) = \sum_{m=1}^{M} F_m \varphi_n^{(1)}(x), \tag{39}$$

We look for an approximate solution of (35) in the form of a truncated series

$$u(x) = \sum_{m=1}^{M} U_m \, \varphi_n^{(1)}(x) \equiv \sum_{m=1}^{M} U_m \, \sin(m\pi x). \tag{40}$$

Substituting (36)–(40), (27) in (35) and integrating on $[0,1]$ with weight functions $\sin(n\pi x)$, $n = 1, \ldots, M$, we obtain the linear system

$$\hat{\mathbf{S}}\bar{\mathbf{U}} = \bar{\mathbf{F}} + q_1\bar{\mathbf{C}}^{(1)} + q_2\bar{\mathbf{C}}^{(2)} \tag{41}$$

with the $M \times M$ system matrix

$$\hat{\mathbf{S}} \equiv \{S_{nm}\}_{n,m=1}^{M}, \qquad S_{nm} = S_{nm}^{a} + S_{nm}^{b} + S_{nm}^{c}\,, \tag{42}$$

$$S_{nm}^{a} = -(m\pi)^2 \sum_{k=0}^{N-1} A_k \int_0^1 \cos(k\pi x)\sin(n\pi x)\,\sin(m\pi x)\,\mathrm{d}x \equiv -(m\pi)^2 \sum_{k=0}^{N-1} A_k Z_{k,n,m}^{(1)}\,, \tag{43}$$

$$S_{nm}^{b} = m\pi \sum_{k=1}^{N} B_k \int_0^1 \cos(m\pi x)\,\sin(k\pi x)\,\sin(n\pi x)\,\mathrm{d}x \equiv m\pi \sum_{k=1}^{N} B_k Z_{m,k,n}^{(1)}\,, \tag{44}$$

$$S_{nm}^{c} = \sum_{k=0}^{N-1} C_k \int_0^1 \cos(k\pi x)\,\sin(n\pi x)\,\sin(m\pi x)\,\mathrm{d}x \equiv \sum_{k=0}^{N-1} C_k Z_{k,n,m}^{(1)}\,. \tag{45}$$

Vector $\bar{\mathbf{U}}$ contains the unknown coefficients of (40). The vectors on the right hand side are

$$\bar{\mathbf{F}} = [F_1 \ldots, F_M], \qquad \bar{\mathbf{C}}^{(i)} = \left[C_1^{(i)}, \ldots, C_M^{(i)}\right], \quad i = 1, 2, \tag{46}$$

$$C_n^{(i)} = c_n(\xi_i), \quad i = 1, 2. \tag{47}$$

The coefficients $c_n(\xi)$ are given by (28).

It is easy to prove that

$$Z_{k,n,m}^{(1)} = 0.25\,[\Delta(k{+}n{-}m) + \Delta(k{+}m{-}n) - \Delta(m{+}n{-}k) - \Delta(k{+}n{+}m)]\,, \tag{48}$$

$$\Delta(k) = \delta_{k0} = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{49}$$

The term $\Delta(k+n+m)$ can be neglected because $n, m > 0$. Moreover, analyzing the expression $Z^{(1)}_{k,n,m}$, one can see that the entries in $\hat{\mathbf{S}}$ will be zero whenever $|n-m| \geq N+1$. This means that matrix $\hat{\mathbf{S}}$ is a banded matrix with $2N+1$ non-zero diagonals.

Solving the linear system (41), we obtain the solution in the form of a linear combination of the free parameters q_1 and q_2,

$$\bar{\mathbf{U}} = \bar{\mathbf{V}} + q_1\bar{\mathbf{D}}^{(1)} + q_2\bar{\mathbf{D}}^{(2)}, \tag{50}$$

where vectors

$$\bar{\mathbf{V}} = [V_1, \dots, V_M], \qquad \bar{\mathbf{D}}^{(1)} = \left[D_1^{(1)}, \dots, D_M^{(1)}\right], \qquad \bar{\mathbf{D}}^{(2)} = \left[D_1^{(2)}, \dots, D_M^{(2)}\right]$$

are the solution of the linear system with the same matrix $\hat{\mathbf{S}}$ and the right hand sides $\bar{\mathbf{F}}$, $\bar{\mathbf{C}}^{(1)}$ and $\bar{\mathbf{C}}^{(2)}$ respectively.

