Computer Assisted Mechanics and Engineering Sciences, 9: 583–595, 2002.

Modified Fourier transform method and its application for solving the problem of large deformations of a beam[1]

Dejan Trajkovski
University "Sv. Kliment Ohridski", Faculty of Technical Sciences
Ivo Ribar Lola bb, 7000 Bitola, Macedonia

Rastko S. Cukic
University of Belgrade, Faculty of Mechanical Engineering
27 Marta 80, 11000 Belgrade, Yugoslavia

(Received Februay 9, 2001)

A modification of the Fourier transform method, which makes feasible transforming products of two functions and/or their derivatives, is described. By application of this method, some kinds of nonlinear differential equations can be transformed and solved. In this paper, the solution of the problem of bending a beam with fixed supports, under continuous transversal loading is given. Equations of the large deformations theory are used. The mutual influence between the deflection and the axial force is taken into account. The problem is mathematically described by a system of three nonlinear differential equations, with the appropriate boundary conditions. The solution is obtained by making use of an iterative procedure, based on the modified Fourier transform method.

1. Introduction

In many problems of mathematical physics we solve boundary value problems of different kind. The solutions should satisfy differential equations and some conditions imposed upon the solutions – boundary and/or initial. These problems are being solved using different methods [5]. The vast majority of analytical methods is suitable only for solving linear differential equations and this limitation forced development of numerical methods.

The Fourier transform method, which is very convenient for solving linear differential equations, cannot be applied in the case of nonlinear equations. The main obstacle is the inability to transform differential expressions that contain products of two functions or derivatives, or the functions are raised to a power different from unity.

In [8] one method of finding approximate solutions for the heat conduction equation in one dimension subject to mixed boundary conditions has been presented. From the results obtained one can see that the solution that the solution of the problem derived by the Fourier cosine series approximates well the solution of the same problem, derived by the method of separation of variables. The boundary conditions have not been satisfied. Paper [9] applies the same method to a certain class of partial differential equations subject to non-Dirichlet boundary conditions, considering the problem of boundary conditions as well, without solving eigenvalue problems. In [10] the Fourier cosine series are applied to solve one heterogeneous thermal problem. These approaches brought solutions of some types of differential equations, particularly the heat transfer differential equation. The main limitations were the order of derivatives and appearance of nonlinear terms in form of products of derivatives.

[1] This is an extended version of the article presented at the 8th International Conference on Numerical Methods in Continuum Mechanics, Liptovský Ján, Low Tatras, Slovakia, September 19–24, 2000.

In this paper, an approximate method for solving some classes of nonlinear differential equations, containing differential expressions in the form of product of two functions or their derivatives of different order, is described. The aim of the present paper is also to give the mathematical grounds for the new method, as well as to introduce mathematical formalism for solving problems in a systematic way. The efficiency of the method is demonstrated on example – the differential equations of large deformations of a beam.

2. FOURIER TRANSFORMS OF PRODUCTS OF TWO FUNCTIONS

The finite sine and cosine Fourier transforms belong to the class of the most widely used integral transforms for solving linear differential equations. The idea of transforming products of two functions, which will be given in more details in this paper for the case of cosine and sine transforms, remains also valid in the case of other kinds of finite and infinite Fourier transformations (for instance, the Hankel transformation etc.).

Given two functions $f_1(x)$ and $f_2(x)$ that satisfy the Dirichlet's conditions [6] in the domain $0 \leq x \leq L$, the finite sine Fourier transforms of these functions are represented by the following integrals,

$$\bar{f}_{1s}^{n} = \mathsf{F}_s\left[f_1(x);\ x \to n\right] = \int_0^L f_1(x) \cdot \sin(\alpha_n x) \cdot \mathrm{d}x\,,$$
$$\bar{f}_{2s}^{n} = \mathsf{F}_s\left[f_2(x);\ x \to n\right] = \int_0^L f_2(x) \cdot \sin(\alpha_n x) \cdot \mathrm{d}x\,; \qquad \alpha_n = \frac{n\pi}{L}\,; \qquad n = 1, 2, \ldots \tag{1}$$

Here F_1 denotes the finite sine Fourier transformation operator. The originals of the functions $f_1(x)$ and $f_2(x)$ can be found by application of the inverse finite sine Fourier transformation,

$$f_1(x) = \mathsf{F}_s^{-1}\left[\bar{f}_{1s}^{n}\,;\ n \to x\right] = \frac{2}{L}\sum_{n=1}^{\infty} \bar{f}_{1s}^{n} \cdot \sin(\alpha_n x),$$
$$f_2(x) = \mathsf{F}_s^{-1}\left[\bar{f}_{2s}^{n}\,;\ n \to x\right] = \frac{2}{L}\sum_{n=1}^{\infty} \bar{f}_{2s}^{n} \cdot \sin(\alpha_n x). \tag{2}$$

The product $g(x) = f_1(x) \cdot f_2(x)$ also satisfies the Dirichlet's conditions in the interval $[0, L]$, and therefore allows expansion into Fourier series,

$$g(x) = f_1(x) \cdot f_2(x) = \frac{4}{L^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \bar{f}_{1s}^{i} \cdot \bar{f}_{2s}^{j} \cdot \sin(\alpha_i x) \cdot \sin(\alpha_j x)$$
$$= \frac{2}{L^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \bar{f}_{1s}^{i} \cdot \bar{f}_{2s}^{j} \cdot \left[\cos(\alpha_{i-j} x) - \cos(\alpha_{i+j} x)\right] = \frac{1}{L}\sum_{k=-\infty}^{+\infty} \bar{g}_c^{k} \cdot \cos(\alpha_k x)$$
$$= \frac{1}{L}\bar{g}_c^{0} + \frac{2}{L}\sum_{k=0}^{\infty} \bar{g}_c^{k} \cdot \cos(\alpha_k x) \tag{3}$$

where

$$\bar{g}_c^{n} = \mathsf{F}_c[g(x);\ x \to n] = \int_0^L g(x) \cdot \cos(\alpha_n x) \cdot \mathrm{d}x\,; \qquad n = 0, 1, 2, \ldots\,, \tag{4}$$

is the finite cosine Fourier transform of the function g(x).

