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Methods for solving systems of linear equations of structure mechanics with interval parameters

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Interval analysis permits to calculate guaranteed *a posteriori* bounds for the solutions of problems with uncertain (interval) input data. Most of the methods of interval analysis assume that all input data vary independently within the given lower and upper bounds. In many practical applications it need not be a case, and the assumption of independence may lead to large overestimation of the set of solutions.

The subject of this work is the problem of solving systems of linear interval equations with coefficients linearly dependent on a set of interval parameters called *coefficient dependence problem*. The purpose of this work is to present methods producing sharp bounds for the set of solutions of systems with dependent input data.

The paper starts with an introduction to systems of linear interval equations and the problem of data dependencies in such systems. A parametric formulation of the coefficient dependence problem follows next.

Finally, three algorithms to calculate tighter bounds for problems with linearly dependent coefficients, namely the Rump's method, its improved version developed by the author, and the IPM method based on the results from Neumaier [8] are presented and discussed. The algorithms are evaluated and compared using some examples of truss structure analysis.

1. Introduction

All structures possess physical and geometrical uncertainties due to physical imperfections, model inaccuracies and system complexities. Modeling these uncertainties is a very important problem considered in structure mechanics [5, 6, 10, 17]. One of the basic tools of solving linear systems of equations with uncertain input data is interval analysis. Interval methods deliver bounds for the solution set, which are verified to be correct. The bounds include measurement imperfections, rounding errors or other procedural errors.

In this paper we investigate the possibilities and problems with application of interval methods to solution of linear interval systems of equations with dependent input data.

A linear interval system of equations of the form

$$[A]x = [b], \tag{1}$$

where $[A] \in \mathbb{IR}^{n\times n}$, $[b] \in \mathbb{IR}^n$ is defined as a family of linear equations

$$Ax = b, \qquad A \in [A], \qquad b \in [b]. \tag{2}$$

To solve Eq. (1) means to compute an interval vector containing the (*united*) solution set

$$\sum([A],[b]) = \sum\nolimits_{\exists\exists}([A],[b]) = \{x \in \mathbb{R}^n \mid \exists A \in [A],\ \exists b \in [b]\ Ax = b\}. \tag{3}$$

For an arbitrary nonempty bounded set $X \subseteq \mathbb{R}^n$ the tightest interval enclosure $\lozenge X$ is defined as

$$\lozenge X = \bigcap\{[Y] \mid X \subseteq [Y],\ [Y] \in \mathbb{IR}^n\} = [\inf[X],\ \sup[X]] \tag{4}$$

and is called the *interval hull* of X.

A square interval matrix $[A] \in \mathbb{IR}^{n\times n}$ is called *regular* (or *non-singular*) if all real matrices $A \in [A]$ are non-singular. If $[A]$ is regular then interval hull, denoted here by $[A]^{\mathrm{H}}[b]$,

$$[A]^{\mathrm{H}}[b] = \Diamond\left(\sum([A],\,[b])\right) \tag{5}$$

for the solution set (3) exists (see Neumaier [8]). Computing $[A]^{\mathrm{H}}[b]$ was proved to be NP-hard [11, 12, 14]. The same is true for computing enclosures with bounded relative overestimation. In [13] Rohn proved that if there exists a polynomial-time algorithm for enclosing solutions of linear interval system of equations with relative overestimation better then $4/n^2$ (where n is a number of equations), then P = NP. The main source of difficulties connetcted with computing the interval hull is the complicated structure of the solution set, which is generally nonconvex.

Various exponential complexity methods for computing the interval hull exist (they can be found in [2, 4, 13]), but for large systems their practical value is small.

It is possible to derive polynomial-complexity algorithms computing interval estimations for (3) (see e.g. [1, 4, 8]). They produce the result faster, but for the price of often large overestimation of the hull, unless special conditions are fulfilled (related to the inverse nonnegavity of the coefficient matrix).

In this paper we consider linear mechanical systems such as truss structures with uncertain parameters. The standard displacement method of truss analysis leads to the system of linear equations with node displacements as unknowns. The system matrix is called a *stiffness matrix*, and the left hand side is a vector of forces. The stiffness matrix has a special structure in that the coefficients of the matrix depend linearly on stiffnesses of the bars (see [4]). Assuming the exact values of stiffnesses of some of the bars to be uncertain, we obtain a linear system of equations with interval parameters.

The main problem with application of standard interval methods to analysis of the linear mechanical systems is that most of these methods implicitly assume that all input data vary independently between the given lower and upper bounds. This causes a large overestimation of the solution, often too large to be practical in real-life applications.

The purpose of this work is to present the problem of data dependencies as well as the methods giving very sharp bounds for the solution sets of systems of structure mechanics which are typical examples of systems with dependent data.