So, an approximate solution can be written in the form

$$u(x \mid q_1, q_2) = v(x) + q_1\Psi_1(x) + q_2\Psi_2(x) \tag{51}$$

where the particular integral is

$$v(x) = \sum_{m=1}^{M} V_m \sin(m\pi x) \tag{52}$$

and the trial functions are

$$\Psi_1(x) = \sum_{m=1}^{M} D_m^{(1)} \sin(m\pi x), \qquad \Psi_2(x) = \sum_{m=1}^{M} D_m^{(2)} \sin(m\pi x). \tag{53}$$

The free parameters q_1 and q_2 should be determined from the boundary conditions. Substituting (51), (52) in (34), we obtain the system of linear equations

$$\hat{\mathbf{A}}\bar{\mathbf{q}} = \bar{\mathbf{b}}, \tag{54}$$

$$\hat{\mathbf{A}} = \begin{pmatrix} \alpha_1\Psi_1(\alpha) + \beta_1 \dfrac{\mathrm{d}\Psi_1(\alpha)}{\mathrm{d}x} & \alpha_1\Psi_2(\alpha) + \beta_1 \dfrac{\mathrm{d}\Psi_2(\alpha)}{\mathrm{d}x} \\ \alpha_2\Psi_1(\beta) + \beta_2 \dfrac{\mathrm{d}\Psi_1(\beta)}{\mathrm{d}x} & \alpha_2\Psi_2(\beta) + \beta_2 \dfrac{\mathrm{d}\Psi_2(\beta)}{\mathrm{d}x} \end{pmatrix}, \tag{55}$$

$$\bar{\mathbf{b}} = \{g_1 - v(\alpha)\,,\ g_2 - v(\beta)\}^T. \tag{56}$$

After solving this system and determining q_1 and q_2 we obtain the coefficients of the approximate solution (40)

$$U_m = V_m + q_1 D_m^{(1)} + q_2 D_m^{(2)}. \tag{57}$$

Let us refer to the choice of free parameters of the algorithm. These are: M, the number of harmonics in (39), (40); N, the number of terms in the *C-expansions* of the coefficients (36)–(38); ξ_1, ξ_2, the coordinates of the source points; l, the parameter of the source function $I(x\,|\,\xi)$, see (27)–(29). As far as M and N are concerned their influence on the solution is quite clear. Their increase reduces, in general, the error in the approximate solution. For example, the increase of N improves the approximation of $a(x)$, $b(x)$, $c(x)$ and that leads to the decrease of the first term of $\epsilon(\mathbf{x})$, which is connected with the approximation of the initial differential operator L, see (10). It is clear that these two parameters should be related. For example, if N is small, then it is impossible to obtain a very accurate solution by increasing M because of the errors in the approximation of $a(x)$, $b(x)$, $c(x)$.

In the present case, the influence of the parameters ξ_1, ξ_2 and l is analogous to the one described in [10], where the particular case of separable PDEs is considered. In particular:

1. The error due to the additional term on the right hand side of (35) can be reduced by removing the source points from the boundary of the solution domain Ω as far as the dimensions of Ω_0 permit. So, in the particular case which is considered, the most accurate solutions are achieved when $\xi_1 \simeq 0$ and $\xi_2 \simeq 1$;

2. For every fixed M, N, ξ_1 and ξ_2 there exists an optimal $l = l_{\text{opt}}$ which provides the minimal error in the solution;

3. The increase of M leads to the increase of the corresponding l_{opt}.

3.2. Numerical applications for one-dimensional problems

In order to illustrate these statements we consider the problem

$$(1+x^2)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \sin^2(x)\,\frac{\mathrm{d}u}{\mathrm{d}x} + \exp(x)\,u = f(x), \qquad u(0.3) = g_1\,, \qquad u(0.7) = g_2\,. \tag{58}$$

In this case, functions $a(x)$, $b(x)$, $c(x)$ are the same as in (23).

The right–hand side term $f(x)$ and the values g_1 and g_2 are chosen here for the exact solution:

$$u_{\text{ex}}(x) = \exp\left(-100(x-0.5)^2\right) \tag{59}$$

In Table 3 we present the scaled maximal absolute error e_{sa} defined as follows,

$$e_{\text{sa}} = \frac{1}{u_{\text{s}}} \max_{j=1,\ldots,N_2} |u_{\text{ex}}(x_j) - u(x_j)|\,, \tag{60}$$

where the scaling value is

$$u_{\text{s}} = \sqrt{\frac{1}{\beta-\alpha}\int_\alpha^\beta u_{\text{ex}}^2(x)\,\mathrm{d}x}. \tag{61}$$

We use $N_2 = 1000$ checking points inside $[\alpha, \beta]$ to compute e_{sa}.

