Computer Assisted Mechanics and Engineering Sciences, 9: 291–308, 2002.

Stability of a circular ring in postcritical equilibrium states with two deformation-dependent loads and geometrical imperfections

Imre Kozák
Department of Mechanics, University of Miskolc
H-3515 Miskolc-Egyetemváros, Hungary

Tamás Szabó
Hungarian Academy of Sciences, University of Miskolc, Numerical Research Group
H-3515 Miskolc-Egyetemváros, Hungary

(Received October 23, 2000)

The circular ring is linearly elastic and its cross-section is rectangular. Two deformation dependent distributed loads, that is follower loads, are applied simultaneously on the outer surface of the ring. The first load is a uniform pressure on the whole outer surface. The second load is uniform normal traction exerted on two surface parts situated in axially symmetric positions. Both loads are selfequilibrated independently from each other. A nonlinear FE program with 3D elements is used for the numerical analysis of a geometrically perfect and two imperfect rings. Displacement control is used in the equilibrium iterations. Equilibrium surfaces are determined in the space of three parameters such as one characteristic displacement coordinate, and two load factors. The stability analysis is performed in the knowledge of the equilibrium surfaces.

Keywords: deformation dependent loads, two-parameter loads, geometric imperfection, displacement control, equilibrium surface, limit point, bifurcation point, unstable region

1. Introduction

Deformation dependent loads can be classified as conservative or non-conservative, Bufler [2], Cohen [3], Kozák [7], Mang [11], Schweizerhof and Ramm [15]. In the FE analysis the influence of the deformation dependent load can be considered with the help of the load stiffness matrix. Numerical experiences show that the application of the load stiffness matrix on the left hand side of the linearized system of equations is advantageous both for conservative loads and non-conservative ones, since it essentially decreases the number of equilibrium iterations, Molk et al. [13], Kozák et al. [8].

There is a number of monographs devoted to the statical stability analysis of equilibriums. The line of thought is: formulation of the problem, stability conditions, possible solution methods and then solved problems; in addition detailed references are included in, for instance, Crisfield [4], Huseyin [5], Kleiber [6], Kurutz [9, 10], Thompson [16], Wriggers [17, 18]. Special attention is paid to taking the geometrical and load imperfections into consideration during the stability analysis.

3D models are analyzed in this paper. The circular ring is linearly elastic and its cross-section is rectangular. By geometrical imperfections we mean elliptical rings with one axis coinciding with the radius of the circle while the other differs from the radius by small value. The primary load is a uniform pressure applied on the whole outer surface of the ring. The secondary load is a uniform normal traction exerted on two small parts of the outer surface in axially symmetric positions. The two loads are in equilibrium independently from each other, and naturally perpendicular to the actual surface. Each load case is associated with a load parameter.

The aim of this paper is to determine the postbuckling equilibrium states and to analyze their static stability. The main features of our investigations are as follows: geometrically non-linear *deformations, deformation dependent loads and two independent load parameters.*

The following mechanical tasks have to be solved: computation of work increments done by equilibrium loads during a displacement increment, determination of the displacement increment in the neighbourhood of a given configuration, determination of equilibrium surfaces and paths, static stability analysis of equilibrium configurations, determination of the unstable points and regions, analysis of the effects of the geometrical imperfections.

In Section 2 the basic formulae are summarized both for a continuum model and for a finite element (FE) discretization in the reference configuration. The deformation dependent load can be given in terms of the deformation dependent vectorial surface element for the problems considered. The incremental form of the principle of virtual work is applied to determine the displacement increments. The Newton-Raphson procedure, as one of the methods, is used to solve the incremental form of the principle of virtual work. The stiffness matrix of the FE discretization includes the load stiffness matrix as well.

In Section 3 an alternative procedure is proposed to solve the incremental form of the principle of virtual work, it is based on the initiative of Marcinowski [12]. This alternative procedure and the calculation of the increments of the strain energy and the external work done by the deformation dependent loads serve as a basis for the stability analysis in Section 4.

Section 5 is devoted to numerical examples and the conclusions that follow from them. Applying a displacement control technique equilibrium surfaces, bifurcation points, limit points, and instability regions are determined for geometrically perfect and imperfect rings.

Indicial notations are used for the description of an appropriate continuum model assuming arbitrary curvilinear coordinate systems. The indices are Latin letters.

The FE quantities are type set by symbolic notation. Square matrices are denoted by bold upper case letters with two lower indices, and column vectors by bold lower case letters with one lower index. Quantities of high order, that is with an order greater than 2, are denoted by italic bold upper case letters and as many lower indexes as the order. The indices of the FE quantities are denoted by Greek letters. Multiplication is denoted by dummy indices.

2. Basic formulae

2.1. Configurations, change of the vectorial surface element and loads

Let us denote the stress and deformation free reference configuration by (B) and the present or instantaneous configuration by $(\overline{B})$ or $(\overline{B}+\Delta\overline{B})$. The initial configuration for the Newton–Raphson iteration is denoted by $(B'_{\underline{1}})$. This configuration is either equilibrated or not equilibrated.

The iteration produces intermediate configurations denoted by $(B'_{\underline{s}})$, where $\underline{s}=\underline{2},\underline{3},\ldots,\underline{n}$, and these configurations are, however, not in equilibrium except the last one $(B'_{\underline{n}})$, therefore it can be denoted by $(\overline{B})$. Then applying a load increment the configuration $(\overline{B})$ is regarded as an initial configuration $(B'_{\underline{1}})$, and as a result of the iteration procedure we shall have a new equilibrium configuration $(B'_{\underline{n}})=(\overline{B}+\Delta\overline{B})$ at the actual load level.

It is assumed that the kinematical boundary conditions are independent of deformations.

Using the total Lagrangian formulation the following notations will be used in the reference configuration (B): u^k, E_{kl}, S^{kl} denote the displacement vector, Green-Lagrangian strain tensor, the II Piola–Kirchhoff stress tensor, respectively; Δu^k, ΔE_{kl}, ΔS^{kl} are the corresponding increments, and $\mathrm{d}A_k$ is the surface element vector. The virtual displacement vector is $\delta u^k=\delta(\Delta u^k)$.

The Green–Lagrangian strain tensor is given by

$$E_{kl}=\frac{1}{2}(u_{k;l}+u_{l;k}+u_{m;k}u^{m}{}_{;l}), \tag{1}$$

where an index preceded by a semicolon (;) denotes the covariant derivatives.

It is supposed that the material obeys the following constitutive equation

$$S^{pq} = E_{kl}C^{klpq} = C^{pqkl}E_{kl}\,. \tag{2}$$

where C^{klpq} is the tensor of elastic coefficients.

The surface element vector $\mathrm{d}\overline{A}_k$ of the present configuration $(\overline{B})$ can be expressed by the surface element vector $\mathrm{d}A_p$ of the reference configuration (B),

$$\mathrm{d}\overline{A}_k = Q_k^{\ p}\,\mathrm{d}A_p\,, \tag{3}$$

where

$$Q_k^{\ p} = \frac{1}{2}\,e_{klm}e^{pqr}(\delta^l_q + u^l_{\ ;q})(\delta^m_r + u^m_{\ ;r}), \tag{4}$$

and e_{klm}, e^{pqr} are permutation symbols, and δ^l_q, δ^m_r are the unit tensors.

For the increment of the surface element vector $\mathrm{d}\overline{A}_k$ the following formulae can be written,

$$\Delta(\mathrm{d}\overline{A}_k) = \Delta Q_k^{\ p}\,\mathrm{d}A_p\,, \qquad \Delta Q_k^{\ p} = \Delta Q^{(1)\ p}_{\ \ k} + \Delta Q^{(2)\ p}_{\ \ k}\,, \tag{5}$$

where

$$\Delta Q^{(1)\ p}_{\ \ k} = e_{klm}e^{pqr}(\delta^l_q + u^l_{\ ;q})(\Delta u^m_{\ ;r}), \tag{6}$$

and

$$\Delta Q^{(2)\ p}_{\ \ k} = \frac{1}{2}e_{klm}e^{pqr}(\Delta u^l_{\ ;q})(\Delta u^m_{\ ;r}). \tag{7}$$

Here and in the sequel a number as an upper index in parentheses denotes to the power of $\Delta u^m_{\ ;r}$.

Body forces are neglected, only distributed normal loads exerted on the instantaneous outer surface are considered. We define two independent loads acting on the two surface parts $(\overline{A}_{t\mathrm{I}})$ and $(\overline{A}_{t\mathrm{II}})$ of an arbitrary present configuration $(\overline{B})$,

$$\widetilde{p}_{\mathrm{I}} = p_{\mathrm{I}}\widetilde{p}_{0\mathrm{I}}\,, \quad x \subset (\overline{A}_{t\mathrm{I}}) \qquad \text{and} \qquad \widetilde{p}_{\mathrm{II}} = p_{\mathrm{II}}\widetilde{p}_{0\mathrm{II}}\,, \quad x \subset (\overline{A}_{t\mathrm{II}}). \tag{8}$$

The forces acting on the corresponding surface elements are given by the formulae

$$\mathrm{d}\overline{F}_k = \widetilde{p}_{\mathrm{I}}\mathrm{d}\overline{A}_k\,, \quad x \subset (\overline{A}_{t\mathrm{I}}) \qquad \text{and} \qquad \mathrm{d}\overline{F}_k = \widetilde{p}_{\mathrm{II}}\mathrm{d}\overline{A}_k\,, \quad x \subset (\overline{A}_{t\mathrm{II}}) \tag{9}$$

where p_{I} and p_{II} are independent load parameters, $\widetilde{p}_{0\mathrm{I}}$ and $\widetilde{p}_{0\mathrm{II}}$ are reference loads regarded as positive if the loads are directed out of the surface.

2.2. Incremental form of the principle of virtual work

Let the characteristic values for the initial configuration B_1' be: u_1^k, E_{1kl}, S_1^{kl}, $Q_{1k}^{\ \ p}$, and p_{I}, p_{II} are two independent load parameters in the equilibrium configuration $(\overline{B})$ while $p_{\mathrm{I}} + \Delta p_{\mathrm{I}}$, $p_{\mathrm{II}} + \Delta p_{\mathrm{II}}$ denote the load parameters in the equilibrium configuration $(\overline{B} + \Delta\overline{B})$. The equilibrium condition is satisfied for the configuration $(\overline{B})$ or $(\overline{B} + \Delta\overline{B})$ if the following incremental form of the principle of virtual work holds for arbitrary δu^k,

$$\begin{aligned}\int_{(B)} \delta(\Delta E_{kl})(S_1^{kl} + \Delta S^{kl})\,\mathrm{d}V = (p_{\mathrm{I}} + \Delta p_{\mathrm{I}})\int_{(A_{t\mathrm{I}})} \widetilde{p}_{0\mathrm{I}}\delta u^k(Q_{1k}^{\ \ p} + \Delta Q_k^{\ p})\,\mathrm{d}A_p \\ + (p_{\mathrm{II}} + \Delta p_{\mathrm{II}})\int_{(A_{t\mathrm{I}})} \widetilde{p}_{0\mathrm{II}}\delta u^k(Q_{1k}^{\ \ p} + \Delta Q_k^{\ p})\,\mathrm{d}A_p\,,\end{aligned} \tag{10}$$

where

$$\Delta E_{kl} = \Delta E^{(1)}{}_{kl} + \Delta E^{(2)}{}_{kl}\,, \qquad \Delta E^{(1)}{}_{kl} C^{klpq} = \Delta u_{m;k}(\delta^m_l + u_{\underline{1}}{}^m{}_{;l}) C^{klpq}\,,$$
$$\Delta S^{kl} = C^{klpq} \Delta E_{pq}\,, \qquad \Delta E^{(2)}{}_{kl} C^{klpq} = \frac{1}{2} \Delta u^m{}_{;k} \Delta u_{m;l} C^{klpq}\,, \tag{11}$$

and $\Delta Q_k{}^p$ can be evaluated using Eq. (5).