After the multiplication of Eq. (3) by $\cos(\alpha_m x) \cdot \mathrm{d}x$ $(m = 0, 1, 2, \ldots)$, integration in the interval $[0, L]$ and taking into account the orthogonality of the trigonometric functions, the expression for the cosine transform of the function $g(x)$ will be obtained,

$$\bar{g}_c^m = \frac{1}{L}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \bar{f}_{1s}^i \cdot \bar{f}_{2s}^j \cdot (\bar{\delta}_{m,|i-j|} - \bar{\delta}_{m,i+j}) = \mathsf{P}_{ss\to c}\left[\bar{f}_{1s}\,,\, \bar{f}_{2s}\right], \tag{5}$$

where $\bar{\delta}_{i,j}$ denotes the modified Kronecker's symbol,

$$\bar{\delta}_{i,j} = \frac{2}{L}\int_0^L \cos(\alpha_i x)\cdot\cos(\alpha_j x)\cdot \mathrm{d}x = \begin{cases} 0, & i \neq j, \\ 1, & i = j \neq 0, \\ 2, & i = j = 0. \end{cases} \tag{6}$$

In Eq. (5) it is possible to substitute the indices and to reduce the double series to single ones,

$$\begin{aligned} \bar{g}_c^0 &= \frac{2}{L}\sum_{i=1}^{\infty} \bar{f}_{1s}^i \cdot \bar{f}_{2s}^i\,, \\ \bar{g}_c^m &= \frac{1}{L}\left[\sum_{j=1}^{\infty} \bar{f}_{1s}^{j+m}\cdot \bar{f}_{2s}^j + \sum_{j=m+1}^{\infty} \bar{f}_{1s}^{j-m}\cdot \bar{f}_{2s}^j - \sum_{\substack{j=1\\(m>1)}}^{\infty} \bar{f}_{1s}^{m-j}\cdot \bar{f}_{2s}^j\right]; \qquad m = 1,2,\ldots \end{aligned} \tag{7}$$

If only the first N transforms ($\bar{f}_{1s}^i\,,\, \bar{f}_{2s}^i\,;\ i = 1,2,\ldots N$) are used in calculations, the following equations for the transform $\bar{g}_c$ can be obtained,

$$\begin{aligned} \bar{g}_c^0 &\approx \frac{2}{L}\sum_{i=1}^{N} \bar{f}_{1s}^i \cdot \bar{f}_{2s}^i\,, \\ \bar{g}_c^m &\approx \frac{1}{L}\left[\sum_{\substack{j=1\\(m<N)}}^{N-m} \bar{f}_{1s}^{j+m}\cdot \bar{f}_{2s}^j + \sum_{\substack{j=m+1\\(m<N)}}^{N} \bar{f}_{1s}^{j-m}\cdot \bar{f}_{2s}^j - \sum_{\substack{j=1\\(m>1)}}^{m-1} \bar{f}_{1s}^{m-j}\cdot \bar{f}_{2s}^j\right]; \qquad m = 1,2,\ldots \end{aligned} \tag{8}$$

Some of the sums in Eq. (8) should be calculated only for the values of the index m given in the parentheses below the sum.

By making use of the multiplying operator $\mathsf{P}_{ss\to c}$, defined in Eq. (5), the cosine Fourier transform of the product $g(x) = f_1(x)\cdot f_2(x)$ can be found, if the sine Fourier transforms $\bar{f}_{1s}\,,\, \bar{f}_{2s}$ of the functions $f_1(x)$ and $f_2(x)$ are known.

The original of the function g in the domain of the variable x can then be found by application of the inverse finite cosine Fourier transformation [6].

$$g(x) = \mathsf{F}_c^{-1}\left[\bar{g}_c^m\,;\ m \to x\right] = \frac{1}{L}\sum_{m=-\infty}^{\infty} \bar{g}_c^m \cdot \cos(\alpha_m x) = \frac{\bar{g}_c^0}{L} + \frac{2}{L}\sum_{m=1}^{\infty} \bar{g}_c^m \cdot \cos(\alpha_m x). \tag{9}$$

Similarly as in Eq. (5), we can also define the product operators $\mathsf{P}_{cc\to c}$ and $\mathsf{P}_{cs\to s}$. The operator $\mathsf{P}_{cc\to c}$ yields the cosine transform $\bar{g}_c$ of the product $g(x) = f_1(x)\cdot f_2(x)$, given the cosine transforms $\bar{f}_{1c}\,,\, \bar{f}_{2c}$ of the multiplying functions,

$$\begin{aligned} g(x) = f_1(x)\cdot f_2(x) &= \left[\frac{\bar{f}_{1c}^0}{L} + \frac{2}{L}\sum_{i=1}^{\infty} \bar{f}_{1c}^i \cdot \cos(\alpha_i x)\right]\left[\frac{\bar{f}_{2c}^0}{L} + \frac{2}{L}\sum_{j=1}^{\infty} \bar{f}_{2c}^j \cdot \cos(\alpha_j x)\right] \\ &= \frac{\bar{f}_{1c}^0 \bar{f}_{2c}^0}{L^2} + \frac{2}{L^2}\bar{f}_{2c}^0 \sum_{i=1}^{\infty} \bar{f}_{1c}^i \cdot \cos(\alpha_i x) + \frac{2}{L^2}\bar{f}_{1c}^0 \sum_{j=1}^{\infty} \bar{f}_{2c}^j \cdot \cos(\alpha_j x) \\ &\quad + \frac{2}{L^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \bar{f}_{1c}^i \bar{f}_{2c}^j \left[\cos(\alpha_{i-j} x) + \cos(\alpha_{i+j} x)\right]. \end{aligned} \tag{10}$$

The cosine transform $\bar{g}_c = \mathsf{P}_{cc\to c}\left[\bar{f}_{1c}\,,\,\bar{f}_{2c}\right]$ of the product is now

$$\begin{aligned}\bar{g}_c^0 &= \frac{1}{L}\,\bar{f}_{1c}^0\bar{f}_{2c}^0 + \frac{2}{L}\sum_{i=1}^{\infty}\bar{f}_{1c}^i\bar{f}_{2c}^i\,,\\ \bar{g}_c^m &= \frac{1}{L}\left(\bar{f}_{1c}^0\bar{f}_{2c}^m + \bar{f}_{1c}^m\bar{f}_{2c}^0\right) + \frac{1}{L}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\bar{f}_{1c}^i\bar{f}_{2c}^j\left(\bar{\delta}_{m,|i-j|} + \bar{\delta}_{m,i+j}\right); \qquad m = 1,2,\ldots\end{aligned} \tag{11}$$