Table 3. Maximal absolute errors e_{sa}

l	$d\xi = 0.05$	$d\xi = 0.10$	$d\xi = 0.15$	$d\xi = 0.20$	$d\xi = 0.25$
0	$1.2\cdot10^{-1}$	$7.2\cdot10^{-2}$	$5.6\cdot10^{-2}$	$5.0\cdot10^{-2}$	$4.7\cdot10^{-2}$
1	$2.2\cdot10^{-2}$	$1.7\cdot10^{-2}$	$1.3\cdot10^{-2}$	$1.1\cdot10^{-2}$	$1.0\cdot10^{-2}$
2	$1.7\cdot10^{-1}$	$5.0\cdot10^{-3}$	$2.0\cdot10^{-3}$	$9.3\cdot10^{-4}$	$6.2\cdot10^{-4}$
3	$3.6\cdot10^{-1}$	$2.1\cdot10^{-2}$	$4.5\cdot10^{-4}$	$1.1\cdot10^{-4}$	$2.0\cdot10^{-4}$
4	$5.5\cdot10^{-1}$	$7.5\cdot10^{-2}$	$2.0\cdot10^{-3}$	$2.4\cdot10^{-4}$	$5.7\cdot10^{-5}$
5	$7.3\cdot10^{-1}$	$1.5\cdot10^{-1}$	$1.4\cdot10^{-2}$	$2.4\cdot10^{-4}$	$1.1\cdot10^{-4}$
6	$9.1\cdot10^{-1}$	$2.4\cdot10^{-1}$	$3.7\cdot10^{-2}$	$3.4\cdot10^{-3}$	$1.2\cdot10^{-4}$

The value $\mathrm{d}\xi$ is the distance between the source and the nearest boundary point

$$\mathrm{d}\xi = a - \xi_1 = \xi_2 - b.$$

The results presented in Table 3 are obtained for $M = 20$ and $N = 20$.

One can see that the optimal values l_{opt} are 1, 2, 3, 3, 4 for distances $\mathrm{d}\xi$ equal to 0.05, 0.10, 0.15, 0.20, 0.25 respectively. Note that $l = 0$ corresponds to the case when the source function $I(x\,|\,\xi)$ is taken in the form of truncated formal series for the Dirac δ-function

$$I(x\,|\,\xi) = \frac{1}{2}\sum_{n=1}^{M} \sin(n\pi\xi)\sin(n\pi x). \tag{62}$$

The numerical experiments show that the optimal parameter l_{opt} is slightly different for different problems. However, the values $l = 4$ for $M \le 20$ and $l = 6$ for $M \ge 30$ when $d\xi \simeq 0.2$ seem suitable in all the cases. These values are also used in all the two-dimensional problems considered in the next subsections.

3.3. Two-dimensional problems

As mentioned above we replace the initial PDE by Eq. (4). We take the following *C-expansions* to approximate the coefficients,

$$a_{1,1}^{(0)}(x,y) = \sum_{k,l=1}^{N} A_{k,l}^{(11)} \varphi_k^{(2)}(x)\, \varphi_l^{(2)}(y), \tag{63}$$

$$a_{1,2}^{(0)}(x,y) = a_{2,1}^{(0)}(x,y) = \sum_{k,l=1}^{N} A_{k,l}^{(12)} \varphi_k^{(1)}(x)\, \varphi_l^{(1)}(y), \tag{64}$$

$$a_{2,2}^{(0)}(x,y) = \sum_{k,l=1}^{N} A_{k,l}^{(22)} \varphi_k^{(2)}(x)\, \varphi_l^{(2)}(y), \tag{65}$$

$$b_1^{(0)}(x,y) = \sum_{k,l=1}^{N} B_{k,l}^{(1)} \varphi_k^{(1)}(x)\, \varphi_l^{(2)}(y), \tag{66}$$

$$b_2^{(0)}(x,y) = \sum_{k,l=1}^{N} B_{k,l}^{(2)} \varphi_k^{(2)}(x)\, \varphi_l^{(1)}(y), \tag{67}$$

$$c^{(0)}(x,y) = \sum_{k,l=1}^{N} C_{k,l}\, \varphi_k^{(2)}(x)\, \varphi_l^{(2)}(y), \tag{68}$$

$$f^{(0)}(x,y) = \sum_{n,m=1}^{M} F_{n,m}\, \varphi_n^{(1)}(x)\, \varphi_m^{(1)}(y), \tag{69}$$

and look for an approximate solution in the form

$$u(x,y) = \sum_{n,m=1}^{M} U_{n,m}\, \varphi_n^{(1)}(x)\, \varphi_m^{(1)}(y). \tag{70}$$