Remark: In accordance with the first paragraph of Section 2.1, Eq. (10) is to be understood in the following sense: if $(B'_{\underline{1}})$ is not an equilibrium configuration then we substitute $\Delta p_{\mathrm{I}} = \Delta p_{\mathrm{II}} = 0$ and seek the equilibrium configuration $(\overline{B})$. If, however, $(B'_{\underline{1}}) \equiv (\overline{B})$ is an equilibrium configuration the whole equation is used and the equilibrium configuration $(\overline{B} + \Delta\overline{B})$ is sought.

2.3. The Newton–Raphson iteration

Equation (10) is a nonlinear one for the unknown displacement increment Δu^k. If it is replaced by a finite element discretization, then for the unknown vector of increments, which includes the displacement parameters $\Delta \mathbf{t}_\beta$ where $\beta = 1, 2, \ldots, M$ and M is the degree of freedom of the system it is also remains a nonlinear one.

The Newton–Raphson iteration is a widely used iteration procedure for the solution of nonlinear FE problems.

For the $\underline{s}^{th}$ step of the Newton-Raphson iteration we can write

- a continuum formula

$$\int_{(B)} \left[\delta \Delta E^{(2)}_{\underline{s}}{}_{kl} S^{kl}_{\underline{s}} + \delta \Delta E^{(1)}_{\underline{s}}{}_{kl} \Delta S^{(1)kl}_{\underline{s}} \right] \mathrm{d}V$$
$$- p_{\mathrm{I}} \int_{(A_{t\mathrm{I}})} \widetilde{p}_{0\mathrm{I}} \delta u^k \Delta Q^{(1)}_{\underline{s}}{}_k{}^p \,\mathrm{d}A_p - p_{\mathrm{II}} \int_{(A_{t\mathrm{II}})} \widetilde{p}_{0\mathrm{II}} \delta u^k \Delta Q^{(1)}_{\underline{s}}{}_k{}^p \,\mathrm{d}A_p$$
$$= - \int_{(B)} \delta \Delta E^{(1)}_{\underline{s}}{}_{kl} S^{kl}_{\underline{s}} \,\mathrm{d}V$$
$$+ (p_{\mathrm{I}} + \Delta p_{\mathrm{I}}) \int_{(A_{t\mathrm{I}})} \widetilde{p}_{0\mathrm{I}} \delta u^k Q_{\underline{s}k}{}^p \,\mathrm{d}A_p + (p_{\mathrm{II}} + \Delta p_{\mathrm{II}}) \int_{(A_{t\mathrm{II}})} \widetilde{p}_{0\mathrm{II}} \delta u^k Q_{\underline{s}k}{}^p \,\mathrm{d}A_p\,; \tag{12}$$

- and the corresponding FE formula

$$\mathbf{K}^{\mathrm{T}}_{\underline{s}\alpha\beta} \Delta \mathbf{t}_{\underline{s}\beta} - p_{\mathrm{I}} \mathbf{K}^{\mathrm{L}}_{\underline{s}\mathrm{I}\alpha\beta} \Delta \mathbf{t}_{\underline{s}\beta} - p_{\mathrm{II}} \mathbf{K}^{\mathrm{L}}_{\underline{s}\mathrm{II}\alpha\beta} \Delta \mathbf{t}_{\underline{s}\beta} = - \mathbf{b}_{\underline{s}\alpha} + (p_{\mathrm{I}} + \Delta p_{\mathrm{I}}) \mathbf{a}_{\underline{s}\mathrm{I}\alpha} + (p_{\mathrm{II}} + \Delta p_{\mathrm{II}}) \mathbf{a}_{\underline{s}\mathrm{II}\alpha}\,. \tag{13}$$

In (13) $\mathbf{K}^{\mathrm{T}}_{\underline{s}\alpha\beta}$ is the usual tangent stiffness matrix made up of the linear and geometrical stiffness matrices (see Eq. (31)), $\mathbf{K}^{\mathrm{L}}_{\underline{s}\mathrm{I}\alpha\beta}$ and $\mathbf{K}^{\mathrm{L}}_{\underline{s}\mathrm{II}\alpha\beta}$ are two load stiffness matrices for the two different loads (see Eq. (22)), $\Delta \mathbf{t}_{\underline{s}\beta}$ is the vector of increments in displacement parameters, and the formula

$$-\mathbf{b}_{\underline{s}\alpha} + (p_{\mathrm{I}} + \Delta p_{\mathrm{I}}) \mathbf{a}_{\underline{s}\mathrm{I}\alpha} + (p_{\mathrm{II}} + \Delta p_{\mathrm{II}}) \mathbf{a}_{\underline{s}\mathrm{II}\alpha}$$

is the unbalanced load in the intermediate configuration of $(B'_{\underline{s}})$(see Eqs. (30) and (21)). The matrix $\mathbf{K}^{\mathrm{T}}_{\underline{s}\alpha\beta}$ is symmetric. The matrices $\mathbf{K}^{\mathrm{L}}_{\underline{s}\mathrm{I}\alpha\beta}$ and $\mathbf{K}^{\mathrm{L}}_{\underline{s}\mathrm{II}\alpha\beta}$ are symmetric if the loads are conservative and are not symmetric if the loads are not conservative. Numerical experiences show that the application of the load stiffness matrix on the left hand side of the linearized system of equations essentially decreases the number of Newton–Raphson iterations: Kozák et al. [8], Molk et al. [13].

Taking the remark made right after the formula (11) into consideration, Eq. (13) is equally applicable with displacement and load controls for the determination of equilibrium paths.

For load control the formula (13) remains unchanged, provided that one of the load parameters $p_{\rm I}$ and $p_{\rm II}$ is constant.

We apply also the displacement control technique proposed by Batoz and Dhatt [1] and developed furthers by Powell and Simon [14]. Let us assume that the initial configuration is in equilibrium, that is, $(B_{\underline{1}}') \equiv (\overline{B})$, one of the different load parameters $p_{\rm I}$ and $p_{\rm II}$ is regarded as unknown (the other is set to zero). In this case $\Delta \mathbf{t}_{\underline{s}B}$ is an appropriate displacement control parameter (the subscript B is a fixed value of β). Then the other increments in displacement parameters $\Delta \mathbf{t}_{\underline{s}\beta}$ $(\beta \neq B)$ and the unknown load parameter increment, that is either $\Delta p_{s\rm I}$ or $\Delta p_{s\rm II}$, can be determined from (13).

The equilibrium paths obtained by the two load parameters give an equilibrium surface in the space of the two load parameters $p_{\rm I}$, $p_{\rm II}$ and the displacement control parameter $\Delta \mathbf{t}_{\underline{s}B}$.

In our numerical examples we determine a series of equilibrium paths each being a section of the equilibrium surface for which a load parameter is fixed. During the computation of points on an equilibrium path it may be necessary to replace the control parameters, since the load control parameter cannot be used at a limit point, and the displacement control parameter is not applicable at a turning point (see Figs. 2 and 3 in Section 5).

3. An alternative procedure to solve the incremental form of the principle of virtual work

The Newton–Raphson iteration gives the vector of increments in displacement parameters $\Delta\mathbf{t}_\beta = \Delta\mathbf{t}_{1\beta} + \Delta\mathbf{t}_{2\beta} + \cdots$ as solutions to algebraic equation systems linearized in the initial and intermediate configurations. The alternative solution method is based on the initiation made by Marcinowski [12] who assumed that the variables of the problem in a small neighbourhood of the equilibrium configuration $(B_{\underline{1}}') \equiv (\overline{B})$ can be given by the truncated Taylor expansion with respect to the control parameter $0 \leq \tau \leq \Delta\tau$. The parameter τ is regarded as quasi-time. In our problem

$$u^k(\tau) = u^k_{\underline{1}} + \Delta u^k = u^k_{\underline{1}} + \dot{u}^k_{\underline{1}}\tau + \frac{1}{2}\ddot{u}^k_{\underline{1}}\tau^2 + \frac{1}{6}\dddot{u}^k_{\underline{1}}\tau^3 + \cdots, \tag{14}$$

$$p_{\rm I}(\tau) = p_{\underline{1}\rm I} + \Delta p_{\rm I} = p_{\underline{1}\rm I} + \dot{p}_{\underline{1}\rm I}\tau + \frac{1}{2}\ddot{p}_{\underline{1}\rm I}\tau^2 + \cdots, \tag{15}$$

$$p_{\rm II}(\tau) = p_{\underline{1}\rm II} + \Delta p_{\rm II} = p_{\underline{1}\rm I} + \dot{p}_{\underline{1}\rm II}\tau + \frac{1}{2}\ddot{p}_{\underline{1}\rm II}\tau^2 + \cdots, \tag{16}$$

$$Q_k^{\ p}(\tau) = Q_{\underline{1}k}^{\ \ p} + \Delta Q_k^{\ p} = Q_{\underline{1}k}^{\ \ p} + \dot{Q}_{\underline{1}k}^{\ \ p}\tau + \frac{1}{2}\ddot{Q}_{\underline{1}k}^{\ \ p}\tau^2 + \frac{1}{6}\dddot{Q}_{\underline{1}k}^{\ \ p}\tau^3 + \cdots, \tag{17}$$

$$\mathbf{t}_\beta(\tau) = \mathbf{t}_{\underline{1}\beta} + \Delta\mathbf{t}_\beta = \mathbf{t}_{\underline{1}\beta} + \dot{\mathbf{t}}_{\underline{1}\beta}\tau + \frac{1}{2}\ddot{\mathbf{t}}_{\underline{1}\beta}\tau^2 + \frac{1}{6}\dddot{\mathbf{t}}_{\underline{1}\beta}\tau^3 + \cdots. \tag{18}$$

where the dot above a variable denotes derivation with respect to a control parameter, $\Delta\tau$ belongs to the displacement increment Δu^k which determines the configuration $(B + \Delta B)$. The control parameter can be either a load parameter ($p_{\rm I}$, $p_{\rm II}$) or a displacement one.

Marcinowski [12] starts from *the principle of virtual work in the equilibrium configuration* $(B_{\underline{1}}') \equiv (\overline{B})$ and takes its successive derivatives with respect to the control parameter. In this way a series of linear equation systems are obtained for $\dot{\mathbf{t}}_{\underline{1}\beta}$, $\ddot{\mathbf{t}}_{\underline{1}\beta}$, $\dddot{\mathbf{t}}_{\underline{1}\beta}$, ... if one applies FE discretization. Marcinowski uses displacement control and applies a concentrated load of constant direction.

The alternative method proposed here is based on *the incremental form of the principle of virtual work applied to the equilibrium configuration* $(\overline{B} + \Delta\overline{B})$ which is loaded by two deformation dependent normal tractions. Before the formulation of the alternative procedure and for the sake of preparation we give the work of the tractions during an increment in the displacements (Δu^k or $\Delta\mathbf{t}_\beta$) and the corresponding strain energy increment as well.

