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A concept of overlapping meshless FEM and its application in experimental mechanics

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In the paper a new meshless FEM method is proposed. The method is physically based and the defined element ensures agreement with equilibrium equations. A special functional is defined which consist of a smoothing term, a boundary term and eventually an experimental one. In one calculation both theoretical and experimental data are used to establish proper solution. The method may be used even in the case when constitutive equation is unknown, what is especially important for residual stress problems.

1. INTRODUCTION

Usually the finite element method is based on approximation of searched field by dividing the whole domain into elements and then the field values at separate points are the quantities to be evaluated. The shape functions proposed in each element usually do not satisfy any physical equation, and their physical meaning do not occur until a suitable functional is defined. Minimisation of that functional under proper boundary conditions leads to solution of the physical problem. For example the displacement field is usually obtained this way. It is observed however that during the subsequent stress calculation some difficulties may occur, that is different traction appears on the left and right side of the element–element boundary what leads to discontinuity of the stress field. The use of T-complete functions forces satisfying of physical equations but the problem is then how to fulfil boundary condition [12]. The other proposal of the diffusive equilibrium model has been presented in [11]. In the method proposed in the present paper satisfying of the equilibrium equations is reached already during defining shape functions and therefore it is physically based. This idea has been proposed in some papers [4, 5, 7] and in the so-called global–local method [1, 2, 3]. The stress field is then calculated at one point only taking information from the whole body. The method has been used successfully e.g. in analysing of residual stress as in railroad rail (above mentioned papers). It is convenient for reasonable experimental data smearing all over the entire domain and even in FEM postprocessing [8].

The methods that simultaneously take into consideration both theoretical and experimental data enable to solve ill-conditioned problems. In general, the theoretical–experimental methods are helpful in solving engineering problems where we can not find all data required for analytical solution of a problem. Moreover, when we have incomplete or suspicious data about boundary shape the use of theoretical approach is impossible or it can create wrong results inside the analysed body.

2. APPROXIMATION FUNCTIONS

The application of the proposed method to plane solid mechanics problems is considered. We introduce a special moving finite element representing the Airy function in a limited region. The function has the polar symmetry and outside of some radius is negligible. It is named hill function. Since the full Airy function is a linear combination of these profiles with centres in an arbitrary set of points this method represents the meshless method. Each of the hill function is of the class C^3 at least, and therefore resulted stress field is continuous. For arbitrary chosen weighting factors the obtained field satisfies equilibrium equation. The weighting factors should be found from variational principle governing a given problem. The physical state of the body appears during minimisation of the functional.

In the absence of volume forces the stress field satisfies equation div $\check{\sigma}$. It leads to the expression

$$\check{\sigma} = \begin{pmatrix} A_{,yy} & -A_{,xy} \\ -A_{,xy} & A_{,xx} \end{pmatrix}, \tag{1}$$

where A is the linear combination of elementary Airy functions. Each of them may be defined as the *hill function*

$$A(\vec{r}) = \begin{cases} \frac{(R^2-r^2)^{p+2}}{2(p+2)}; & r < R, \\ 0; & r \geq R. \end{cases} \tag{2}$$

Parameter p is a natural number and is a radius of non-zero domain. The image of this function for $p = 4$ is shown in Fig.1.

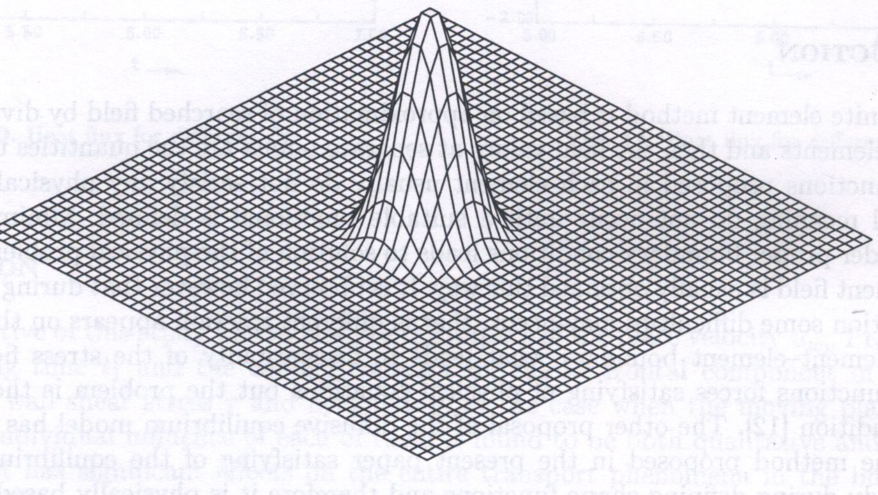

Fig. 1

The Airy function (2) is of the class C^{p+2} and is at least twice differentiable. The relevant stress field may be presented as

$$\check{s}(\vec{r}) = \begin{cases} (R^2-r^2)^p \left[(2p+3)r^2 - R^2 - 2(p+1)\vec{r}\,\vec{r}^t\right] & r < R, \\ 0; & r \geq R. \end{cases} \tag{3}$$