Multiplying Eq. (4) by $\varphi_n^{(1)}(x)\varphi_m^{(1)}(y)$ and integrating the resulting equation over the square Ω_0, we obtain a linear system similar to (41),

$$\sum_{i,j=1}^{M} S_{n,m}^{i,j} U_{i,j} = F_{n,m} + \sum_{k=1}^{K} q_k c_{n,m}\,. \tag{71}$$

Matrix $S_{n,m}^{i,j}$ is the sum of the 7 terms,

$$S_{n,m}^{i,j} = S_{n,m}^{i,j}(a_{1,1}) + S_{n,m}^{i,j}(a_{1,2}) + S_{n,m}^{i,j}(a_{2,1}) + S_{n,m}^{i,j}(a_{2,2}) + S_{n,m}^{i,j}(b_1) + S_{n,m}^{i,j}(b_2) + S_{n,m}^{i,j}(c), \tag{72}$$

each of which corresponds to one of the terms in (4). For example

$$\begin{aligned} S_{n,m}^{i,j}(a_{1,1}) &= -\int_0^1\int_0^1 \frac{\partial}{\partial x}\left[a_{1,1}^{(0)}(x,y)\,\frac{\partial}{\partial x}\left(\varphi_i^{(1)}(x)\,\varphi_j^{(1)}(y)\right)\right]\varphi_n^{(1)}(x)\,\varphi_m^{(1)}(y) \\ &= -\sum_{k,l=1}^{N} A_{k,l}^{(11)}\left(\int_0^1 \frac{\partial}{\partial x}\left[\varphi_k^{(2)}(x)\,\frac{\partial\varphi_i^{(1)}(x)}{\partial x}\right]\varphi_n^{(1)}(x)\,\mathrm{d}x\right)\left(\int_0^1 \varphi_l^{(2)}(y)\,\varphi_j^{(1)}(y)\,\varphi_m^{(1)}(y)\,\mathrm{d}y\right). \end{aligned} \tag{73}$$

The second integral is defined by (48)–(49),

$$\int_0^1 \varphi_l^{(2)}(y)\,\varphi_j^{(1)}(y)\,\varphi_m^{(1)}(y)\,\mathrm{d}y = \int_0^1 \cos[(l-1)\pi y]\,\sin(j\pi y)\,\sin(m\pi y)\,\mathrm{d}y = Z_{l-1,j,m}^{(1)}. \tag{74}$$

The first integral is

$$\begin{aligned} &\int_0^1 \frac{\partial}{\partial x}\left[\cos((k-1)\pi x)\,\frac{\partial}{\partial x}\sin(i\pi x)\right]\sin(n\pi x)\,\mathrm{d}x \\ &\quad = -in\pi^2\int_0^1 \cos((k-1)\pi x)\,\cos(i\pi x)\,\cos(n\pi x)\,\mathrm{d}x \stackrel{\text{def}}{=} -in\pi^2 Z_{k-1,i,n}^{(2)}. \end{aligned} \tag{75}$$

It is easy to prove the following result, where $\Delta(k)$ is defined by (49)

$$Z_{k,n,m}^{(2)} = 0.25\,[\Delta(k+n-m) + \Delta(k+m-n) + \Delta(m+n-k), +\Delta(k+n+m)]. \tag{76}$$

Therefore, the first addend in the matrix term $S_{n,m}^{i,j}$ can be written in the form

$$S_{n,m}^{i,j}(a_{1,1}) = in\pi^2\sum_{k,l=1}^{N} A_{k,l}^{(11)}\,Z_{k-1,i,n}^{(2)}\,Z_{l-1,j,m}^{(1)}. \tag{77}$$

We obtain the other terms in (72) in a similar way,

$$S_{n,m}^{i,j}(a_{1,2}) = jn\pi^2\sum_{k,l=1}^{N} A_{k,l}^{(12)}\,Z_{n,k,i}^{(1)}\,Z_{j,l,m}^{(1)}, \tag{78}$$

$$S_{n,m}^{i,j}(a_{2,1}) = im\pi^2\sum_{k,l=1}^{N} A_{k,l}^{(21)}\,Z_{i,k,n}^{(1)}\,Z_{m,l,j}^{(1)}, \tag{79}$$

$$S_{n,m}^{i,j}(a_{2,2}) = jm\pi^2\sum_{k,l=1}^{N} A_{k,l}^{(22)}\,Z_{k-1,i,n}^{(1)}\,Z_{l-1,j,m}^{(2)}, \tag{80}$$

$$S_{n,m}^{i,j}(b_1) = i\pi\sum_{k,l=1}^{N} B_{k,l}^{(1)}\,Z_{i,k,n}^{(1)}\,Z_{l-1,j,m}^{(1)}, \tag{81}$$

$$S_{n,m}^{i,j}(b_2) = j\pi\sum_{k,l=1}^{N} B_{k,l}^{(2)}\,Z_{k-1,i,n}^{(1)}\,Z_{j,l,m}^{(1)}, \tag{82}$$

$$S_{n,m}^{i,j}(c) = \sum_{k,l=1}^{N} C_{k,l}\,Z_{k-1,i,n}^{(1)}\,Z_{l-1,j,m}^{(1)}. \tag{83}$$