3.1. Calculation of the work of deformation dependent tractions

Using the formulae (3) and (14)–(17) the power of the tractions (8) and (9) in the equilibrium configuration $(B_{\underline{1}}')\equiv(\overline{B})$ can be written in the interval $0\le\tau\le\Delta\tau$ as follows,

$$
\begin{aligned}
P(\tau) &= p_{\underline{1}\mathrm{I}}\int_{(A_{t\mathrm{I}})}\widetilde{p}_{0\mathrm{I}}\dot{u}^k(\tau)Q_k^{\ p}(\tau)\,\mathrm{d}A_p + p_{\underline{1}\mathrm{II}}\int_{(A_{t\mathrm{II}})}\widetilde{p}_{0\mathrm{II}}\dot{u}^k(\tau)Q_k^{\ p}(\tau)\,\mathrm{d}A_p\\
&= p_{\underline{1}\mathrm{I}}\int_{(A_{t\mathrm{I}})}\widetilde{p}_{0\mathrm{I}}\left(\dot{u}_{\underline{1}}^k+\ddot{u}_{\underline{1}}^k\tau^2+\cdots\right)\left(Q_{\underline{1}k}^{\ \ p}+\dot{Q}_{\underline{1}k}^{\ \ p}\tau+\frac{1}{2}\ddot{Q}_{\underline{1}k}^{\ \ p}\tau^2+\cdots\right)\mathrm{d}A_p\\
&\quad + p_{\underline{1}\mathrm{II}}\int_{(A_{t\mathrm{II}})}\widetilde{p}_{0\mathrm{II}}\left(\dot{u}_{\underline{1}}^k+\ddot{u}_{\underline{1}}^k\tau^2+\cdots\right)\left(Q_{\underline{1}k}^{\ \ p}+\dot{Q}_{\underline{1}k}^{\ \ p}\tau+\frac{1}{2}\ddot{Q}_{\underline{1}k}^{\ \ p}\tau^2+\cdots\right)\mathrm{d}A_p\,.
\end{aligned}
\tag{19}
$$

The work increment of the traction in the interval $0\le\tau\le\Delta\tau$ can be given by the integration of the power $P(\tau)$ with respect to τ,

$$
\Delta W = \int_{\tau=0}^{\Delta\tau}P(\tau)\mathrm{d}\tau. \tag{20}
$$

For the sake of the FE discretization let us introduce the following notation,

$$
\int_{(A_{t\mathrm{N}})}\widetilde{p}_{0\mathrm{N}}\dot{u}_{\underline{1}}^kQ_{\underline{1}k}^{\ \ p}\mathrm{d}A_p = \dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{a}_{\underline{1}\mathrm{N}\alpha}\,,\qquad \mathrm{N}=\mathrm{I},\mathrm{II}, \tag{21}
$$

$$
\begin{aligned}
\int_{(A_{t\mathrm{N}})}\widetilde{p}_{0\mathrm{N}}\dot{u}_{\underline{1}}^k\dot{Q}_{\underline{1}k}^{\ \ p}\,\mathrm{d}A_p &= \int_{(A_{t\mathrm{N}})}\widetilde{p}_{0\mathrm{N}}\dot{u}_{\underline{1}}^ke_{klm}e^{pqr}(\delta_q^l+u^l_{\ ;q})\dot{u}_{\underline{1}\ ;r}^{\ m}\,\mathrm{d}A_p\\
&= \dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta}\,,\qquad \mathrm{N}=\mathrm{I},\mathrm{II},
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\int_{(A_{t\mathrm{N}})}\widetilde{p}_{0\mathrm{N}}\dot{u}_{\underline{1}}^k\ddot{Q}_{\underline{1}k}^{\ \ p}\,\mathrm{d}A_p &= \int_{(A_{t\mathrm{N}})}\widetilde{p}_{0\mathrm{N}}\dot{u}_{\underline{1}}^ke_{klm}e^{pqr}\left[(\delta_q^l+u^l_{\ ;q})\ddot{u}_{\underline{1}\ ;r}^{\ m}+\dot{u}_{\underline{1}\ ;q}^{\ l}\dot{u}_{\underline{1}\ ;r}^{\ m}\right]\mathrm{d}A_p\\
&= \dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\ddot{\mathbf{t}}_{\underline{1}\beta}+\dot{\mathbf{t}}_{\underline{1}\alpha}\boldsymbol{L}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta\gamma}\dot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}\,,\qquad \mathrm{N}=\mathrm{I},\mathrm{II},
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\int_{(A_{t\mathrm{N}})}\widetilde{p}_{0\mathrm{N}}\dot{u}_{\underline{1}}^k\dddot{Q}_{\underline{1}k}^{\ \ p}\,\mathrm{d}A_p &= \int_{(A_{t\mathrm{N}})}\widetilde{p}_{0\mathrm{N}}\dot{u}_{\underline{1}}^ke_{klm}e^{pqr}\left[(\delta_q^l+u^l_{\ ;q})\dddot{u}_{\underline{1}\ ;r}^{\ m}+3\ddot{u}_{\underline{1}\ ;q}^{\ l}\dot{u}_{\underline{1}\ ;r}^{\ m}\right]\mathrm{d}A_p\\
&= \dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\dddot{\mathbf{t}}_{\underline{1}\beta}+3\dot{\mathbf{t}}_{\underline{1}\alpha}\boldsymbol{L}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta\gamma}\ddot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}\,,\qquad \mathrm{N}=\mathrm{I},\mathrm{II}.
\end{aligned}
\tag{24}
$$

In these formulae $\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}$, $(\mathrm{N}=\mathrm{I},\mathrm{II})$ are two stiffness matrices, and $\boldsymbol{L}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta\gamma}$, $(\mathrm{N}=\mathrm{I},\mathrm{II})$ are two three-dimensional matrices.

Using the formulae (21)–(24) and (19) and performing the integration in (20) we have ΔW (see the terms with positive sign in (34)).

3.2. Strain energy increment

The strain energy increment, as the negative work increment of the internal forces, is written in the interval $0\le\tau\le\Delta\tau$,

$$
\Delta U = \dot{U}_{\underline{1}}(\Delta\tau)+\frac{1}{2}\ddot{U}_{\underline{1}}(\Delta\tau)^2+\frac{1}{6}\dddot{U}_{\underline{1}}(\Delta\tau)^3+\frac{1}{24}\ddddot{U}_{\underline{1}}(\Delta\tau)^4+\cdots, \tag{25}
$$

where

$$
\dot{U}_{\underline{1}} = \int_{(B)}\dot{E}_{\underline{1}kl}C^{klpq}E_{\underline{1}pq}\mathrm{d}V,\qquad \ddot{U}_{\underline{1}} = \int_{(B)}\left(\ddot{E}_{\underline{1}kl}C^{klpq}E_{\underline{1}pq}+\dot{E}_{\underline{1}kl}C^{klpq}\dot{E}_{\underline{1}pq}\right)\mathrm{d}V,\qquad \text{and so on.} \tag{26}
$$

First let us derive the following quantities,

$$
\begin{aligned}
\dot{E}_{\underline{1}kl}C^{klpq} &= \dot{u}_{\underline{1}m;k}(\delta_l^m + u_{\underline{1}}{}^m{}_{;l})C^{klpq}, \\
\ddot{E}_{\underline{1}kl}C^{klpq} &= \left[\ddot{u}_{\underline{1}m;k}(\delta_l^m + u_{\underline{1}}{}^m{}_{;l}) + \dot{u}_{\underline{1}m;k}\dot{u}_{\underline{1}}{}^m{}_{;l}\right] C^{klpq}, &(27)\\
\dddot{E}_{\underline{1}kl}C^{klpq} &= \left[\dddot{u}_{\underline{1}m;k}(\delta_l^m + u_{\underline{1}}{}^m{}_{;l}) + 3\ddot{u}_{\underline{1}m;k}\dot{u}_{\underline{1}}{}^m{}_{;l}\right] C^{klpq}, &(28)\\
\ddddot{E}_{\underline{1}kl}C^{klpq} &= \left[\ddddot{u}_{\underline{1}m;k}(\delta_l^m + u_{\underline{1}}{}^m{}_{;l}) + 4\dddot{u}_{\underline{1}m;k}\dot{u}_{\underline{1}}{}^m{}_{;l} + 3\ddot{u}_{\underline{1}m;k}\ddot{u}_{\underline{1}}{}^m{}_{;l}\right] C^{klpq}, &(29)
\end{aligned}
$$

and then we introduce some new notations so that we can change to the finite element discretization,

$$
\int_{(B)} \dot{u}_{\underline{1}m;k}(\delta_l^m + u_{\underline{1}}{}^m{}_{;l})C^{klpq}E_{\underline{1}pq}\,\mathrm{d}V = \dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{b}_{\underline{1}\alpha}\,, \tag{30}
$$

$$
\int_{(B)} \Big[\dot{u}_{\underline{1}m;k}(\delta_l^m + u_{\underline{1}}{}^m{}_{;l})C^{klpq}(\delta_p^s + u_{\underline{1}}{}^s{}_{;p})\dot{u}_{\underline{1}s;q} + \dot{u}_{\underline{1}m;k}\dot{u}_{\underline{1}}{}^m{}_{;l}C^{klpq}E_{\underline{1}pq}\Big]\,\mathrm{d}V = \dot{\mathbf{t}}_{\underline{1}\alpha}\dot{\mathbf{b}}_{\underline{1}\alpha} = \dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{T}}_{\underline{1}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta}\,, \tag{31}
$$

$$
3\int_{(B)} \dot{u}_{\underline{1}m;k}\dot{u}_{\underline{1}}{}^m{}_{;l}C^{klpq}(\delta_p^s + u_{\underline{1}}{}^s{}_{;p})\ddot{u}_{\underline{1}s;q}\,\mathrm{d}V = \dot{\mathbf{t}}_{\underline{1}\alpha}\boldsymbol{M}_{\underline{1}\alpha\beta\gamma}\ddot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}\,, \tag{32}
$$

$$
3\int_{(B)} \dot{u}_{\underline{1}m;k}\dot{u}_{\underline{1}}{}^m{}_{;l}C^{klpq}\dot{u}_{\underline{1}}{}^s{}_{;p}\dot{u}_{\underline{1}s;q}\,\mathrm{d}V = \dot{\mathbf{t}}_{\underline{1}\alpha}\boldsymbol{P}_{\underline{1}\alpha\beta\gamma\vartheta}\dot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}\dot{\mathbf{t}}_{\underline{1}\vartheta}\,. \tag{33}
$$

Thereafter the value of ΔU can be calculated from (25) by utilizing Eqs. (26)–(29) and (30)–(33). As to the results see the negative terms in (34).