The operator $\mathsf{P}_{cs\to s}$ yields the sine transform $\bar{g}_s$ of the product $g(x) = f_1(x)\cdot f_2(x)$, given the cosine and sine transforms $\bar{f}_{1c}\,,\,\bar{f}_{2s}$ of the multiplying functions,

$$\begin{aligned}g(x) = f_1(x)\cdot f_2(x) &= \left[\frac{\bar{f}_{1c}^0}{L} + \frac{2}{L}\sum_{i=1}^{\infty}\bar{f}_{1c}^i\cdot\cos(\alpha_i x)\right]\cdot\frac{2}{L}\sum_{j=1}^{\infty}\bar{f}_{2s}^j\cdot\sin(\alpha_j x)\\ &= \frac{2}{L^2}\,\bar{f}_{1c}^0\sum_{j=1}^{\infty}\bar{f}_{2s}^j\cdot\sin(\alpha_j x) + \frac{2}{L^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\bar{f}_{1c}^i\bar{f}_{2s}^j\left[\sin(\alpha_{i+j}x) - \cos(\alpha_{i-j}x)\right].\end{aligned} \tag{12}$$

The sine transform $\bar{g}_s = \mathsf{P}_{cs\to s}\left[\bar{f}_{1c}\,,\,\bar{f}_{2s}\right]$ of the product $g(x) = f_1(x)\cdot f_2(x)$ will be

$$\bar{g}_s^m = \frac{1}{L}\,\bar{f}_{1c}^0\bar{f}_{2s}^m + \frac{1}{L}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\bar{f}_{1c}^i\bar{f}_{2s}^j\left(\bar{\delta}_{m,i+j} + \mathrm{sign}(i-j)\cdot\bar{\delta}_{m,|i-j|}\right); \qquad m = 1,2,\ldots \tag{13}$$

Which transformation of the three ones (Eqs. (5), (11), (13)) will be used depends on the transform types of the multiplying functions as well as on the boundary conditions.

3. Retransforming the sine and cosine transforms

It is often useful to have a method for transition from a cosine Fourier transform into a sine transform and vice versa. For example, in order to apply the multiplying operator $\mathsf{P}_{ss\to c}$, it is necessary to have the sine transforms of the multipliers. If the cosine transforms are given instead, the sine transforms can be obtained by means of the retransformation operator $\mathsf{R}_{c\to s}$, which is defined below. For the opposite transition the retransforming operator $\mathsf{R}_{s\to c}$ should be used.

In order to find the expression for transition ("retransformation") from the cosine transform $\bar{f}_c$ to the sine transform $\bar{f}_s$ or vice versa, the function $f(x)$ will be presented in the form of cosine and sine Fourier series,

$$f(x) = \frac{2}{L}\sum_{i=1}^{\infty}\bar{f}_s^i\sin(\alpha_i x) = \frac{1}{L}\bar{f}_c^0 + \frac{2}{L}\sum_{j=1}^{\infty}\bar{f}_c^j\cos(\alpha_j x). \tag{14}$$

We multiply the last equation by $\sin(\alpha_k x)\mathrm{d}x$ $(k = 1,2,\ldots)$ and integrate it in the interval $[0,L]$. Due to the orthogonality of the trigonometric functions,

$$\frac{2}{L}\int_0^L \sin(\alpha_i x)\,\sin(\alpha_k x)\,\mathrm{d}x = \begin{cases}0, & i\neq k,\\ 1, & i = k,\end{cases} = \delta_{ik}\,, \tag{15}$$

we arrive to

$$\bar{f}_s^k = \frac{1}{2}\,a_{0k}\bar{f}_c^0 + \sum_{j=1}^{\infty}a_{jk}\bar{f}_c^j\,; \qquad k = 1,2,\ldots, \tag{16}$$

where a_{jk} are the retransforming coefficients

$$a_{jk} = \frac{2}{L}\int_0^L \cos(\alpha_j x)\sin(\alpha_k x)\,\mathrm{d}x = \frac{2}{L}\int_0^L \cos\left(\frac{j\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x\,; \qquad j = 0,1,\ldots; \quad k = 1,2,\ldots \tag{17}$$

It should be noted that generally $a_{jk} \neq 0$, since the interval of integration L differs from the basic period $2L$ of the trigonometric functions. It is always

$$\int_0^{2L} \cos(\alpha_j x)\,\sin(\alpha_k x)\,\mathrm{d}x = 0$$

(due to the orthogonality), but

$$\int_0^{L} \cos(\alpha_j x)\,\sin(\alpha_k x)\,\mathrm{d}x \neq 0.$$

Equation (16) can be formally written as

$$\bar{f_s} = \mathsf{R}_{c\to s}\left[\bar{f_c}\right]. \tag{18}$$

In analogous way we can obtain the sine-to-cosine retransformation, if we multiply Eq. (14) by $\cos(\alpha_k x)\,\mathrm{d}x$ $(k = 0,1,2,\ldots)$ and integrate it in the interval $[0, L]$,

$$\bar{f}_c^k = \sum_{i=1}^{\infty} a_{ki}\bar{f}_s^i\,; \qquad k = 0,1,2,\ldots \tag{19}$$

The above retransformation can be written in operator form as

$$\bar{f_s} = \mathsf{R}_{c\to s}\left[\bar{f_c}\right]. \tag{20}$$

4. Matrix formulations

In practical computations, instead of Fourier series we use finite sums of N terms. The cosine and sine transform of a given function can be presented in vectorized form. The dimensions of the vectors of cosine and sine transforms are $N+1$ and, respectively,

$$\bar{\mathbf{f}}_c = \begin{bmatrix} \bar{f}_c^0 \\ \bar{f}_c^1 \\ \vdots \\ \bar{f}_c^N \end{bmatrix}_{N+1}, \qquad \bar{\mathbf{f}}_s = \begin{bmatrix} \bar{f}_s^1 \\ \bar{f}_c^2 \\ \vdots \\ \bar{f}_c^N \end{bmatrix}_{N}. \tag{21}$$

For transforming the derivatives, the following vectors are also useful,

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix}_N = \begin{bmatrix} \pi/L \\ 2\pi/L \\ \vdots \\ N\pi/L \end{bmatrix}_N, \qquad \beta = \begin{bmatrix} \alpha_1^2 \\ \alpha_2^2 \\ \vdots \\ \alpha_N^2 \end{bmatrix}_N = \overrightarrow{\alpha\cdot\alpha}, \tag{22}$$

where the arrow $\overrightarrow{()\cdot()}$ denotes the component-by-component product of two vectors.