System (71) can be written in a form similar to (41), where $\bar{\mathbf{F}}$ and $\bar{\mathbf{C}}^{(k)}$ are known $M \times M$ vectors and q_k are free parameters,

$$\hat{\mathbf{S}}\bar{\mathbf{U}} = \bar{\mathbf{F}} + \sum_{k=1}^{K} q_k \bar{\mathbf{C}}^{(k)}. \tag{84}$$

Solving (84), we obtain a solution in the form of a linear combination,

$$\bar{\mathbf{U}} = \bar{\mathbf{V}} + \sum_{k=1}^{K} q_k \bar{\mathbf{D}}^{(k)}, \tag{85}$$

where $\bar{\mathbf{V}}$ and $\bar{\mathbf{D}}^{(k)}$ are the solution of the linear system with the same matrix $\hat{\mathbf{S}}$ and right hand sides $\bar{\mathbf{F}}$ and $\bar{\mathbf{C}}^{(k)}$, respectively.

Therefore, an approximate solution of (4) can be written in the form,

$$u(x, y \mid q_1, \dots, q_K) = v(x, y) + \sum_{k=1}^{K} q_k \Psi_k(x, y), \tag{86}$$

where the particular solution is

$$v(x, y) = \sum_{n,m=1}^{M} V_{n,m} \sin(n\pi x) \sin(m\pi y) \tag{87}$$

and the trial functions are

$$\Psi_k(x, y) = \sum_{n,m=1}^{M} D_{n,m}^{(k)} \sin(n\pi x) \sin(m\pi y). \tag{88}$$

The free parameters q_k are determined from the boundary condition (3), written at the collocation points (x_j, y_j), $j = 1, \dots, N_c$, which are distributed uniformly on $\partial\Omega$. The linear system with N_c equations and K unknowns is solved by the least squares procedure.

Note that, in general, any products of the basis functions $\varphi_k^{(r)}(x)\, \varphi_l^{(s)}(y)$, $r, s = 1, 2$, can be used in the *C-expansions* as (63)–(70). We obtain the matrix elements $S_{n,m}^{i,j}$ from analytical expressions, without numerical integration. However, the *C-expansions* of this particular form lead to a band matrix in system (85). This follows from the fact that

$$Z_{k,i,n}^{(s)} = 0, \quad s = 1, 2, \qquad \text{when } |i - n| > N.$$

3.4. Numerical results for two-dimensional problems

In this subsection, we present the results of the numerical experiments conducted on a representative sample of 12 boundary value problems listed below. In all the examples the solution domain Ω is the disk of the radius 0.25 centered at point (0.5, 0.5). The number of harmonics N in the *C-expansions* is $N = 10$ for $M = 10$ and $N = 15$ for $M > 10$. To approximate the boundary conditions we take $K = 40$ source points and $N_c = 100$ collocation points uniformly placed on boundary $\partial\Omega$.

In Table 4 we present the maximal absolute error (24) and the mean square root error,

$$e_{\text{sr}} = \sqrt{\frac{1}{N_2} \sum_{j=1}^{N_2} \left[u(x_j, y_j) - u_{\text{ex}}(x_j, y_j)\right]^2}, \tag{89}$$