3.3. The total work increment on an arbitrary displacement increment

With regard to the foregoing, the work increment done by the external forces exerted on the equilibrium configuration $(B_{\underline{1}}') \equiv (\overline{B})$ and the internal forces throughout the displacement increment Δu^k is

$$
\begin{aligned}
\Delta L = \Delta W - \Delta U = &\left[\left(\sum\nolimits_{\mathrm{N=I}}^{\mathrm{II}} p_{\underline{1}\mathrm{N}}\dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{a}_{\underline{1}\mathrm{N}\alpha}\right) - \dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{b}_{\underline{1}\alpha}\right](\Delta\tau)\\
&+ \left[\sum\nolimits_{\mathrm{N=I}}^{\mathrm{II}} p_{\underline{1}\mathrm{N}}\left(\ddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{a}_{\underline{1}\mathrm{N}\alpha} + \dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta}\right) - \left(\ddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{b}_{\underline{1}\alpha} + \dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{T}}_{\underline{1}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta}\right)\right]\frac{1}{2}(\Delta\tau)^2\\
&+ \left[\sum\nolimits_{\mathrm{N=I}}^{\mathrm{II}} p_{\underline{1}\mathrm{N}}\left(\dddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{a}_{\underline{1}\mathrm{N}\alpha} + 2\ddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta} + \dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\ddot{\mathbf{t}}_{\underline{1}\beta} + \dot{\mathbf{t}}_{\underline{1}\alpha}\boldsymbol{L}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta\gamma}\dot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}\right)\right]\frac{1}{6}(\Delta\tau)^3\\
&- \left(\dddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{b}_{\underline{1}\alpha} + 2\ddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{T}}_{\underline{1}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta} + \dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{T}}_{\underline{1}\alpha\beta}\ddot{\mathbf{t}}_{\underline{1}\beta} + \dot{\mathbf{t}}_{\underline{1}\alpha}\boldsymbol{M}_{\underline{1}\alpha\beta\gamma}\dot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}\right)\frac{1}{6}(\Delta\tau)^3\\
&+ \left[\sum\nolimits_{\mathrm{N=I}}^{\mathrm{II}} p_{\underline{1}\mathrm{N}}\left(\ddddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{a}_{\underline{1}\mathrm{N}\alpha} + 3\dddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta} + 3\ddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\ddot{\mathbf{t}}_{\underline{1}\beta}\right)\right]\frac{1}{24}(\Delta\tau)^4\\
&+ \left[\sum\nolimits_{\mathrm{N=I}}^{\mathrm{II}} p_{\underline{1}\mathrm{N}}\left(3\ddot{\mathbf{t}}_{\underline{1}\alpha}\boldsymbol{L}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta\gamma}\dot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma} + \dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\dddot{\mathbf{t}}_{\underline{1}\beta} + 3\dot{\mathbf{t}}_{\underline{1}\alpha}\boldsymbol{L}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta\gamma}\ddot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}\right)\right]\frac{1}{24}(\Delta\tau)^4\\
&- \left(\ddddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{b}_{\underline{1}\alpha} + 3\dddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{T}}_{\underline{1}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta} + \dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{T}}_{\underline{1}\alpha\beta}\dddot{\mathbf{t}}_{\underline{1}\beta} + 3\ddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{T}}_{\underline{1}\alpha\beta}\ddot{\mathbf{t}}_{\underline{1}\beta} + 3\ddot{\mathbf{t}}_{\underline{1}\alpha}\boldsymbol{M}_{\underline{1}\alpha\beta\gamma}\dot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}\right)\frac{1}{24}(\Delta\tau)^4\\
&- \left(\dot{\mathbf{t}}_{\underline{1}\alpha}\boldsymbol{M}_{\underline{1}\alpha\beta\gamma}\dot{\mathbf{t}}_{\underline{1}\beta}\ddot{\mathbf{t}}_{\underline{1}\gamma} + 2\dot{\mathbf{t}}_{\underline{1}\alpha}\boldsymbol{M}_{\underline{1}\alpha\beta\gamma}\ddot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma} + \dot{\mathbf{t}}_{\underline{1}\alpha}\boldsymbol{P}_{\underline{1}\alpha\beta\gamma\theta}\dot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}\dot{\mathbf{t}}_{\underline{1}\vartheta}\right)\frac{1}{24}(\Delta\tau)^4 + \cdots\,.
\end{aligned} \tag{34}
$$

3.4. Formulation of the alternative iteration procedure

In Section 2.3 the Newton-Raphson iteration has been applied for the solution of the incremental form of the principle of virtual work. An alternative procedure is obtained by substituting truncated series into Eq. (10),

$$\int_{(B)}\left\{\delta\left[\dot{E}_{\underline{1}kl}(\Delta\tau)+\frac{1}{2}\ddot{E}_{\underline{1}kl}(\Delta\tau)^2+\frac{1}{6}\dddot{E}_{\underline{1}kl}(\Delta\tau)^3+\cdots\right]\right.$$
$$\left.\times C^{klpq}\left[E_{\underline{1}pq}+\dot{E}_{\underline{1}pq}(\Delta\tau)+\frac{1}{2}\ddot{E}_{\underline{1}pq}(\Delta\tau)^2+\frac{1}{6}\dddot{E}_{\underline{1}pq}(\Delta\tau)^3+\cdots\right]\right\}\mathrm{d}V=$$
$$=\sum_{\mathrm{N=I}}^{\mathrm{II}}\left\{\left[p_{\underline{1}\mathrm{N}}+\dot{p}_{\underline{1}\mathrm{N}}(\Delta\tau)+\frac{1}{2}\ddot{p}_{\underline{1}\mathrm{N}}(\Delta\tau)^2+\frac{1}{6}\dddot{p}_{\underline{1}\mathrm{N}}(\Delta\tau)^3+\cdots\right]\right.$$
$$\left.\times\int_{(A_{t\mathrm{N}})}\widetilde{p}_{0\mathrm{N}}\delta u^k\left[Q_{\underline{1}k}{}^p+\dot{Q}_{\underline{1}k}{}^p(\Delta\tau)+\frac{1}{2}\ddot{Q}_{\underline{1}k}{}^p(\Delta\tau)^2+\frac{1}{6}\dddot{Q}_{\underline{1}k}{}^p(\Delta\tau)^3+\cdots\right]\mathrm{d}A_p\right\}.$$

Proceeding to the FE discretization we shall take the relations

$$\delta\left[\dot{u}_{k;l}(\Delta\tau)\right]=\delta(\Delta u_{k;l})=\delta u_{k;l}\qquad\text{and}\qquad\delta\left[\ddot{u}_{k;l}(\Delta\tau)\right]=\delta\Delta\dot{u}_{k;l}=0$$

into account and shall employ the notations introduced by means of the Eqs. (27)–(33) and (21)–(24). In this way we have

$$\delta\mathbf{t}_\alpha\left[\mathbf{b}_{\underline{1}\alpha}-\sum_{\mathrm{N=I}}^{\mathrm{II}}p_{\underline{1}\mathrm{N}}\mathbf{a}_{\underline{1}\mathrm{N}\alpha}\right]+\delta\mathbf{t}_\alpha\left[\mathbf{K}^{\mathrm{T}}_{\underline{1}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta}-\sum_{\mathrm{N=I}}^{\mathrm{II}}\left(p_{\underline{1}\mathrm{N}}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta}+\dot{p}_{\underline{1}\mathrm{N}}\mathbf{a}_{\underline{1}\mathrm{N}\alpha}\right)\right](\Delta\tau)$$
$$+\,\delta\mathbf{t}_\alpha\left[\mathbf{K}^{\mathrm{T}}_{\underline{1}\alpha\beta}\ddot{\mathbf{t}}_{\underline{1}\beta}+\boldsymbol{M}_{\underline{1}\alpha\beta\gamma}\dot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}\right]\frac{1}{2}(\Delta\tau)^2$$
$$-\,\delta\mathbf{t}_\alpha\left[\sum_{\mathrm{N=I}}^{\mathrm{II}}\left(p_{\underline{1}\mathrm{N}}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\ddot{\mathbf{t}}_{\underline{1}\beta}+p_{\underline{1}\mathrm{N}}\boldsymbol{L}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta\gamma}\dot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}+2\dot{p}_{\underline{1}\mathrm{N}}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta}+\ddot{p}_{\underline{1}\mathrm{N}}\mathbf{a}_{\underline{1}\mathrm{N}\alpha}\right)\right]\frac{1}{2}(\Delta\tau)^2+\cdots \tag{35}$$

The spacious term multiplied by $(\Delta\tau)^3$ is not written here (see formula (39)).

Since the incremental form of the principle of virtual work (35) should hold for arbitrary $(\Delta\tau)$ and $\delta\mathbf{t}_\alpha$ the following equations are obtained,

$$\sum_{\mathrm{N=I}}^{\mathrm{II}}p_{\underline{1}\mathrm{N}}\mathbf{a}_{\underline{1}\mathrm{N}\alpha}-\mathbf{b}_{\underline{1}\alpha}=0, \tag{36}$$

$$\sum_{\mathrm{N=I}}^{\mathrm{II}}\left(p_{\underline{1}\mathrm{N}}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta}+\dot{p}_{\underline{1}\mathrm{N}}\mathbf{a}_{\underline{1}\mathrm{N}\alpha}\right)-\mathbf{K}^{\mathrm{T}}_{\underline{1}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta}=0, \tag{37}$$

$$\sum_{\mathrm{N=I}}^{\mathrm{II}}\left(p_{\underline{1}\mathrm{N}}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\ddot{\mathbf{t}}_{\underline{1}\beta}+p_{\underline{1}\mathrm{N}}\boldsymbol{L}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta\gamma}\dot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}+2\dot{p}_{\underline{1}\mathrm{N}}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta}+\ddot{p}_{\underline{1}\mathrm{N}}\mathbf{a}_{\underline{1}\mathrm{N}\alpha}\right)$$
$$-\,\mathbf{K}^{\mathrm{T}}_{\underline{1}\alpha\beta}\ddot{\mathbf{t}}_{\underline{1}\beta}+\boldsymbol{M}_{\underline{1}\alpha\beta\gamma}\dot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}=0, \tag{38}$$

$$\left[\sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\underline{1}\mathrm{N}}\left(\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\dddot{\mathbf{t}}_{\underline{1}\beta}+3\boldsymbol{L}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta\gamma}\ddot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}\right)\right]$$
$$+\left[\sum_{\mathrm{N=I}}^{\mathrm{II}} 3\dot{p}_{\underline{1}\mathrm{N}}\left(\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\ddot{\mathbf{t}}_{\underline{1}\beta}+\boldsymbol{L}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta\gamma}\dot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}\right)+3\ddot{p}_{\underline{1}\mathrm{N}}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta}+\dddot{p}_{\underline{1}\mathrm{N}}\mathbf{a}_{\underline{1}\mathrm{N}\alpha}\right]$$
$$-\left[\mathbf{K}^{\mathrm{T}}_{\underline{1}\alpha\beta}\dddot{\mathbf{t}}_{\underline{1}\beta}+2\boldsymbol{M}_{\underline{1}\alpha\beta\gamma}\ddot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}+\boldsymbol{M}_{\underline{1}\alpha\beta\gamma}\dot{\mathbf{t}}_{\underline{1}\beta}\ddot{\mathbf{t}}_{\underline{1}\gamma}+\boldsymbol{P}_{\underline{1}\alpha\beta\gamma\vartheta}\dot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}\dot{\mathbf{t}}_{\underline{1}\vartheta}\right]=0, \quad \text{and so on.} \tag{39}$$

Equation (36) is the equilibrium condition for the configuration $(B'_{\underline{1}}) \equiv (\overline{B})$. If load control is employed the unknown $\dot{\mathbf{t}}_{\underline{1}\beta}$ can be determined from (37) then the unknown $\ddot{\mathbf{t}}_{\underline{1}\beta}$ from (38) and so on. In the case of displacement control one should appropriately follow what has been written at the end of Section 2.3 regarding, in addition to the derivatives of noncontrolled parameters, the derivatives $\dot{p}_{\underline{1}\mathrm{N}}$, $\ddot{p}_{\underline{1}\mathrm{N}}$, $\dddot{p}_{\underline{1}\mathrm{N}}$, $(\mathrm{N}=\mathrm{I},\mathrm{II})$ of the load parameters as unknowns.

The alternative solution that has been presented is also an iterative procedure for the incremental form of the principle of virtual work. Observe that (36) together with the form of (37) multiplied by $(\Delta\tau)$ yield the first step of the Newton–Raphson iteration, for the latter see (13). Both algorithms are applicable for investigating geometrically imperfect continua.