The above stress function is of the class at least C^0. The meaning of the parameter p is the same as in the adaptive method (hp). The stress at arbitrary point may be presented as linear combination of these functions in some set of nodal points $\vec{r}_n$,

$$\check{\sigma}(\vec{r}) = \sum_n P_n \check{s}(\vec{r} - \vec{r}_n). \tag{4}$$

The coefficients P_n define the stress field approximation. The expression (4) realises an arbitrary stress field in the admissible functions class. These coefficients may be established by introducing expression (4) to some variational principle.

This method is meshless. We should create a set of centres at arbitrary points and the *hill functions* are spread around them. These nodes must be located suitably close to avoid gaps between them and the hills must mutually overlap. The mean distance between nodes is analogous to the parameter h in the adaptive method so that in the method we have the three parameters i.e. h, p, R (hpR method). The stress function (4) may be differentiated. The entire domain of the hills must be larger than relevant body. An optimal choice of the parameters h, p, R is rather difficult and therefore some experience may help to found them.

3. FUNCTIONAL DEFINITION

We have three kinds of data:

- internal theoretical equation (I)
- boundary conditions (B)
- experimental data (E)

The functional may be defined as follows,

$$\Phi(P) = c_I \Phi^I(P) + c_B \Phi^B(P) + c_E \Phi^E(P) . \tag{5}$$

Each of the terms may be present in this functional or may be omitted. The problem how to establish parameters c_I, c_B, c_E properly will be discussed later. The array P can be found by minimisation of the functional

$$\min_P \Phi(P) . \tag{6}$$

Now the three terms of the functional will be defined.

3.1. Internal theoretical equation

The mean square of the constitutive equation residuum over the entire domain (e.g. Beltrami–Mitchell equation) may be used as the Φ^I. However, we often do not know proper equation. This situation appears e.g. in the residual stress case. The extent of the plastic zone is difficult to establish and temperature history is completely unknown (the yield stress is strongly dependent on temperature). In this case we may use heuristic internal equation as a specially defined smoothing expression. It uses the concept of a tensor curvature defined [6, 7] as a square root of the trace of the mean square of the second stress directional derivatives

$$\varkappa = \left(\frac{1}{2} \int_0^{2\pi} \mathrm{d}\varphi \, \mathrm{tr} \left(\frac{\partial^2 \check{\sigma}}{\partial \vec{v}^2} \right)^2 \right)^{\frac{1}{2}} , \tag{7}$$

where $\vec{v}$ is a directional vector $(\cos\varphi, \sin\varphi)$. The internal equation term is now $\Phi^I = \kappa^2$. Using expression (4) it may be presented as

$$\Phi^I = \sum_n \sum_{n'} P_n P_{n'} \frac{1}{8} \left[3(s_{ij,xx} \, s'_{ij,xx} + s_{ij,yy} \, s'_{ij,yy}) + (s_{ij,xx} \, s'_{ij,yy} + s_{ij,yy} \, s'_{ij,xx}) + 4 s_{ij,xy} \, s'_{ij,xy} \right] . \tag{8}$$

The quantities without prime are related to $\vec{\mathbf{r}} - \vec{\mathbf{r}}_n$ and those with prime refer to $\vec{\mathbf{r}} - \vec{\mathbf{r}}_{n'}$.

3.2. Boundary conditions

The boundary conditions may appear as a traction or displacement condition. In this paper the finite element method is of a stress type (shape functions are used to model a stress field), and for the hookean body a form of the displacement condition is possible. By use of the Hooke law we may obtain strain field. Defining rotation quantity as

$$\omega = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right), \tag{9}$$

the unknown displacement field may be expressed as follows,

$$u_{x,x} = \varepsilon_{xx}\,, \qquad u_{x,y} = \varepsilon_{xy} + \omega, \qquad u_{y,x} = \varepsilon_{xy} - \omega, \qquad u_{y,y} = \varepsilon_{yy}\,. \tag{10}$$

Now the rotation ω may be calculated as

$$\omega(l) = \omega_o + \int_0^l (\varepsilon_{xx,y}\,\mathrm{d}x - \varepsilon_{xy,x}\,\mathrm{d}x + \varepsilon_{xy,y}\,\mathrm{d}y - \varepsilon_{yy,x}\,\mathrm{d}y)\,. \tag{11}$$

(l is a path along boundary). This last expression enables to determine the displacement field

$$u(l) = \vec{u}_o + \int_0^l (\varepsilon_{xx}\,\mathrm{d}x + (\varepsilon_{xy} + \omega)\,\mathrm{d}y + (\varepsilon_{xy} - \omega)\,\mathrm{d}x + \varepsilon_{yy}\,\mathrm{d}y)\,. \tag{12}$$

The constant quantities ω_o and $\vec{u}_o$ that result from possible rigid motion may be omitted. The strain field in (12) is given explicit and may be differentiated and integrated analytically. It is rather tedious but special computer programmes allows to do it automatically.