Table 4. Maximal absolute error e_a and mean square root error e_{sr}

Problem	$M = 10$		$M = 20$		$M = 30$	
	e_a	e_{sr}	e_a	e_{sr}	e_a	e_{sr}
1	$8.2 \cdot 10^{-4}$	$3.8 \cdot 10^{-4}$	$2.3 \cdot 10^{-6}$	$1.4 \cdot 10^{-6}$	$1.6 \cdot 10^{-8}$	$5.4 \cdot 10^{-9}$
2	$1.3 \cdot 10^{-3}$	$6.4 \cdot 10^{-4}$	$3.5 \cdot 10^{-6}$	$1.5 \cdot 10^{-6}$	$2.7 \cdot 10^{-8}$	$8.5 \cdot 10^{-9}$
3	$5.6 \cdot 10^{-4}$	$2.5 \cdot 10^{-4}$	$1.4 \cdot 10^{-6}$	$5.7 \cdot 10^{-7}$	$1.1 \cdot 10^{-8}$	$3.7 \cdot 10^{-9}$
4	$1.1 \cdot 10^{-3}$	$5.7 \cdot 10^{-4}$	$1.5 \cdot 10^{-6}$	$8.0 \cdot 10^{-7}$	$5.2 \cdot 10^{-8}$	$1.9 \cdot 10^{-8}$
5	$3.4 \cdot 10^{-4}$	$2.0 \cdot 10^{-4}$	$6.4 \cdot 10^{-6}$	$3.6 \cdot 10^{-6}$	$3.0 \cdot 10^{-9}$	$1.5 \cdot 10^{-9}$
6	$3.1 \cdot 10^{-3}$	$1.7 \cdot 10^{-3}$	$6.4 \cdot 10^{-6}$	$4.7 \cdot 10^{-6}$	$5.3 \cdot 10^{-8}$	$1.6 \cdot 10^{-8}$
7	$2.9 \cdot 10^{-3}$	$1.3 \cdot 10^{-3}$	$2.0 \cdot 10^{-5}$	$1.1 \cdot 10^{-5}$	$2.5 \cdot 10^{-7}$	$1.3 \cdot 10^{-7}$
8	$3.8 \cdot 10^{-2}$	$3.7 \cdot 10^{-2}$	$8.2 \cdot 10^{-5}$	$7.2 \cdot 10^{-5}$	$7.8 \cdot 10^{-7}$	$7.5 \cdot 10^{-7}$
9	$5.2 \cdot 10^{-3}$	$2.8 \cdot 10^{-3}$	$3.7 \cdot 10^{-6}$	$2.1 \cdot 10^{-6}$	$1.2 \cdot 10^{-7}$	$6.3 \cdot 10^{-8}$
10	$8.0 \cdot 10^{-1}$	$7.8 \cdot 10^{-1}$	$3.4 \cdot 10^{-4}$	$3.0 \cdot 10^{-4}$	$3.7 \cdot 10^{-6}$	$3.5 \cdot 10^{-6}$
11	$3.0 \cdot 10^{-2}$	$3.7 \cdot 10^{-2}$	$3.2 \cdot 10^{-4}$	$3.0 \cdot 10^{-4}$	$3.3 \cdot 10^{-6}$	$3.2 \cdot 10^{-6}$
12	$8.0 \cdot 10^{-1}$	$7.8 \cdot 10^{-1}$	$1.3 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$	$1.5 \cdot 10^{-5}$	$1.5 \cdot 10^{-5}$

where u_{ex} denotes the exact solution of the problem considered. We use $N_2 = 1000$ test points inside Ω to calculate e_a and e_{sr}. All the calculations were performed with double precision.

Problems 1 to 7 correspond to Dirichlet conditions,

$$u(x,y) = g(x,y), \tag{90}$$

where function $g(x,y)$ is taken in accordance with a given exact solution $u_{ex}(x,y)$.

Problem 1: $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = f(x,y)$ $\qquad u_{ex}(x,y) = e^x \sin(y)$

Problem 2: $-\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} + x^2y^2u = f(x,y)$ $\qquad u_{ex}(x,y) = e^{x+y}\sin(x-y)$

Problem 3: $-\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial}{\partial y}\left(\cos(y)\dfrac{\partial u}{\partial y}\right) - e^y u = f(x,y)$ $\qquad u_{ex}(x,y) = \cos(10x)(y^2 - y)$

Problem 4: $-\dfrac{\partial}{\partial x}\left(e^{x+y}\dfrac{\partial u}{\partial y}\right) - \dfrac{\partial}{\partial y}\left(e^{x+y}\dfrac{\partial u}{\partial y}\right) = f(x,y)$ $\qquad u_{ex}(x,y) = 1 - 3x^3y^3 + x^5$

Problem 5: $-\dfrac{\partial}{\partial x}\left(e^{x+y}\dfrac{\partial u}{\partial y}\right) - \dfrac{\partial}{\partial y}\left(e^{x+y}\dfrac{\partial u}{\partial y}\right) + (1+x^2+y^2)u = f(x,y)$ $\qquad u_{ex}(x,y) = xy$

Problem 6: $-\dfrac{\partial}{\partial x}\left(e^x\dfrac{\partial u}{\partial y}\right) - \dfrac{\partial}{\partial y}\left((1+y)\dfrac{\partial u}{\partial y}\right) + e^x\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = f(x,y)$ $\qquad u_{ex}(x,y) = 1 + x + y + xy + x^2 + y^2$