3.5. The total work increment on a given equilibrium path

Formula (34) gives the work increment done by the tractions and the internal forces of the equilibrium configuration $(B'_{\underline{1}}) \equiv (\overline{B})$ *during an arbitrary kinematically admissible displacement increment.* The same quantity can be obtained on a *given equilibrium path*, i.e., making use of the derivatives $\dot{p}_{\underline{1}\mathrm{N}}$, $\ddot{p}_{\underline{1}\mathrm{N}}$, $\dddot{p}_{\underline{1}\mathrm{N}}, \ldots$, $\mathrm{N}=\mathrm{I},\mathrm{II}$, and $\dot{\mathbf{t}}_{\underline{1}\alpha}$, $\ddot{\mathbf{t}}_{\underline{1}\alpha}$, $\dddot{\mathbf{t}}_{\underline{1}\alpha}, \ldots$ which are determined from Eqs. (37)–(39). These parameters throughout Eqs. (16) and (18) give the equilibrium configuration $(\overline{B}+\Delta\overline{B})$. After substitutions we have

$$\Delta L=\Delta W-\Delta U=-\left(\sum_{\mathrm{N=I}}^{\mathrm{II}}\dot{p}_{\underline{1}\mathrm{N}}\dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{a}_{\underline{1}\mathrm{N}\alpha}\right)\frac{1}{2}(\Delta\tau)^2$$
$$-\left[\sum_{\mathrm{N=I}}^{\mathrm{II}}2\dot{p}_{\underline{1}\mathrm{N}}\left(\ddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{a}_{\underline{1}\mathrm{N}\alpha}+\dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta}\right)+\ddot{p}_{\underline{1}\mathrm{N}}\dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{a}_{\underline{1}\mathrm{N}\alpha}\right]\frac{1}{6}(\Delta\tau)^3$$
$$-\left[\sum_{\mathrm{N=I}}^{\mathrm{II}}3\dot{p}_{\underline{1}\mathrm{N}}\left(\dddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{a}_{\underline{1}\mathrm{N}\alpha}+2\ddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta}+\dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\ddot{\mathbf{t}}_{\underline{1}\beta}+\dot{\mathbf{t}}_{\underline{1}\alpha}\boldsymbol{L}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta\gamma}\dot{\mathbf{t}}_{\underline{1}\beta}\dot{\mathbf{t}}_{\underline{1}\gamma}\right)\right]\frac{1}{24}(\Delta\tau)^4$$
$$-\left[\sum_{\mathrm{N=I}}^{\mathrm{II}}3\ddot{p}_{\underline{1}\mathrm{N}}\left(\ddot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{a}_{\underline{1}\mathrm{N}\alpha}+\dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{K}^{\mathrm{L}}_{\underline{1}\mathrm{N}\alpha\beta}\dot{\mathbf{t}}_{\underline{1}\beta}\right)+\dddot{p}_{\underline{1}\mathrm{N}}\dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{a}_{\underline{1}\mathrm{N}\alpha}\right]\frac{1}{24}(\Delta\tau)^4-\cdots$$

or

$$\Delta L=\Delta W-\Delta U=\Delta L^{(2)}+\Delta L^{(3)}+\Delta L^{(4)}+\cdots, \tag{40}$$

where

$$\Delta L^{(2)}=-\frac{1}{2}\left\{\left[\dot{p}_{\underline{1}\mathrm{I}}(\Delta\tau)\right]\left[\dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{a}_{\underline{1}\mathrm{I}\alpha}(\Delta\tau)\right]+\left[\dot{p}_{\underline{1}\mathrm{II}}(\Delta\tau)\right]\left[\dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{a}_{\underline{1}\mathrm{II}\alpha}(\Delta\tau)\right]\right\}, \tag{41}$$

$$\Delta L^{(3)}=-\frac{1}{3}\left\{\left[\dot{p}_{\underline{1}\mathrm{I}}(\Delta\tau)\right]\left[(\dot{\mathbf{t}}_{\alpha}\mathbf{a}_{\mathrm{I}\alpha})^{\cdot}_{\underline{1}}(\Delta\tau)^2\right]+\left[\dot{p}_{\underline{1}\mathrm{II}}(\Delta\tau)\right]\left[(\dot{\mathbf{t}}_{\alpha}\mathbf{a}_{\mathrm{II}\alpha})^{\cdot}_{\underline{1}}(\Delta\tau)^2\right]\right\}$$
$$-\frac{1}{3}\left\{\left[\frac{1}{2}\ddot{p}_{\underline{1}\mathrm{I}}(\Delta\tau)^2\right]\left[(\dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{a}_{\underline{1}\mathrm{I}\alpha})(\Delta\tau)\right]+\left[\frac{1}{2}\ddot{p}_{\underline{1}\mathrm{II}}(\Delta\tau)^2\right]\left[(\dot{\mathbf{t}}_{\underline{1}\alpha}\mathbf{a}_{\underline{1}\mathrm{II}\alpha})(\Delta\tau)\right]\right\}, \tag{42}$$

and

$$\Delta L^{(4)} = -\frac{1}{4}\left\{[\dot{p}_{\underline{1}\mathrm{I}}(\Delta\tau)]\left[\frac{1}{2}(\dot{\mathbf{t}}_\alpha \mathbf{a}_{\mathrm{I}\alpha})^{\cdot\cdot}_{\underline{1}}(\Delta\tau)^3\right] + [\dot{p}_{\underline{1}\mathrm{II}}(\Delta\tau)]\left[\frac{1}{2}(\dot{\mathbf{t}}_\alpha \mathbf{a}_{\mathrm{II}\alpha})^{\cdot\cdot}_{\underline{1}}(\Delta\tau)^3\right]\right\}$$
$$-\frac{1}{4}\left\{\left[\frac{1}{2}\ddot{p}_{\underline{1}\mathrm{I}}(\Delta\tau)^2\right]\left[(\dot{\mathbf{t}}_\alpha \mathbf{a}_{\mathrm{I}\alpha})^{\cdot}_{\underline{1}}(\Delta\tau)^2\right] + \left[\frac{1}{2}\ddot{p}_{\underline{1}\mathrm{II}}(\Delta\tau)^2\right]\left[(\dot{\mathbf{t}}_\alpha \mathbf{a}_{\mathrm{II}\alpha})^{\cdot}_{\underline{1}}(\Delta\tau)^2\right]\right\}$$
$$-\frac{1}{4}\left\{\left[\frac{1}{6}\dddot{p}_{\underline{1}\mathrm{I}}(\Delta\tau)^3\right]\left[(\dot{\mathbf{t}}_{\underline{1}\alpha} \mathbf{a}_{\underline{1}\mathrm{I}\alpha})(\Delta\tau)\right] + \left[\frac{1}{6}\dddot{p}_{\underline{1}\mathrm{II}}(\Delta\tau)^3\right]\left[(\dot{\mathbf{t}}_{\underline{1}\alpha} \mathbf{a}_{\underline{1}\mathrm{II}\alpha})(\Delta\tau)\right]\right\}. \tag{43}$$

Writing the equations in continuum form we get

$$\Delta L^{(2)} = -\frac{1}{2}\left\{[\dot{p}_{\underline{1}\mathrm{I}}(\Delta\tau)]\int_{(A_{t\mathrm{I}})}\widetilde{p}_{0\mathrm{I}}\,\dot{u}^k_{\underline{1}}(\Delta\tau)Q_{\underline{1}k}{}^p\,\mathrm{d}A_p + [\dot{p}_{\underline{1}\mathrm{II}}(\Delta\tau)]\int_{(A_{t\mathrm{II}})}\widetilde{p}_{0\mathrm{II}}\,\dot{u}^k_{\underline{1}}(\Delta\tau)Q_{\underline{1}k}{}^p\,\mathrm{d}A_p\right\}, \tag{44}$$

$$\Delta L^{(3)} = -\frac{1}{3}\left\{[\dot{p}_{\underline{1}\mathrm{I}}(\Delta\tau)]\int_{(A_{t\mathrm{I}})}\widetilde{p}_{0\mathrm{I}}\left[\dot{u}^k(\Delta\tau)Q_k{}^p\right]^{\cdot}_{\underline{1}}(\Delta\tau)\,\mathrm{d}A_p\right.$$
$$\left.+ [\dot{p}_{\underline{1}\mathrm{II}}(\Delta\tau)]\int_{(A_{t\mathrm{II}})}\widetilde{p}_{0\mathrm{II}}\left[\dot{u}^k(\Delta\tau)Q_k{}^p\right]^{\cdot}_{\underline{1}}(\Delta\tau)\,\mathrm{d}A_p\right\}$$
$$-\frac{1}{3}\left\{\left[\frac{1}{2}\ddot{p}_{\underline{1}\mathrm{I}}(\Delta\tau)^2\right]\int_{(A_{t\mathrm{I}})}\widetilde{p}_{0\mathrm{I}}\,\dot{u}^k_{\underline{1}}(\Delta\tau)Q_{\underline{1}k}{}^p\,\mathrm{d}A_p\right.$$
$$\left.+\left[\frac{1}{2}\ddot{p}_{\underline{1}\mathrm{II}}(\Delta\tau)^2\right]\int_{(A_{t\mathrm{II}})}\widetilde{p}_{0\mathrm{II}}\,\dot{u}^k_{\underline{1}}(\Delta\tau)Q_{\underline{1}k}{}^p\,\mathrm{d}A_p\right\}, \tag{45}$$

and

$$\Delta L^{(4)} = -\frac{1}{4}\left\{[\dot{p}_{\underline{1}\mathrm{I}}(\Delta\tau)]\int_{(A_{t\mathrm{I}})}\widetilde{p}_{0\mathrm{I}}\,\frac{1}{2}\left[\dot{u}^k(\Delta\tau)Q_k{}^p\right]^{\cdot\cdot}_{\underline{1}}(\Delta\tau)^2\,\mathrm{d}A_p\right.$$
$$\left.+ [\dot{p}_{\underline{1}\mathrm{II}}(\Delta\tau)]\int_{(A_{t\mathrm{II}})}\widetilde{p}_{0\mathrm{II}}\,\frac{1}{2}\left[\dot{u}^k(\Delta\tau)Q_k{}^p\right]^{\cdot\cdot}_{\underline{1}}(\Delta\tau)^2\,\mathrm{d}A_p\right\}$$
$$-\frac{1}{4}\left\{\left[\frac{1}{2}\ddot{p}_{\underline{1}\mathrm{I}}(\Delta\tau)^2\right]\int_{(A_{t\mathrm{I}})}\widetilde{p}_{0\mathrm{I}}\left[\dot{u}^k(\Delta\tau)Q_k{}^p\right]^{\cdot}_{\underline{1}}(\Delta\tau)\,\mathrm{d}A_p\right.$$
$$\left.+\left[\frac{1}{2}\ddot{p}_{\underline{1}\mathrm{II}}(\Delta\tau)^2\right]\int_{(A_{t\mathrm{II}})}\widetilde{p}_{0\mathrm{II}}\left[\dot{u}^k(\Delta\tau)Q_k{}^p\right]^{\cdot}_{\underline{1}}(\Delta\tau)\,\mathrm{d}A_p\right\}$$
$$-\frac{1}{4}\left\{\left[\frac{1}{6}\dddot{p}_{\underline{1}\mathrm{I}}(\Delta\tau)^3\right]\int_{(A_{t\mathrm{I}})}\widetilde{p}_{0\mathrm{I}}\,\dot{u}^k_{\underline{1}}(\Delta\tau)Q_{\underline{1}k}{}^p\,\mathrm{d}A_p\right.$$
$$\left.+\left[\frac{1}{6}\dddot{p}_{\underline{1}\mathrm{II}}(\Delta\tau)^2\right]\int_{(A_{t\mathrm{II}})}\widetilde{p}_{0\mathrm{II}}\,\dot{u}^k_{\underline{1}}(\Delta\tau)Q_{\underline{1}k}{}^p\,\mathrm{d}A_p\right\}. \tag{46}$$

4. An alternative stability analysis method

In this study an equilibrium configuration is regarded as stable if the work done by the total force system by which we mean tractions and the internal forces during any arbitrary kinematically admissible small displacement increment is negative.