Given the cosine and sine transform of an arbitrary function $f(x)$, for zero boundary conditions at the boundaries $x = 0$ and $x = L$, the sine and cosine transforms of the first and second derivatives $\mathrm{d}f/\mathrm{d}x$ are [2, 5]:

$$\begin{aligned}
\mathsf{F}_s[\mathrm{d}f/\mathrm{d}x;\ x \to n] &= -\alpha_n \bar{f}_c^n\,,\\
\mathsf{F}_c[\mathrm{d}f/\mathrm{d}x;\ x \to n] &= \alpha_n \bar{f}_s^n\,,\\
\mathsf{F}_s[\mathrm{d}^2 f/\mathrm{d}x^2;\ x \to n] &= -\alpha_n^2 \cdot \bar{f}_s^n = -\beta_n \cdot \bar{f}_s^n\,,\\
\mathsf{F}_c[\mathrm{d}^2 f/\mathrm{d}x^2;\ x \to n] &= -\alpha_n^2 \cdot \bar{f}_c^n = -\beta_n \cdot \bar{f}_c^n\,.
\end{aligned} \tag{23}$$

The upper transforms in matrix form receive the following form,

$$\begin{aligned}
&\mathsf{F}_s[\mathrm{d}f/\mathrm{d}x;\ x \to n] = -\overrightarrow{\alpha \cdot \bar{f}_c} && (\bar{f}_c^0 \text{ is removed}),\\
&\mathsf{F}_c[\mathrm{d}m/\mathrm{d}x;\ x \to n] = \begin{bmatrix} 0 \\ \overrightarrow{\alpha \cdot \bar{f}_c} \end{bmatrix}_{N+1} && (\text{zero added at position } 0),\\
&\mathsf{F}_s[\mathrm{d}^2 f/\mathrm{d}x^2;\ x \to n] = -\overrightarrow{\beta \cdot \bar{f}_s}\,,\\
&\mathsf{F}_c[\mathrm{d}^2 f/\mathrm{d}x^2;\ x \to n] = -\overrightarrow{\begin{bmatrix} 0 \\ \beta \end{bmatrix} \cdot \bar{f}_c}\,.
\end{aligned} \tag{24}$$

The retransformations, defined in Eqs. (16), (18), in matrix form become quite simple,

$$\bar{f}_s = \mathsf{R}_{c\to s} \cdot \bar{f}_c\,, \qquad \bar{f}_c = \mathsf{R}_{s\to c} \cdot \bar{f}_s\,. \tag{25}$$

Here, $\mathsf{R}_{c\to s}$ and $\mathsf{R}_{s\to c}$ are matrices of dimensions $N \times (N+1)$ and $(N+1) \times N$ respectively, which consist of the retransforming coefficients a_{jk} (Eq. (17)),

$$\mathsf{R}_{c\to s} = \frac{2}{L}\begin{bmatrix} a_{10}/2 & a_{12} & \cdots & a_{1N} \\ a_{20}/2 & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N0}/2 & a_{N1} & \cdots & a_{NN} \end{bmatrix}, \qquad \mathsf{R}_{s\to c} = \frac{2}{L}\begin{bmatrix} a_{10} & a_{20} & \cdots & a_{N0} \\ a_{12} & a_{22} & \cdots & a_{N1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & \cdots & a_{NN} \end{bmatrix} = \frac{2}{L}\mathbf{a}^T\,. \tag{26}$$

In Eq. (26), $\mathbf{a}$ is a matrix of retransforming coefficients

$$\mathbf{a} = \begin{bmatrix} a_{10} & a_{12} & \cdots & a_{1N} \\ a_{20} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N0} & a_{N1} & \cdots & a_{NN} \end{bmatrix}. \tag{27}$$

It should be noted that $\mathsf{R}_{c\to s}$ differs from the matrix $\mathbf{a}$.

The multiplying operators are nonlinear. They cannot be presented in matrix form and the Fourier transformation of product cannot be performed by matrix multiplication as the retransformation can. The multiplying operators are vector functions of two arguments – the vectors of transforms of both multiplying functions,

$$\begin{aligned}
\bar{g}_c &= \mathsf{F}_c[f_1(x) \cdot f_2(x);\ x \to n] = \mathsf{P}_{ss\to c}\left[\bar{f}_{1s}\,, \bar{f}_{2s}\right],\\
\bar{g}_c &= \mathsf{F}_c[f_1(x) \cdot f_2(x);\ x \to n] = \mathsf{P}_{cc\to c}\left[\bar{f}_{1c}\,, \bar{f}_{2c}\right],\\
\bar{g}_s &= \mathsf{F}_s[f_1(x) \cdot f_2(x);\ x \to n] = \mathsf{P}_{cs\to s}\left[\bar{f}_{1c}\,, \bar{f}_{2s}\right].
\end{aligned} \tag{28}$$

The operators $\mathsf{P}_{cc\to c}$, $\mathsf{P}_{ss\to c}$, $\mathsf{P}_{cs\to s}$ are practically implemented as computational procedures, based on the formulas given in Eqs. (5), (11) and (13) respectively.

The application of the operators P and R will be demonstrated on the large deformation problem of a simply supported beam.

5. Solving the equations of large deformations of a beam by application of the modified Fourier transformation method

5.1. Governing equations of the problem

The differential equation of bending, the equation of equilibrium of forces in the axial direction of the beam expressed in displacements, and the equation for the axial (normal) force according to the large deformation theory have the following form [1],

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - N\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = q, \tag{29}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = -\frac{\mathrm{d}w}{\mathrm{d}x}\cdot\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}, \tag{30}$$

$$N = EA\cdot\left[\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{2}\cdot\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right], \tag{31}$$

where u and w are the axial and transversal displacements; E is the Young's modulus of elasticity of the material; A and I are the cross-section and axial momentum of inertia with respect of the axis y of the beam (Fig. 1); N is the axial (normal) force; q is the transversal load.

Let us consider a simply supported beam, of length L, which is connected by means of joints to the fixed supports A and B (Fig. 1.).

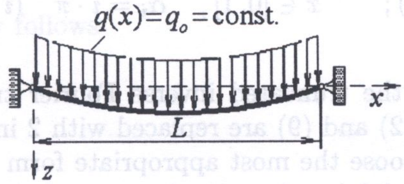

Fig. 1. Transversally loaded simple beam with fixed supports

The boundary conditions for the differential equations (29)–(31), based on the means of supporting of the beam, have the form

$$w(x{=}0) = w(x{=}L) = 0, \tag{32}$$

$$\left.\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right|_{x=0} = \left.\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right|_{x=L} = 0, \tag{33}$$

$$u(x{=}0) = u(x{=}L) = 0. \tag{34}$$

It is more convenient if the differential equations (29)–(31) and the boundary conditions (32)–(34) are reduced to a dimensionless form,

$$\frac{\mathrm{d}^4 \bar{w}}{\mathrm{d}\bar{x}^4} - \bar{N}\frac{\mathrm{d}^2 \bar{w}}{\mathrm{d}\bar{x}^2} = \bar{q}, \tag{35}$$