The term Φ^B may be now expressed as

$$\Phi^B = \int_{\Gamma_q} \mathrm{d}l(\check{\sigma}\vec{\nu} - \vec{q})^2 + c_u \int_{\Gamma_u} \mathrm{d}l(\vec{u} - \vec{u}_l)^2\,, \tag{13}$$

where $\vec{\nu}$ is a vector normal to the boundary, $\vec{q}$ is the traction and $\vec{u}$ is the boundary displacement. The parameter c_u is a weighting factor of both terms. It may be chosen as E/L (L is a full path). The integration is performed only along a part of the boundary (conditions on remain part may be unknown). The functional term may be expressed as

$$\Phi^B(P) = \Phi^B_o + \sum_n \sum_{n'} P_n P_{n'} T_{nn'} - 2\sum_n P_n Q_n\,, \tag{14}$$

where

$$\begin{aligned} \Phi^B_o &= \int_\Gamma \mathrm{d}l\,\vec{q}^t\vec{q}\,, & \check{s}_n &= \check{s}\left(\vec{r}^B - \vec{r}_n\right) \\ T_{nn'} &= \int_\Gamma \mathrm{d}l\,\vec{\nu}^t\,\check{s}_n\check{s}_{n'}\,\vec{\nu}\,, & Q_n &= \int_\Gamma \mathrm{d}l\,\vec{q}^t\,\check{s}_n\vec{\nu}\,. \end{aligned} \tag{15}$$

3.3. Experimental data

The interpretation of the experimental data from mathematical point of view was considered for example in [9, 10]. Now data is taken into consideration as a sum of the squares of norms of differences between approximated and experimental values. That allows to obtain a linear term after differentiation. In this paper it is assumed that the experimental data is taken from strain gauges,

$$\Phi^E = \sum_E \mathrm{tr}\left(\check{\sigma}(\vec{r}^E) - \check{\sigma}^E\right)^2. \tag{16}$$

Defining

$$\Phi_o^E = \sum_E \operatorname{tr}\left(\breve{\sigma}^E \breve{\sigma}^E\right),$$

$$L_{nn'} = \sum_E \operatorname{tr}\left(\breve{s}(\vec{r}^E - \vec{r}_n)\,\breve{s}(\vec{r}^E - \vec{r}_{n'})\right), \tag{17}$$

$$R_n = \frac{1}{2}\sum_E \operatorname{tr}\left(\breve{s}(\vec{r}^E - \vec{r}_n)\,\breve{\sigma}^E + \breve{\sigma}^E \breve{s}(\vec{r}^E - \vec{r}_n)\right),$$

the experimental term of the functional has a form

$$\Phi^E = \Phi_o^E + \sum_n \sum_{n'} P_n P_{n'} L_{nn'} - 2\sum_n P_n R_n\,. \tag{18}$$

3.4. Choice of the functional parameters

To establish functional parameters a new criterion should be used, but there is no physical reason to do it. It seems that the best method is a self-control of boundary and experimental data. One non-zero parameter may be equal to 1. The remaining weighting factors may be chosen by minimisation of the following additional functional

$$\min_{c_B, c_E} \sum_k \Phi\left(x^{(k)}, y^{(k)}\right), \tag{19}$$

where k numerates both boundary and experimental points. However, in this technique the non-linear optimisation must be used.

4. Searching for the minimum of quadratic form

The minimum of the functional (5) may be found by creation of the sparse set of linear equations

$$\frac{\partial \Phi}{\partial P_n} = 0\,. \tag{20}$$

We propose to solve it by the following algorithm dedicated for not too large systems. There is an equation

$$Ax = b\,, \tag{21}$$

where the following signature is used: capital letter – matrix, small Latin letter – vector and small Greek letters – real number. This set of equations may be solved by subsequent iterations. The solution at the next step of iteration is

$$x' = x + \lambda(b - Ax)\,. \tag{22}$$

The number λ was introduced to ensure physical compatibility and a proper scale of sum terms. Defining residuum $r = b - Ax$ one finds for the subsequent steps,

$$x' = x + \lambda r \qquad r' = r - \lambda A r\,. \tag{23}$$

At the each step of iteration λ can be found from the condition

$$\min_\lambda r'^2. \tag{24}$$

If the matrix A is positive the convergence is ensured and the measure of it can be defined as

$$\kappa = \min_r \frac{|r'|}{|r|} \tag{25}$$

and then

$$\kappa = \left(1 - \frac{A_{\min}}{A_{\max}}\right)^{\frac{1}{2}}, \tag{26}$$

where $A_{\min}$ and $A_{\max}$ are minimal and maximal eigenvalues of A, respectively.