This statement can be better understood if it is assumed that there is some (but very small) kinetic energy in the equilibrium state of continuum and then the kinetic energy is decreased on arbitrary small kinematically admissible displacements in accordance with the principle of the mechanical work and energy. That is, the movement of the continuum is restricted to a small neighborhood of the equilibrium configuration.

Otherwise if the work of the total force system during arbitrary kinematically admissible small displacement increments is positive or zero then the equilibrium state is unstable.

According to the stability definition given above, the work increment $\Delta L = \Delta W - \Delta U$ is applicable to characterize the static stability of the equilibrium configuration $(B'_{\underline{1}}) \equiv (\overline{B})$ in both pre- and postbuckling states. The stability criteria can be summarized as follows:

The equilibrium of the configuration $(B'_{\underline{1}}) \equiv (\overline{B})$ is

$$\left\{\begin{array}{c}\text{stable}\\ \text{transitional (critical)}\\ \text{unstable}\end{array}\right\} \text{ if } \left\{\begin{array}{ll}\Delta L^{(2)} < 0 & \text{for any } \Delta u^k\\ \Delta L^{(2)} = 0 & \text{for some } \Delta u^k\\ \Delta L^{(2)} > 0 & \text{for some } \Delta u^k\end{array}\right\} \tag{47}$$

The transitional (critical) equilibrium configuration is characterized by

$$\left\{\begin{array}{c}\text{a limit point}\\ \text{a bifurcation point}\end{array}\right\} \text{ if } \left\{\begin{array}{l}\Delta L^{(3)} \neq 0 \text{ for any } \Delta u^k\\ \Delta L^{(3)} = 0 \text{ for a given } \Delta u^k \text{ and } \Delta L^{(4)} \neq 0 \text{ for any } \Delta u^k\end{array}\right\}. \tag{48}$$

When a computed equilibrium path is considered on a displacement increment Δu^k, the stability definitions are modified as follows:

The equilibrium of the configuration $(B'_{\underline{1}}) \equiv (\overline{B})$ is

$$\left\{\begin{array}{c}\text{stable}\\ \text{transitional (critical)}\\ \text{unstable}\end{array}\right\} \text{ if } \left\{\begin{array}{l}\Delta L^{(2)} < 0\\ \Delta L^{(2)} = 0\\ \Delta L^{(2)} > 0\end{array}\right\}. \tag{49}$$

In this case, the transitional (critical) equilibrium configuration is characterized by

$$\left\{\begin{array}{c}\text{a limit point}\\ \text{a bifurcation point}\end{array}\right\} \text{ if } \left\{\begin{array}{l}\Delta L^{(3)} \neq 0\\ \Delta L^{(3)} = 0\end{array}\right\}. \tag{50}$$

It is obvious that the bifurcation point must be situated on the equilibrium path.

The set of formulae (41)–(43) and (37)–(39) together with the arbitrary parameters $\dot{p}_{\underline{1}\mathrm{I}}$, $\ddot{p}_{\underline{1}\mathrm{I}}$, $\dddot{p}_{\underline{1}\mathrm{I}}$, $\dot{p}_{\underline{1}\mathrm{II}}$, $\ddot{p}_{\underline{1}\mathrm{II}}$, $\dddot{p}_{\underline{1}\mathrm{II}}$ can be used for calculating $\Delta L^{(2)}$, $\Delta L^{(3)}$ and $\Delta L^{(4)}$ for the stability analysis of any equilibrium configuration $(B'_{\underline{1}}) \equiv (\overline{B})$.

In concrete numerical computations one should use a set of the derivatives of the load parameters with respect to the control parameter $\dot{p}_{\underline{1}\mathrm{I}}$, $\ddot{p}_{\underline{1}\mathrm{I}}$, $\dddot{p}_{\underline{1}\mathrm{I}}$, $\dot{p}_{\underline{1}\mathrm{II}}$, $\ddot{p}_{\underline{1}\mathrm{II}}$, $\dddot{p}_{\underline{1}\mathrm{II}}$. At a limit case this set can be formed by one of the derivatives.

Especially this is the case if the whole equilibrium surface is required in order to analyze equilibrium configurations in a region and not only in a few points. It is worthy of mention that the method requires some intuition from the user when he selects an appropriate secondary load in addition to the primary one. It is recommended to select a secondary load $\widetilde{p}_{\mathrm{II}}$ in addition to the primary $\widetilde{p}_{\mathrm{I}}$ which itself produces displacements similar to the postbuckling displacements. In Section 5 we present examples for postcritical stability analysis of circular rings.

5. Equilibrium configurations of a circular ring and their stability

5.1. Problem definition

Consider a circular ring of rectangular cross-section loaded by deformation dependent surface loads given by Eqs. (8) and (9) (see Fig. 1). Knowing that the first buckling mode is symmetric it is enough to investigate a quarter of the ring for which symmetric boundary conditions are imposed on the edges that lie on the axes of symmetry of the deformation. It is worthy of mention that, in addition to the primary load $\widetilde{p}_{\mathrm{I}}$, the secondary load $\widetilde{p}_{\mathrm{II}}$ is selected in a way that the displacements due to this load are similar to the postcritical displacements due to the primary load. The two load parameters p_{I}, p_{II}, and the displacement parameter v_A are shown in Fig. 1, the area of (A_{tII}) is one tenth of (A_{tI}).

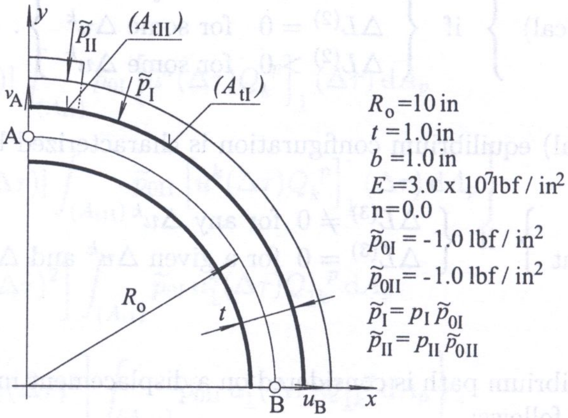

Fig. 1. Elastic circular ring

For the primary load parameter $\widetilde{p}_{\mathrm{I}}$ ($\widetilde{p}_{\mathrm{II}} = 0$) Kozák et al. [8] determined the critical load parameter $p_{cr} = 7422.8$, at the bifurcation on the fundamental path by applying a 3D FE code with 10 hexahedral elements using p-extension ($p = 5$). The algorithm is a path following method which rests on the Newton–Raphson iteration. The load stiffness matrix is non-symmetric. In this study there are also examples for geometrically imperfect rings modeled by elliptic rings. The horizontal major and vertical minor axes for the ellipses considered are $2a = 2R_o = 20$ in, $2b = 19.9$ in or $2b = 19.99$ in. Reducing the problem to a linear eigenvalue problem $p_{cr} = 7424.9$ was found for the critical load parameter. The result of beam model was: $p_{cr} = 7500$. The authors later performed computations with symmetrized load stiffness matrix as it was recommended by Mang [11], and $p_{cr} = 7408$ was obtained using our 3D FE code.

The same FE mesh is used in this paper as in [8] applying the Newton–Raphson iteration. Equilibrium configurations are considered in pre- and postbuckling states. Our primary aim is to determine the equilibrium surface $p_{\mathrm{II}} = f_1(p_{\mathrm{I}}, v_A)$, from which partial derivatives of p_{II} with respect to the displacement control parameter v_A can be obtained as surfaces

$$\dot{p}_{\mathrm{II}} = \frac{\partial p_{\mathrm{II}}}{\partial v_A} = f_2(p_{\mathrm{I}}, v_A), \qquad \ddot{p}_{\mathrm{II}} = \frac{\partial^2 p_{\mathrm{II}}}{\partial v_A^2} = f_3(p_{\mathrm{I}}, v_A).$$

These three surfaces serve as a basis for the stability analysis, i.e., to find the limit points, to determine the unstable region and the bifurcation points as well. In addition to the geometrically perfect circular ring, two elliptical rings have also been calculated as examples for the geometrically imperfect rings.

5.2. Circular ring

As a results of preliminary computations two equilibrium paths are shown in Figs. 2 and 3. The first is computed with a constant primary load parameter p_{I} and one variable secondary load parameter p_{II}, the second is computed by interchanging the two parameters. Both paths have been obtained with the mixed control method.

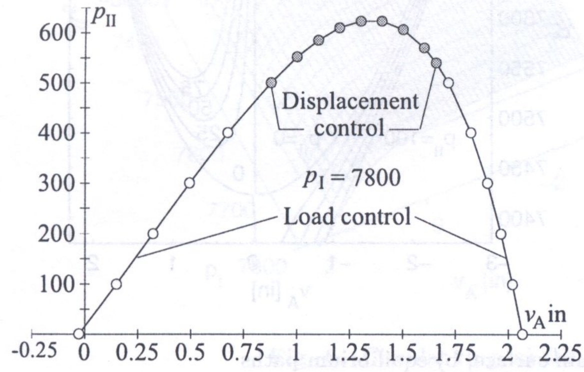

Fig. 2. Equilibrium path, $p_{\mathrm{I}} = \mathrm{const}$

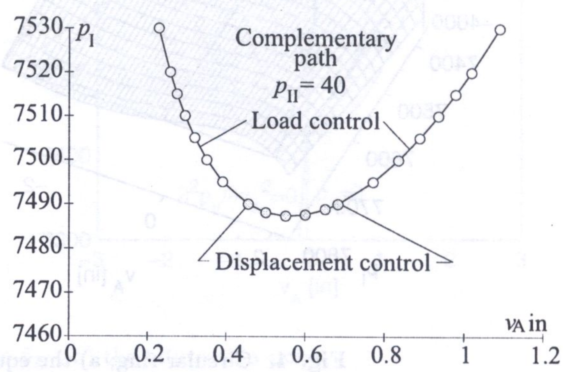

Fig. 3. Equilibrium path, $p_{\mathrm{II}} = \mathrm{const}$

Further computations have been performed on the regions

$$-3.0 \le v_A \le 3.0 \text{ in} \qquad \text{and} \qquad 7300 \le p_{\mathrm{I}} \le 7800$$

using constant increments $\Delta v_A = 0.1$ and $\Delta p_{\mathrm{I}} = 25$ and applying displacement control at the constant values of the load parameter p_{I}. In this way numerous equilibrium paths are determined which form the equilibrium surface $p_{\mathrm{II}} = f_1(p_{\mathrm{I}}, v_A)$ and it holds that $\dot{p}_{\mathrm{I}} = \ddot{p}_{\mathrm{I}} = \dddot{p}_{\mathrm{I}} = 0$ at every point of the surface.