$$\frac{\mathrm{d}^2 \bar{u}}{\mathrm{d}\bar{x}^2} = -\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{x}}\cdot\frac{\mathrm{d}^2 \bar{w}}{\mathrm{d}\bar{x}^2}, \tag{36}$$

$$\bar{N} = \bar{A}\cdot\left[\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{x}} + \frac{1}{2}\left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{x}}\right)^2\right], \tag{37}$$

$$\bar{w}(\bar{x}{=}0) = \bar{w}(\bar{x}{=}1) = 0, \tag{38}$$

$$\left.\frac{\mathrm{d}^2 \bar{w}}{\mathrm{d}\bar{x}^2}\right|_{\bar{x}=0} = \left.\frac{\mathrm{d}^2 \bar{w}}{\mathrm{d}\bar{x}^2}\right|_{\bar{x}=1} = 0, \tag{39}$$

$$\bar{u}(\bar{x}{=}0) = \bar{u}(\bar{x}{=}1) = 0. \tag{40}$$

Here $\bar{x} = x/L$ is the dimensionless coordinate $(0 \leq \bar{x} \leq 1)$; $\bar{w} = w/L$ and $\bar{u} = u/L$ are dimensionless deflection and axial displacement; $\bar{N} = NL^2/(EI)$ is a dimensionless axial force; $\bar{q} = qL^3/(EI)$ is a dimensionless transversal load; $\bar{A} = AL^2/I$ is a dimensionless area of the cross section (flexibility of the beam).

5.2. Transforming the system of differential equations

Taking into account the boundary conditions (38)–(40), the functions for the displacements $\bar{u}$, $\bar{w}$, as well as for the load $\bar{q}$ will be taken in form of sine series (actually, in form of finite sums), and the axial force $\bar{N}$ will be in form of cosine sum of finite number of terms N.

$$\begin{aligned}
\bar{u}(x) &\approx 2\sum_{i=1}^{N} \bar{u}_s^i \sin(\alpha_i \bar{x}) , \\
\bar{w}(x) &\approx 2\sum_{i=1}^{N} \bar{w}_s^i \sin(\alpha_i \bar{x}) , \\
\bar{q}(x) &\approx 2\sum_{i=1}^{N} \bar{q}_s^i \sin(\alpha_i \bar{x}) , \\
\bar{N}(x) &\approx \bar{N}_c^0 + 2\sum_{i=1}^{N} \bar{N}_c^i \cos(\alpha_i \bar{x}) ; \qquad \bar{x} \in [0,1] , \quad \alpha_i = i \cdot \pi \quad (i = 1, 2, \ldots).
\end{aligned} \tag{41}$$

The sums in Eq. (41) represent the truncated inverse Fourier transforms (Eqs. 2 and 9). Since $\bar{x} \in [0,1]$, the terms $2/L$ in Eqs. (2) and (9) are replaced with 2 in Eq. (41).

Some intuition is needed to choose the most appropriate form of the solutions (41). We expect a uniform distribution for the axial force along the beam, and the sum of cosines should work the best. On the other hand, both the axial and transversal displacements are equal to zero at both ends of the beam ($\bar{x} = 0; 1$). Accordingly, there are no more choices for the solutions but sums of sines.

The expressions $(41)_2$ and $(41)_3$ will be substituted into Eq. (35). After the multiplication of this equation by $\sin(\alpha_i \bar{x}) \cdot d\bar{x}$ $(i = 1, 2, \ldots N)$ and the integration in the interval [0,1], we arrive to the following matrix equation,

$$\mathbf{K} \cdot \bar{\mathbf{w}}_s = \bar{\mathbf{q}}_s , \tag{42}$$

where $\mathbf{K} = [K_{ij}]_{N\times N}$ is a matrix of dimensions $N\times N$, $\bar{\mathbf{w}}_s = [\bar{w}_s^i]_N$ and $\bar{\mathbf{q}}_s = [\bar{q}_s^i]_N$ are vectors of finite sine Fourier transforms for the dimensionless deflection $\bar{w}$ and dimensionless loading $\bar{q}$. The upper dash $\overline{(\ldots)}$ here denotes the transform symbol. It is understood that all quantities are dimensionless and, for the sake of simplicity, the second upper dash is omitted.

$$K_{ij} = \alpha_j^2 \cdot (\alpha_j^2 + \bar{N}_c^0) \cdot \delta_{i,j} + \alpha_j^2 \cdot \sum_{k=1}^{N} \bar{N}_c^k \cdot [\delta_{i,k+j} - \delta_{i,|k-j|} \cdot \operatorname{sign}(k-j)]; \qquad i, j = 1, 2, \ldots N, \tag{43}$$

$$\bar{q}_s^i = \int_0^1 \bar{q}(x) \cdot \sin(\alpha_i \bar{x}) \cdot d\bar{x} = \bar{q}_o \int_0^1 \sin(\alpha_i \bar{x}) \cdot d\bar{x} = \begin{cases} 2\bar{q}_o, & i = 1, 3, 5, \ldots, N, \\ 0, & i = 2, 4, 6, \ldots, N, \end{cases} \quad \text{for } \bar{q}(\bar{x}) = \bar{q}_o = \text{const.} \tag{44}$$

In Eq. (43) $\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$ denotes the Kronecker's symbol.

By application of the finite sine Fourier transformation (1) on Eq. (36), we obtain its transformed form,

$$-\overrightarrow{\beta \cdot \bar{u}_s} = -\mathsf{P}_{cs\to s}\left[\overrightarrow{\alpha \cdot \bar{w}_s}\,,\ \overrightarrow{-\beta \cdot \bar{w}_s}\right]. \tag{45}$$

The multiplying operator $\mathsf{P}_{cs\to s}$ accepts two arguments – the cosine transform vector $\overrightarrow{\alpha \cdot \bar{w}_s}$ of dimensions $N+1$, and the sine transform vector $-\overrightarrow{\beta \cdot \bar{w}_s}$ of dimension N. The output from the function is a vector of sine transform of dimension N. As stated before (Eq. (22)), the symbol $\overrightarrow{(\dots)\cdot(\dots)}$ denotes a component by component multiplication of two operands.