5. ADAPTIVITY OF THE METHOD

The final value of the functional (5) is a measure of the committed a posteriori error. This information may be used to create a new set of nodes and to perform an adaptive procedure. Overlapping and meshless approach enables an easy application of this technique. The nodes may be easily created and cancelled.

6. TESTS

A circle (Fig. 2) was used as a test domain.

In the domain a set of nodes have been placed (small circles). They are localised at the following points,

$$\vec{r}_{mj} = \frac{m}{M} \begin{pmatrix} \cos\left(\frac{2\pi}{N_m} j\right) \\ \sin\left(\frac{2\pi}{N_m} j\right) \end{pmatrix}, \tag{27}$$

where M is a density parameter, m runs natural numbers from 1 to M, and j belongs to $[0, N_m]$. In order to achieve a homogeneous distribution of nodes the quantity N_m is chosen as

$$N_m = \left[\frac{\pi}{\arcsin(1/2m)} + 1\right]. \tag{28}$$

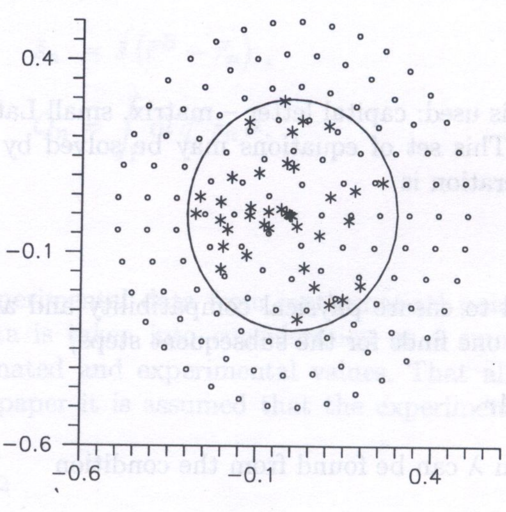

Fig. 2

A set of strain gauge pseudo-experimental points (signed by $*$) have been placed in the domain. Airy function was defined as follows,

$$A(x,y) = A_0 x^2 + 2A_1 xy + A_2 y^2 . \tag{29}$$

This function guarantees that the stress field is uniform. Figure 3 shows diagram of distribution of the stress components σ_{xx} , σ_{xy} σ_{yy} , along the crossing horizontal line. The continuous line represents the theoretical function (constant) whereas the dashed line the approximated one.

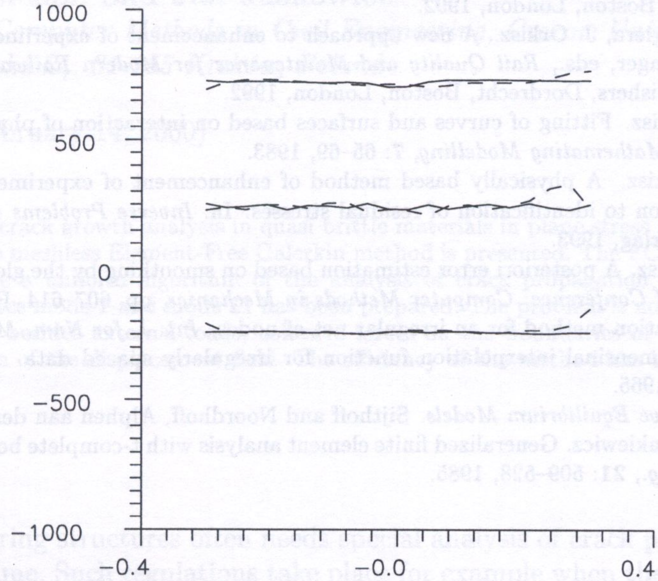

Fig. 3

7. Conclusions

- The presented above a new method of stress approximation allows to manage with the hill oriented finite elements and create a class of statically admissible stress fields that are true for all materials and their states.
- The constitutive equation may be omitted, what is very important e.g. in residual stress problems.
- The optimisation of the problem oriented functional allows to obtain the stress field. This method may be very helpful in solving combined theoretical–experimental problems, where the agreement of measurements, boundary condition and equilibrium equation (TEDI) should be found.
- This approach to experimental data interpretation will be especially suitable for problems ill conditioned from theoretical point of view.

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