It is easy to show that the surfaces of $\Delta L^{(2)} = \Delta L^{(2)}(p_{\mathrm{I}}, v_A)$, $\Delta L^{(3)} = \Delta L^{(3)}(p_{\mathrm{I}}, v_A)$, and $\Delta L^{(4)} = \Delta L^{(4)}(p_{\mathrm{I}}, v_A)$ are not required directly, what we need for the stability is only the sign of these quantities. Since v_A is the control parameter and p_{I} is constant we assume that $\Delta\tau = \Delta v_A$ in the finite element solution. It also follows from the nature of the problem, see Fig. 1 for details, that we can use the approximations $\Delta\tau\, \dot{u}^k = \Delta u^k$ where $\Delta u^1 \approx \Delta v_A > 0$ and $\Delta u^2 = \Delta u^3 \approx 0$ for any values of p_{I} and v_A, and the sign of the second integral in Eq. (44) is negative

$$\int_{(A_{t\mathrm{II}})} \widetilde{p}_{0\mathrm{II}}\, \dot{u}^k(\Delta\tau) Q_k{}^p\, \mathrm{d}A_p = \int_{(A_{t\mathrm{II}})} \widetilde{p}_{0\mathrm{II}}\, \Delta u^k Q_k{}^p\, \mathrm{d}A_p < 0. \tag{51}$$

Therefore the sign of $\Delta L^{(2)}$ is determined by the sign of the first derivative $\dot{p}_{\mathrm{II}} = f_2(p_{\mathrm{I}}, v_A)$. However, if $\dot{p}_{\mathrm{II}} = 0$, the sign of $\Delta L^{(3)}$ is required; it follows from (45) that the latter sign is determined by the second derivative $\ddot{p}_{\mathrm{II}} = f_3(p_{\mathrm{I}}, v_A)$.

The surfaces $p_{\mathrm{II}} = f_1(p_{\mathrm{I}}, v_A)$, $\dot{p}_{\mathrm{II}} = f_2(p_{\mathrm{I}}, v_A)$, $\ddot{p}_{\mathrm{II}} = f_3(p_{\mathrm{I}}, v_A)$, and the corresponding intersections with the horizontal zero plane are respectively shown in Figs. 4a, 5b and 6a. A number of equilibrium paths can be produced by further intersections of $p_{\mathrm{II}} = f_1(p_{\mathrm{I}}, v_A)$ with the planes determined by different but fixed values of p_{II} (see Figs. 4b, 5a).

The curves for which

$$p_{\mathrm{II}} = 0 = g_1(p_{\mathrm{I}}, v_A), \qquad \dot{p}_{\mathrm{II}} = \frac{\partial p_{\mathrm{II}}}{\partial v_A} = 0 = g_2(p_{\mathrm{I}}, v_A) \qquad \text{and} \qquad \ddot{p}_{\mathrm{II}} = \frac{\partial^2 p_{\mathrm{II}}}{\partial v_A^2} = 0 = g_3(p_{\mathrm{I}}, v_A)$$

are also shown in Fig. 6b.

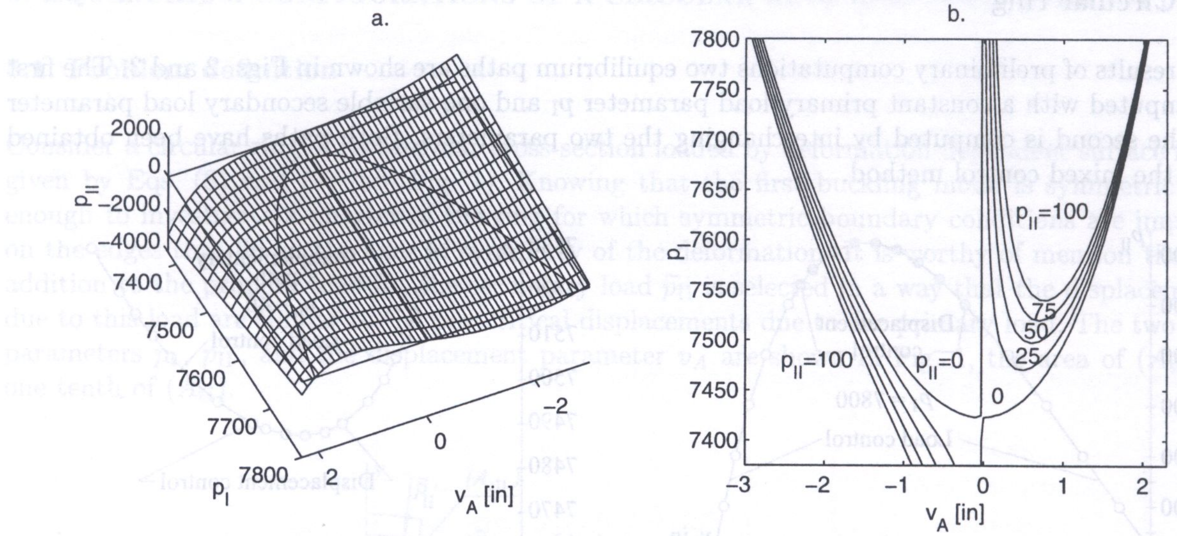

Fig. 4. Circular ring; a) the equilibrium surface, b) equilibrium paths

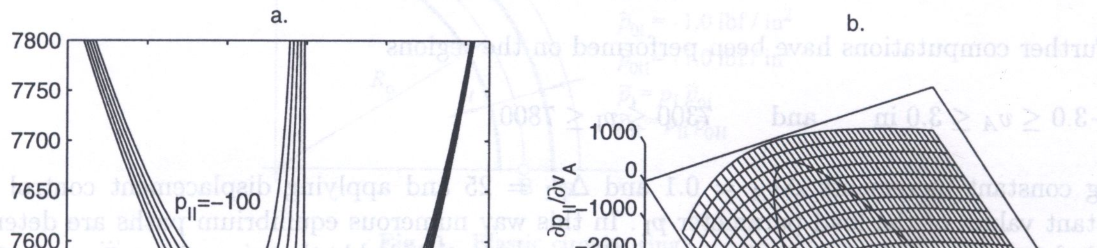

Fig. 5. Circular ring; a) equilibrium paths, b) the surface $\dot{p}_{II}$

Taking the above remarks concerning (44) into considerations we can conclude that at those places where $g_2(p_I, v_A) = 0$ it holds that $\Delta L^{(2)} > 0$ and where $p_{II} = f_2(p_I, v_A) < 0$ there also $\Delta L^{(2)} > 0$, i.e., the equilibrium is unstable at the latter places. The unstable regions for which $p_I = \text{const}$ is the equilibrium path, are situated between the two branches of the parabolic type curve $\dot{p}_{II}$ (see Fig. 6b).

According to (50) and (45) the bifurcation point is situated at the common point of the fundamental equilibrium path $p_{II} = g_1(p_I, v_A) = 0$ and the curves $g_2(p_I, v_A) = 0$ and $g_3(p_I, v_A) = 0$ that is where $\Delta L^{(3)} = 0$ but the fourth order term is non zero $\Delta L^{(4)} \neq 0$. It holds for every point and also at the common point that $\dddot{p}_{II} \neq 0$ (see Fig. 6a), consequently it really follows from (46) and (51) that there $\Delta L^{(4)} \neq 0$.

Figure 6b shows the bifurcation path and the critical point for which $p_I = p_{cr} = 7422.8$ and $p_{II} = 0$ are the load parameters.

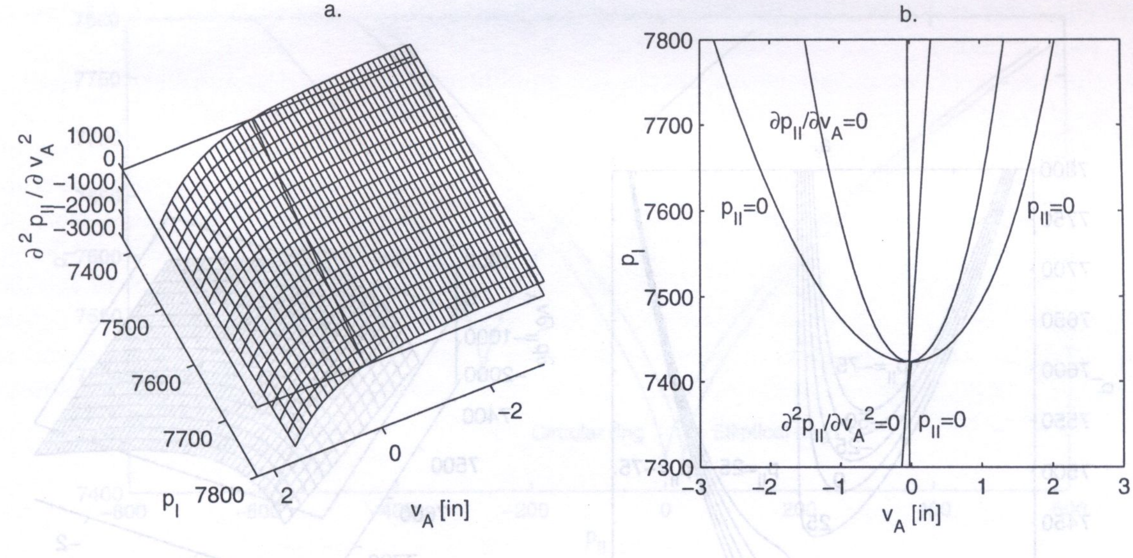

Fig. 6. Circular ring; a) the surface $\dot{p}_{II}$, b) the bifurcation point

5.3. Elliptical rings

Two elliptical rings, which differ only slightly from the circular ring, have been analyzed. For the elliptical ring1 $2a = 19.99$ in and $2b = 20$ in are the horizontal and vertical axes, respectively. However, for the other $2a = 20$ in and $2b = 19.9$ in.

Results for the elliptical ring1 can be seen in Figs. 7a, 8b, 9a, which represent the surfaces $f_1(p_I, v_A)$, $f_2(p_I, v_A)$, $f_3(p_I, v_A)$, respectively. Equilibrium paths obtained by intersecting the equilibrium surface $p_{II} = f_1(p_I, v_A)$ with $p_{II} = \text{const}$ planes can be seen in Fig. 7b. and 8a. The curves $p_{II} = f_1(p_I, v_A) = 26.9$, $g_2(p_I, v_A) = 0$, $g_3(p_I, v_A) = 0$ are in Fig. 9b. Here the unstable region is situated between the branches of the curve $\dot{p}_{II}$. It is worthy of mention that the bifurcation point presented is the common point of the equilibrium path $p_{II} = f_1(p_I, v_A) = 26.9$ and curves $g_2(p_I, v_A) = 0$, $g_3(p_I, v_A) = 0$. At the bifurcation point $p_I = 7427.2$ and $p_{II} = 26.9$ are the load parameters. Figure 9b also shows the fundamental and the bifurcation paths too.