Problem 7: $-\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial}{\partial x}\left(e^{-(x+y)}\dfrac{\partial u}{\partial y}\right) - \dfrac{\partial}{\partial y}\left(e^{-(x+y)}\dfrac{\partial u}{\partial x}\right) - \dfrac{\partial^2 u}{\partial y^2} + \sin(x+y)\dfrac{\partial u}{\partial x} + e^x\sin(y)\dfrac{\partial u}{\partial y} + (1+x^2+y^2)u = f(x,y)$ $\qquad u_{ex}(x,y) = 1 + x + y$

Problems 8 to 12 correspond to mixed boundary conditions (3) which include the normal derivative. For problems 8 to 10 the governing equation and the exact solution are the same as in problems 1, 3 and 6 respectively. The functions $\alpha(x,y)$ and $\beta(x,y)$ in boundary conditions are taken as constant,

$$\frac{\partial u(x,y)}{\partial n} - u(x,y) = g(x,y) \tag{91}$$

and the function $g(x,y)$ is derived from the exact solution $u_{\text{ex}}(x,y)$.

The last two problems, 11 and 12, include the governing equations and the exact solutions of problems 1 and 6. The boundary conditions are

$$\cos(x+y)\frac{\partial u(x,y)}{\partial n} - e^{(x+y)}u(x,y) = g(x,y). \tag{92}$$

When M increases, the errors decrease rapidly as it is typical of spectral methods. Note that in problem 12 the errors become $e_a = 3.7\cdot 10^{-7}$ and $e_{sr} = 3.3\cdot 10^{-7}$ when we take $M = 40$ and $l = 8$.

4. Particular Cases

The algorithm described above can be considerably simplified for separable PDEs and for PDEs with constant coefficients. Note that the heaviest part of the algorithm is solving linear systems (71) to obtain the trial functions $\Psi_k(x,y)$. In both cases we can avoid solving these systems and the trial functions $\Psi_k(x,y)$ can be defined analytically.

If a separable PDE is considered, the initial operator L, as well as the operator $L^{(0)}$, is a sum of one-dimensional operators

$$L^{(0)}(x,y) = L_1^{(0)}(x) + L_2^{(0)}(y), \tag{93}$$

$$L_i^{(0)} = -\frac{1}{r_i^{(0)}(x_i)}\left\{\frac{\partial}{\partial x_i}\left(p_i^{(0)}(x_i)\frac{\partial}{\partial x_i}\right) - q_i^{(0)}(x_i)\right\}, \qquad i = 1,2. \tag{94}$$

Let $\psi_n^{(i)}(x_i)$, $i = 1,2$, be eigenfunctions of $L_i^{(0)}$, i.e., they satisfy the equations

$$L_i^{(0)}(\psi_n^{(i)}(x_i)) = \lambda_n^{(i)}\psi_n^{(i)}(x_i), \qquad n = 1,\dots,\infty, \qquad i = 1,2. \tag{95}$$

The source functions can be represented in the form of truncated series,

$$I(x,y \mid \xi,\eta) = \sum_{n,m=1}^{M} c_{n,m}(\xi,\eta)\,\psi_n^{(1)}(x)\,\psi_m^{(2)}(y), \tag{96}$$

as it is described in [10]. So, the trial function $\Psi_k(x,y)$ is a solution of the PDE,

$$\left\{L_1^{(0)}(x) + L_2^{(0)}(y)\right\}(\Psi_k) = \sum_{n,m=1}^{M} c_{n,m}(\xi,\eta)\,\psi_n^{(1)}(x)\,\psi_m^{(2)}(y). \tag{97}$$

Using (95), we obtain the trial function also in the form of a truncated series,

$$\Psi_k(x,y) = \sum_{n,m=1}^{M}\frac{c_{n,m}(\xi_k,\eta_k)}{\lambda_n^{(1)} + \lambda_m^{(2)}}\,\psi_n^{(1)}(x)\,\psi_m^{(2)}(y). \tag{98}$$

The numerical examples illustrating the use of such trial functions are presented in [10].