With respect to the rules for transforming derivatives, Eqs. (23)–(25) [2, 3], in Eq. (45) we have

$$\begin{aligned}
\mathsf{F}_c[\mathrm{d}\bar{w}/\mathrm{d}\bar{x};\ \bar{x}\to n] &= \alpha_n \cdot \bar{w}_s^n + (-1)^n \cdot \bar{w}(1) - \bar{w}(0) = \alpha_n \cdot \bar{w}_s\,,\\
\mathsf{F}_s[\mathrm{d}^2\bar{w}/\mathrm{d}\bar{x}^2;\ \bar{x}\to n] &= -\alpha_n^2 \cdot \bar{w}_s^n + \alpha_n \cdot \left[(-1)^{n+1}\cdot \bar{w}(1) + \bar{w}(0)\right] = -\alpha_n^2 \cdot \bar{w}_s^n\,,\\
\mathsf{F}_s[\mathrm{d}^2\bar{u}/\mathrm{d}\bar{x}^2;\ \bar{x}\to n] &= -\alpha_n^2 \cdot \bar{u}_s^n + \alpha_n \cdot \left[(-1)^{n+1}\cdot \bar{u}(1) + \bar{u}(0)\right] = -\alpha_n^2 \cdot \bar{u}_s^n\,.
\end{aligned} \tag{46}$$

The above equations take simpler form in matrix notation:

$$\mathsf{F}_c[\mathrm{d}\bar{w}/\mathrm{d}\bar{x}] = \overrightarrow{\alpha \cdot \bar{w}_s}\,,\qquad \mathsf{F}_s[\mathrm{d}^2\bar{w}/\mathrm{d}\bar{x}^2] = -\overrightarrow{\beta \cdot \bar{w}_s}\,,\qquad \mathsf{F}_s[\mathrm{d}^2\bar{u}/\mathrm{d}\bar{x}^2] = -\overrightarrow{\beta \cdot \bar{u}_s}\,. \tag{47}$$

From Eq. (45) it immediately follows

$$\bar{u}_s = \overrightarrow{\frac{\mathsf{P}_{cs\to s}\left[\overrightarrow{\alpha \cdot \bar{w}_s}\,, \overrightarrow{-\beta \cdot \bar{w}_s}\right]}{\beta}}\,. \tag{48}$$

In the latter equation, the symbol $\overrightarrow{\frac{(\dots)}{(\dots)}}$ denotes a component-by-component division of two vectors.

By application of the finite cosine transform (Eq. (4)) on Eq. (37), the cosine transform of the axial force is obtained in the following form,

$$\bar{N}_c = \bar{A}\cdot\left\{\alpha \cdot \bar{u}_s + \frac{1}{2}\mathsf{P}_{cc\to c}\left[\overrightarrow{\alpha \cdot \bar{w}_s}\,, \overrightarrow{\alpha \cdot \bar{w}_s}\right]\right\}. \tag{49}$$

The multiplying operators P are optimally used in Eqs. (48) and (49). There are alternative, though not optimal forms for these equations. Below are given two examples:

$$\bar{u}_s = \overrightarrow{\frac{\mathsf{R}_{c\to s}\cdot \mathsf{P}_{ss\to c}\left[\mathsf{R}_{c\to s}\cdot \overrightarrow{\alpha \cdot \bar{w}_s}\,, \overrightarrow{-\beta \cdot \bar{w}_s}\right]}{\beta}} \tag{50}$$

$$\bar{N}_c = \bar{A}\cdot\left\{\alpha \cdot \bar{u}_s + \frac{1}{2}\mathsf{P}_{ss\to c}\left[\mathsf{R}_{c\to s}\cdot \overrightarrow{\alpha \cdot \bar{w}_s}\,,\ \mathsf{R}_{c\to s}\cdot \overrightarrow{\alpha \cdot \bar{w}_s}\right]\right\}. \tag{51}$$

Needless to say, due to the extensive use of the retransforming operators, Eqs. (50) and (51) would produce greater errors compared to Eqs. (48) and (49).

5.3. Solving the transformed system of differential equations

The system of differential equations (35)–(37) is nonlinear and its solution is to be obtained by successive approximations. The algorithm of the iterative procedure, which is based on the use of the matrix equations (42), (48), (49), consists of the following 6 steps:

1. Setting the transforms in the initial ($i = 0$) iteration,

$$\bar{N}_c^{(0)} = [0]_{N+1}, \qquad \bar{w}_s^{(0)} = f_1\left(\bar{q}_s, \bar{N}_c^{(0)}\right) \quad \text{– according to Eq. (42);}$$

2. Computing the transforms $\bar{u}_s^{(i)}$ in the i-th iteration,

$$\bar{u}_s^{(i)} = f_2\left(\bar{w}_s^{(i-1)}\right) \quad \text{– according to Eq. (48);}$$

3. Computing the transform $\bar{N}_c^{(i)}$ in the i-th iteration,

$$\bar{N}_c^{(i)} = f_3\left(\bar{u}_s^{(i)}, \bar{w}_s^{(i-1)}\right) \quad \text{– according to Eq. (49);}$$

4. Computing new transform for the axial force,

$$\bar{N}_c^{(i)} = \frac{\bar{N}_c^{(i)} + \bar{N}_c^{(i-1)}}{2};$$

5. Computing the transform $\bar{w}_s^{(i)}$ in the i-th iteration,

$$\bar{w}_s^{(i)} = f_1\left(\bar{q}_s, \bar{N}_c^{(i)}\right) \quad \text{– according to Eq. (42);}$$

6. Checking the criterion for stopping the iterative process. Steps 2–5 are to be repeated until the following condition is fulfilled,

$$\left|\frac{\bar{w}^{(i)}(\bar{x}=0.5) - \bar{w}^{(i-1)}(\bar{x}=0.5)}{\bar{w}^{(i)}(\bar{x}=0.5) + \bar{w}^{(i-1)}(\bar{x}=0.5)}\right| \le \varepsilon.$$

In the upper expression i denotes the ordinal number of the iteration.

The displacements $\bar{u}$, $\bar{w}$ and the force $\bar{N}$ can be obtained by application of the finite inverse Fourier transforms (Eqs. (2), (9)).

6. Iterative and analytical solution of the problem of large deformations of a beam

The example of a simple beam is very useful for testing the efficiency of the modified Fourier transform method earlier described, since it is possible to compare the results of computations with another solution, obtained in a different, semi-analytical way.