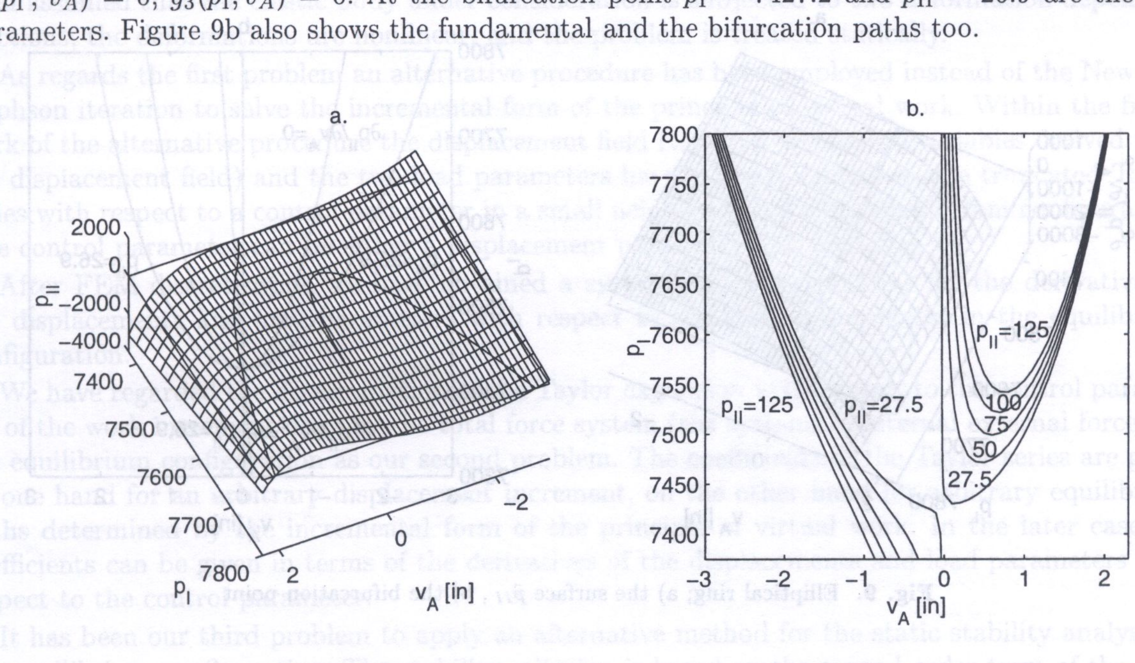

Fig. 7. Elliptical ring; a) the equilibrium surface, b) equilibrium paths

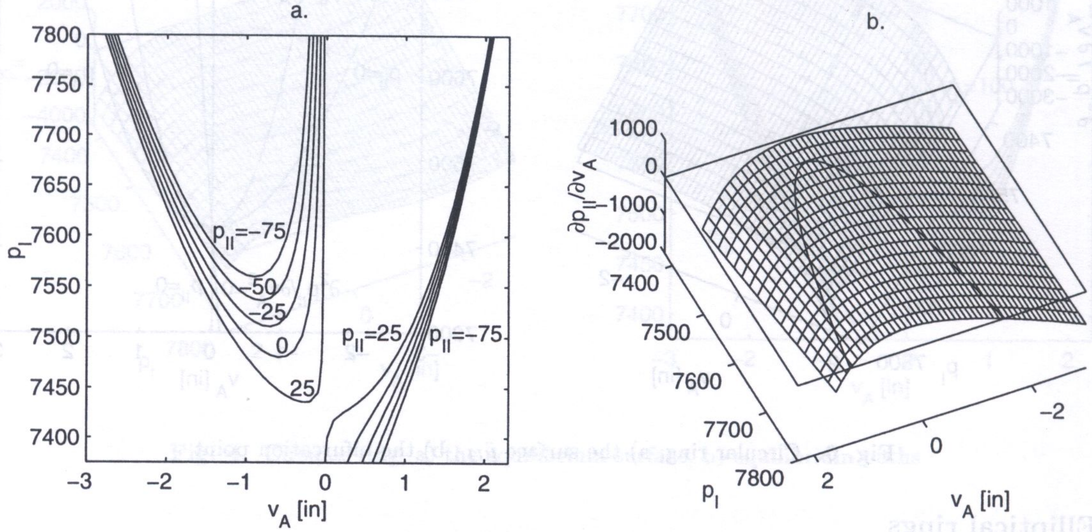

Fig. 8. Elliptical ring; a) equilibrium paths, b) the surface $\dot{p}_{II}$

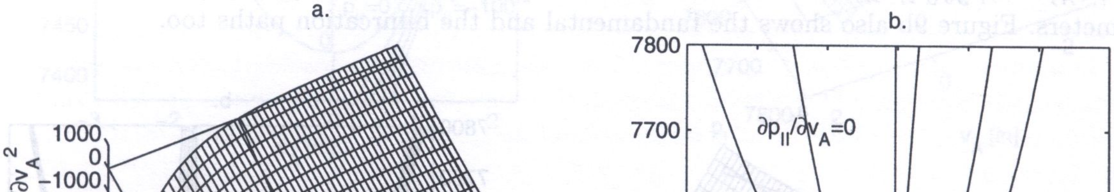

Fig. 9. Elliptical ring; a) the surface $\ddot{p}_{II}$, b) the bifurcation point

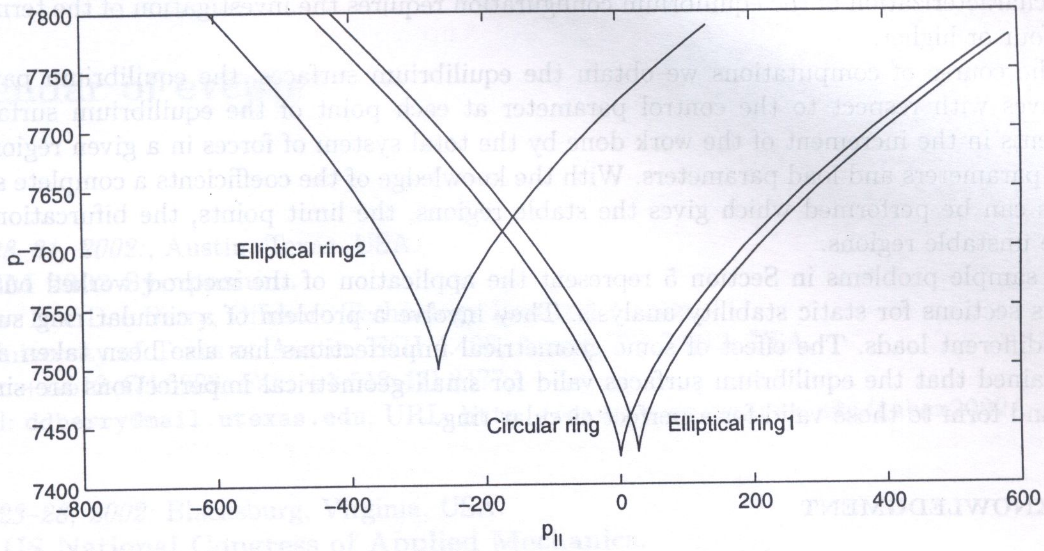

Fig. 10. Stability boundaries for all the three examples

Here we do not show the appropriate figures obtained for the elliptical ring2 since those figures are similar in shape and form to Figs. 7–9 concerning with the elliptical ring1.

However, we give the values of the parameter p_{II} at the limit points (as a function of p_{I}) are shown in Fig. 10 for all the three examples. The regions between the corresponding two branches of the curves shown are unstable. We note that the curves are obtained from the condition $\dot{p}_{\mathrm{II}} = 0$.

6. Conclusions

The paper has solved three general problems concerning the investigations given in the title. It has been assumed that the elastic body under consideration is subjected to two deformation dependent tractions, the deformations are nonlinear, and the problem is treated statically.

As regards the first problem an alternative procedure has been employed instead of the Newton–Raphson iteration to solve the incremental form of the principle of virtual work. Within the frame work of the alternative procedure the displacement field (together with those variables derived from the displacement field) and the two load parameters have all been expanded into truncated Taylor series with respect to a control parameter in a small neighborhood of the equilibrium configuration. The control parameter can be either a displacement parameter or a load one.

After FEM discretization we have obtained a system of linear equations for the derivatives of the displacements and load parameters with respect to the control parameter in the equilibrium configuration.

We have regarded the determination of the Taylor expansion with respect to the control parameter of the work increment done by the total force system (the systems of internal external forces) in the equilibrium configuration as our second problem. The coefficients of the Taylor series are given on one hand for an arbitrary displacement increment, on the other hand for arbitrary equilibrium paths determined by the incremental form of the principle of virtual work. In the later case the coefficients can be given in terms of the derivatives of the displacements and load parameters with respect to the control parameter.

It has been our third problem to apply an alternative method for the static stability analysis of the equilibrium configuration. The stability criterion is based on the second order term of the Taylor expansion: the equilibrium configuration under consideration is stable, unstable or transitional (critical) if the second order term is positive, negative or zero, respectively. If the second order term is zero characterization of the equilibrium configuration requires the investigation of the terms order three, four or higher.

In the course of computations we obtain the equilibrium surfaces, the equilibrium paths, the derivatives with respect to the control parameter at each point of the equilibrium surface, and coefficients in the increment of the work done by the total system of forces in a given region of the control parameters and load parameters. With the knowledge of the coefficients a complete stability analysis can be performed which gives the stable regions, the limit points, the bifurcation points and the unstable regions.

The sample problems in Section 5 represent the application of the method worked out in the previous sections for static stability analysis. They involve a problem of a circular ring subjected to two different loads. The effect of some geometrical imperfections has also been taken account. We obtained that the equilibrium surfaces valid for small geometrical imperfections are similar in shape and form to those valid for a perfect circular ring.

7. Acknowledgment

The support provided by the Hungarian National Research Foundation (OTKA T031998) is gratefully acknowledged. The authors would like to express their gratitude to Prof. Dr. Gy. Szeidl for reading the manuscript and making several valuable suggestions.

References

[1] J.L. Batoz, G. Dhatt. Incremental displacement algorithms nonlinear problems. *Int. J. Num. Meths. Engng.*, **14**: 1262–1267, 1979.

[2] H. Bufler. Pressure loaded structures under large deformations. *ZAMM*, **64**: 287–295, 1984.

[3] G.A. Cohen. Conservativeness of normal pressure field acting on a shell. *AIAA J.*, **4**: 1886, 1966.

[4] M.A. Crisfield. *Non-linear Finite Element Analysis of Solids and Structures, Volume 2.* Wiley, London, 1997.

[5] K. Huseyin. *Nonlinear theory of elastic stability.* Nordhoff, 1975.

[6] M. Kleiber. Stability problems and methods. In: M. Kleiber, ed., *Handbook of Computational Solid Mechanics*, 253–333. Springer-Verlag, 1998.

[7] I. Kozák. Conservativeness of follower surface loads and the stiffness matrix of FE analysis. *Publ. Univ. Miskolc. Series C, Mechanical Engineering*, **50**: 41–54, 1999.

[8] I. Kozák, F. Nándori, T. Szabó. FE analysis of geometrically nonlinear static problems with follower loads. *CAMES*, **6**: 369–383, 1999.

[9] M. Kurutz. Modification of the structural tangent stiffness due to nonlinear configuration-dependent conservative loading. *CAMES*, **3**: 367–388, 1996.

[10] M. Kurutz. Postbifurcation equilibrium paths modified by nonlinear configuration-dependent conservative loading using non smooth analysis. *Mech. Struct. and Mach.*, **25**(4): 445–476, 1997.

[11] H.A. Mang. Symmetricability of pressure stiffness matrices of shells with loaded free edges. *Int. J. Num. Meths. Engng.*, **15**: 15–30, 1981.

[12] M. Marcinowsky. Large deflections of shells subjected to external load and temperature changes. *Int. J. Solids Structures*, **34**(6): 755–768, 1997.

[13] D.P. Molk, W.A. Wall, M. Bischoff, E. Ramm. Algorithmic aspects of deformation dependent loads in non-linear static finite element analysis. *Engineering Computations*, **16**(5): 601–618, 1999.

[14] G. Powell, J. Simons. Improved iteration strategy for nonlinear structures. *Int. J. Num. Meths. Engng.*, **17**: 1455–1467, 1981.

[15] K.H. Schweizerhof, E. Ramm. Displacement dependent pressure loads in nonlinear finite element analysis. *Comp. Struct.*, **18**: 1099–1114, 1984.

[16] J.M.T. Thompson, G.W. Hunt. *Elastic Instability Phenomena.* Wiley, London, 1984.

[17] P. Wriggers. Numerical methods in nonlinear finite element procedures. In: P. Wriggers, W. Wagner, eds., *Nonlinear Computational Mechanics*, 47–192. Springer-Verlag, 1991.

[18] P. Wriggers. Continuum mechanics, nonlinear finite element techniques and computational stability. In: E. Stein, ed., *Progress in computational analysis of inelastic structures*, 245–287. Springer-Verlag, 1993.