In the case of PDE with constant coefficients,

$$L(u) = a_{1,1}\frac{\partial^2 u}{\partial x^2} + 2a_{1,2}\frac{\partial^2 u}{\partial x \partial y} + a_{2,2}\frac{\partial^2 u}{\partial y^2} + b_1\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y} + c\,u = f(x,y), \tag{99}$$

the trial functions satisfy the following equation, when the source functions (31), (32) are used,

$$L(\Psi_k) = I(x,y\,|\,\xi_k,\eta_k) = \sum_{n,m=1}^{M} c_{nm}(\xi_k,\eta_k)\sin(n\pi x)\sin(m\pi y). \tag{100}$$

If we denote

$$\begin{aligned} L^{-1}\left\{e^{i\pi(nx+my)}\right\} &= Z_{n,m}e^{i\pi(nx+my)} \\ &\equiv \left[-\pi^2(a_{1,1}n^2 + 2a_{1,2}nm + a_{2,2}m^2) + i(b_1 n + b_2 m) + c\right]^{-1} e^{i\pi(nx+my)}, \end{aligned} \tag{101}$$

the trial function can be written in the form

$$\begin{aligned} \Psi_k(x,y) &= \sum_{n,m=1}^{M} c_{nm}(\xi_k,\eta_k)\,L^{-1}\{\sin(n\pi x)\sin(m\pi y)\} \\ &= -\frac{1}{4}\sum_{n,m=1}^{M} c_{nm}(\xi_k,\eta_k)\Big[Z_{n,m}e^{i\pi(nx+my)} - Z_{n,-m}e^{i\pi(nx-my)} \\ &\qquad - Z_{-n,m}e^{i\pi(-nx+my)} + Z_{-n,-m}e^{-i\pi(nx+my)}\Big]. \end{aligned} \tag{102}$$

It is easy to prove that this is a real function.

In Table 5 we present the results of applying these trial functions to the approximate solution of the PDE,

$$2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} - 5\,u = f(x,y), \tag{103}$$

with different boundary conditions. To approximate these boundary conditions we use $K = 50$ sources placed on a circle with the radius 0.49 and $N_c = 100$ collocation points on $\partial\Omega$. The exact solution is $u_{\text{ex}}(x,y) = 1+x+y+xy+x^2+y^2$. The calculations are performed with double precision.

Table 5. Maximal absolute error e_{a} and mean square root error e_{sr}.

(M,l)	$u = g$		$\partial u/\partial n - u = g$	
	e_{a}	e_{sr}	e_{a}	e_{sr}
(10,2)	$8.8\cdot10^{-3}$	$2.1\cdot10^{-3}$	$2.5\cdot10^{-1}$	$2.1\cdot10^{-1}$
(15,4)	$2.7\cdot10^{-4}$	$4.6\cdot10^{-5}$	$1.5\cdot10^{-3}$	$6.9\cdot10^{-4}$
(20,5)	$6.0\cdot10^{-5}$	$1.6\cdot10^{-5}$	$6.8\cdot10^{-4}$	$5.6\cdot10^{-4}$
(30,6)	$5.7\cdot10^{-7}$	$1.1\cdot10^{-7}$	$1.1\cdot10^{-5}$	$8.9\cdot10^{-6}$
(40,7)	$1.4\cdot10^{-8}$	$2.5\cdot10^{-9}$	$3.2\cdot10^{-7}$	$2.7\cdot10^{-7}$
(50,10)	$7.6\cdot10^{-10}$	$5.4\cdot10^{-11}$	$6.3\cdot10^{-9}$	$1.7\cdot10^{-9}$

The data presented in Table 6 show the dependence of exactness of the approximate solution on parameter l. The problem is the same as above. The Dirichlet boundary conditions are used. The number of harmonics sources and collocation points is the same in all the cases, $M = 20$, $K = 50$ and $N_c = 100$, respectively. One can see that in this case $l_{\text{opt}} = 5$.

Table 6. Maximal absolute error e_a as a function of parameter l

l	0	1	...	4	5	6	...	10
e_a	$2.6 \cdot 10^{-1}$	$2.1 \cdot 10^{-2}$	...	$6.0 \cdot 10^{-5}$	$2.5 \cdot 10^{-5}$	$7.6 \cdot 10^{-5}$	...	$6.4 \cdot 10^{-3}$

5. CONCLUSIONS

A new numerical method for elliptic PDEs with varying coefficients in a complex geometry has been presented and tested. It follows the general scheme of Trefftz type methods. The main difference is that the trial functions satisfy the PDE only approximately. This reduction of the requirements to the trial functions extends the field of application of the method presented, as these trial functions can be found for a broad class of differential operators.

Although the method is developed for 2D problems, the ideas extend quite simply to the 3D case and also to time–dependent problems as it is shown in [7]. The method presented can also be used as a basis for the so called Trefftz–elements technique, as it has been done for the classic Trefftz method, see e.g. [3], [12]– [15].

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