One alternative form of Eq. (36) is the following one,

$$\frac{d^2\bar{u}}{d\bar{x}^2} + \frac{d\bar{w}}{d\bar{x}} \cdot \frac{d^2\bar{w}}{d\bar{x}^2} = \frac{d^2\bar{u}}{d\bar{x}^2} + \frac{1}{2}\frac{d}{d\bar{x}}\left(\frac{d\bar{w}}{d\bar{x}}\right)^2 = \frac{1}{\bar{A}}\frac{d\bar{N}}{d\bar{x}} = 0. \tag{52}$$

Equation (52) after integration yields

$$\bar{N} = \text{const}. \tag{53}$$

The axial force can be found by integrating the expression in Eq. (37),

$$\begin{aligned} &\int_0^1 \bar{N}\,d\bar{x} = \bar{A} \cdot \left[\int_0^1 \frac{d\bar{u}}{d\bar{x}}\,dx + \frac{1}{2}\int_0^1 \left(\frac{d\bar{w}}{d\bar{x}}\right)^2 d\bar{x}\right], \\ &\bar{N} = \bar{A} \cdot \left[\bar{u}(1) - \bar{u}(0) + \frac{1}{2}\int_0^1 \left(\frac{d\bar{w}}{d\bar{x}}\right)^2 d\bar{x}\right] = \frac{\bar{A}}{2}\int_0^1 \left(\frac{d\bar{w}}{d\bar{x}}\right)^2 d\bar{x}. \end{aligned} \tag{54}$$

The boundary conditions, Eq. (39), are taken into account here.

Formula (54) would yield the exact solution if the exact field for the deflection $\bar{w}$ would have been known. The deflection can be presumed in form of sine Fourier series,

$$\bar{w}(\bar{x}) = 2\sum_{i=1}^{\infty} \bar{w}_s^i \cdot \sin(\alpha_i \bar{x})\,; \qquad \alpha_i = i\cdot\pi. \tag{55}$$

The constant continuous loading can also be expanded into sine Fourier series,

$$\bar{q}(\bar{x}) = \bar{q}_o = 2\sum_{i=1}^{\infty} \bar{q}_s^i \cdot \sin(\alpha_i \bar{x})\,; \quad \bar{q}_s^i = \frac{1-(-1)^i}{\alpha_i}\,\bar{q}_o = \begin{cases} 2\bar{q}_o/\alpha_i; & i = 1,3,5,\ldots \\ 0; & i = 2,4,6,\ldots \end{cases} \tag{56}$$

After the substitution of expressions (55), (56) into Eq. (35) and solving it with respect to $\bar{w}_s^i$, the deflection becomes

$$\bar{w}_s^i = \frac{2\bar{q}_o}{\alpha_i^3(\alpha_i^2+\bar{N})}\,; \qquad i = 1,3,\ldots, \tag{57}$$

$$\bar{w}(\bar{x}) = 4\bar{q}_o \sum_{i=1,3,\ldots}^{\infty} \frac{\sin(\alpha_i \bar{x})}{\alpha_i^3(\alpha_i^2+\bar{N})}\,. \tag{58}$$

All terms of the series (58) decrease very rapidly as the index i increases, and all of them can be neglected with respect to the first term,

$$\bar{w}(\bar{x}) \cong 4\bar{q}_o \frac{\sin(\alpha_1 \bar{x})}{\alpha_1^3(\alpha_1^2+\bar{N})} = 4\bar{q}_o \frac{\sin(\pi \bar{x})}{\pi^3(\pi^2+\bar{N})}\,. \tag{59}$$

If the approximate solution (59) is inserted into Eq. (54), we arrive to the following recursive formula for the axial force,

$$\bar{N}^{(i)} = \frac{4\bar{A}\bar{q}_o^2}{\pi^4\left(\pi^2+\bar{N}^{(i-1)}\right)^2}\,; \qquad i = 1,2,\ldots; \qquad \bar{N}^{(0)} = 0. \tag{60}$$

The recursion is always stable and there is a limit value $\lim_{i\to\infty} \bar{N}^{(i)} = \bar{N}$.

The formula (60) is easy for deriving and quite simple for programming. Below is also given an analytical solution. Its derivation is more difficult, but takes less computation time.

Based on the theory of orders,

$$\lim_{i\to\infty} \frac{\bar{N}^{(i)}}{\bar{N}^{(i-1)}} = 1$$

and Eq. (60) can be written as an algebraic equation of third order with respect to $\bar{N}$,

$$\pi^4\bar{N}^3 + 2\pi^6\bar{N}^2 + \pi^8\bar{N} - 4\bar{A}\bar{q}_o^2 = 0. \tag{61}$$

This equation has only one real and positive root. It can be found by making use of the Cardano's formula,

$$\bar{N} = \frac{u}{6\pi^2} + \frac{2}{3}\pi^2\left(\frac{\pi^4}{u} - 1\right) \quad \text{where} \quad u = 2\cdot\sqrt[3]{\pi^2}\cdot\sqrt[3]{\pi^{10} + 54\bar{A}\bar{q}_o^2 + 6\sqrt{3}\sqrt{\bar{A}\bar{q}_o^2\left(\pi^{10}+27\bar{A}\bar{q}_o^2\right)}}\,. \tag{62}$$

With the computed value for the axial force, the deflection can be found from Eq. (57). Deflection in the center of the beam is

$$\bar{w}(\bar{x}=0.5) \cong \frac{4\bar{q}_o^2}{\pi^3(\pi^2+\bar{N})}\,. \tag{63}$$

7. NUMERICAL EXAMPLE AND CONCLUSION

The system of differential equations (35)–(37) of large deformations of a beam is solved by making use of the modified Fourier transform method. All computations are carried out in MathCAD. The following values for the dimensionless coefficients have been used: $\bar{A} = 10^4$, $\bar{q}_o = 2$. These values correspond to a steel beam which length is $L = 1$ m, of circular shape and diameter $d = 40$ mm, under constant transversal load $q_o = 53$ kN/m. The number of terms in the Fourier sums is $N = 4$.

In diagram in Fig. 2., the dimensionless deflection $\bar{w}$, the dimensionless axial displacement $\bar{u}$ and the dimensionless normal force $\bar{N}$ are shown. The algorithm described in Section 4.3, based on Eqs. (48), (49), have been used. Due to the optimal use of the multiplying operators **P**, very smooth and numerically stable solutions are obtained. For comparison, if Eqs. (50), (51) would have been used in lieu of Eqs. (48), (49), the solutions deteriorate considerably (dotted line in Fig. 3).

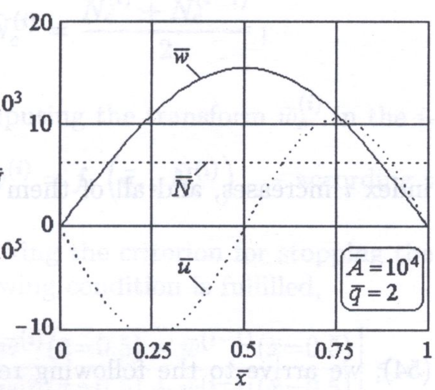

Fig. 2. Dimensionless displacements $\bar{u}$, $\bar{w}$ and axial force $\bar{N}$ along the beam

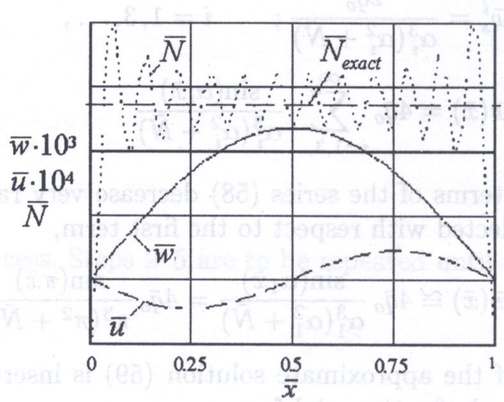

Fig. 3. Non-optimal use of the multiplying and retransforming operators (Eqs. (50), (51))

The solution for the force (curve $\bar{N}$ in Fig. 3) deviates around the exact solution (curve $\bar{N}_{\text{exact}}$ in Fig. 3). The deviations decrease with the increase of the number of terms. These deviations, however, never vanish in the vicinity of the boundaries ($\bar{x} = 0, \bar{x} = 1$). This error cannot be eliminated, ever if the number of terms in the Fourier sums increases indefinitely. It is the well known Gibs' phenomenon in the theory of Fourier series [4].

In diagrams in Fig. 4 and Fig. 5 dimensionless displacement in the center of the beam and dimensionless axial force are depicted.

Exactly the same values for the displacement in the center of the beam ($\bar{w} = 0.016$) and for the axial force ($\bar{N} = 6.289$) have been obtained using the modified Fourier method (Section 5.3), as well as according to the approximate Eqs. (62), (63) and exact formula (58), (60).

As the dimensionless flexibility $\bar{A} = AL^2/I$ increases, deflection in the center of the beam shows more and more nonlinear behaviour (Fig. 4). The normal force tends to increase in linear manner for large dimensionless loads (Fig. 5). Both phenomena imply that at heavier loads the beam behaves as a momentless flexible rope, i.e. the transversal load is being dominantly carried by the axial force and less by the bending moment.

The number of iterations in the iterative procedure (Section 5.3) depends both on the dimensionless flexibility $\bar{A}$ and the dimensionless load $\bar{q}_o$. Diagram in Fig. 6. represents this correlation.

Besides the ordinary differential equations, by application of the modified Fourier transform method it is also possible to solve some partial differential equations in the domain of two or three variables. In [7] the Karman's differential equations of Large deformations of a rectangular plate are solved by application of this method.

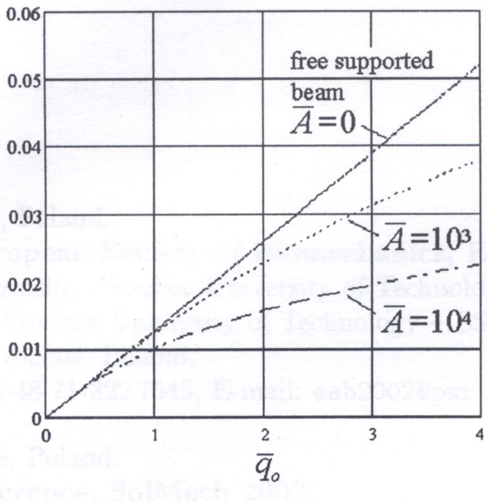

Fig. 4. Dimensionless displacement in the center of the beam versus load

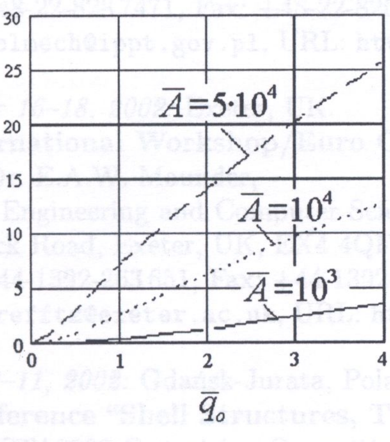

Fig. 5. Dimensionless normal force versus load

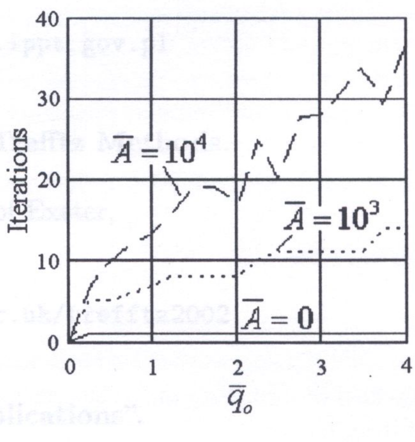

Fig. 6. Number of iterations in the iterative process as function of the dimensionless load and flexibility

REFERENCES

[1] I.A. Birger. *Sterzhni, Plastinki, Obolochki* (in Russian). Nauka, Moscow, 1992.

[2] R.V. Churchill, J.W. Brown. *Fourier Series and Boundary Value Problems.* McGraw–Hill Book Company, New York, 1978.

[3] R. Cukic. *Solutions of Some Problems of Thermoelasticity by Application of Integral Transforms* (in Serbian). Naucna knjiga, Belgrade, 1979.

[4] R. Kurant, D. Gilbert. *Metody Matematicheskoy Fiziki* (in Russian). GITTL, Moscow, 1951.

[5] T.Y. Na. *Computational Methods in Engineering Boundary Value Problems.* Academic Press, New York, 1979.

[6] I.N. Sneddon. *Preobrazovaniya Furye* (in Russian). Izdat. Innostrannoy Literatury, Moscow, 1955.

[7] D. Trajkovski, R. Cukic. Fourier transform method in solving the nonlinear Karman's equations in the theory of thermoelastic plates. *Proceedings of the 16th Canadian Congress of Applied Mechanics CANCAM'97*, Vol. 1, p. 3. University Laval, Quebec City, 1997.

[8] Z. Turek. A new method of finding approximate solutions of the heat conduction equation. *Engineering Trans.*, 44(2): 295–301, 1996.

[9] Z. Turek. A new method of finding strong approximation to solutions to some IBVPs. *Arch. Mech.*, **49**: 573–587, 1997.

[10] Z. Turek. Application of the Fourier cosine series to continuous approximation to solutions of heterogeneous and nonlinear thermal problems. *Proc. of the 2nd Intl. Symp. on Thermal Stresses and Related Topics "Thermal Stresses '97"*, 611–614. Rochester Institute of Technology, 1997.