2. Interval expressions: the problem of dependence

In this section we briefly discuss the problem of dependence (fully described in [8]) which arises from evaluation of interval expressions. As we will show, dependence may lead to large overestimation for apparently simple expressions.

Consider the arithmetical expression $f(x) = 1/(1 - x + x^2)$. For real values the expression describes the real function with values in $(0,\, 4/3]$, and the expression is numerically stable for real arguments. On the other hand, for an interval argument $[0,\, 2]$ an interval evaluation of f gives

$$f([0,\,2]) = 1/(1 - [0,\,2] + [0,\,4]) = 1/[-1,\,5] = \mathrm{NaN}$$

(NaN means "not a number").

For intervals close to $[0,\,1]$ we get

$$f([0,\,t]) = [1/(1+t^2),\, 1/(1-t)], \qquad 0 \le t \le 1.$$

Since $\Diamond\{1/(1 - x + x^2) \mid x \in [0,\,t]\} \subseteq [1, 4/3]$, $1/2 \le t \le 1$ then the overestimation of the upper bound becomes arbitrarily large when $t \to 1$. This overestimation is due to dependence of the variables. Once the subexpressions $1 - x$ and x^2 are computed, interval arithmetic effectively computes in the next step the interval hull

$$\Diamond\left\{1 - x_1 + x_2^2,\;\; x_1, x_2 \in [0,\,t]\right\} = [-1,\,5]$$

instead of

$$\Diamond\{1 - x + x^2,\ \ x \in [0,\ t]\} = [3/4,\ 1].$$

Dependence is always present, when one or several variables repeatedly occur in an expression. In some cases we can transform an expression to the equivalent form without dependence. For example let

$$g(x) = 4/\left(3 + (2x - 1)^2\right)$$

then f and g are equivalent for real arguments. The interval evaluation of g for $x = [0,\ 1]$

$$g([0,\ 1]) = [1,\ 4/3]$$

gives exactly the range.

Dependence is mostly due to the fact that some of the algebraic laws valid for real numbers are no longer valid for intervals, they only hold in a weaker form. For example interval arithmetic lacks the distributivity law. For arbitrary $[a]$, $[b]$, $[c] \in \mathbb{IR}$ we have:

$$\left.\begin{array}{l}[a]\,([b] \pm [c]) \subseteq [a][b] \pm [a][c] \\ ([a] \pm [b])\,[c] \subseteq [a][c] \pm [b][c]\end{array}\right\}\ \text{subdistributivity.}$$

To summarize: evaluation of interval expressions properly done avoids overestimations, and leads to reliable algorithms for the enclosure of the solutions to most of the problems with uncertain data.

3. Linear interval systems with dependencies in data

The problem of linear interval equations with dependencies in data was first considered by Jansson [3]. Jansson proposed an algorithm computing very sharp bounds for the solution sets of symmetric, respectively skew-symmetric, matrices with interval input data and dependencies in the right hand vector.

Rump [16] gave straightforward generalization of these dependencies to affine-linear dependencies of the matrix and the right hand vector coefficients on a set of interval parameters. This covers various problems from many fields.

Let us consider the system of linear equations of the form

$$A(p)x = b(p) \tag{6}$$

and assume $A(p)$ and $b(p)$ depend linearly on $p \in \mathbb{R}^k$, that is there are vectors $\omega(i,\ j) \in \mathbb{R}^k$ with

$$\{A(p)\}_{ij} = \omega(i,\ j)^{\mathrm{T}} \cdot p \quad \text{and} \quad \{b(p)\}_j = \omega(0,\ j)^{\mathrm{T}} \cdot p, \tag{7}$$

$i = 1, \ldots, n$; $j = 1, \ldots, n$. If p is allowed to vary within an interval $[p] \in \mathbb{IR}^k$, we may ask for the enclosure of the set

$$\sum(A(p),\ b(p),\ [p]) = \{x \in \mathbb{R}^n \mid \exists p \in [p],\ A = A(p),\ b = b(p)\ :\ Ax = b\}. \tag{8}$$

Theorem 1 (Rump 1994) *Let $A(p) \cdot x = b(p)$, $A(p) \in \mathbb{R}^{n \times n}$, $b(p) \in \mathbb{R}^n$, $[p] \in \mathbb{IR}^k$, $p \in [p]$ be a parametrized linear system, where $A(p)$, $b(p)$ are given by Eq. (7). Let $R \in \mathbb{R}^{n \times n}$, $[Y] \in \mathbb{IR}^n$, $\widetilde{x} \in \mathbb{R}^n$, and define $[Z] \in \mathbb{IR}^n$, $[C] \in \mathbb{IR}^{n \times n}$ by*

$$\begin{aligned}[Z]_i &= \left(\sum_{j=1}^{n}\left\{R_{ij} \cdot \left(\omega(0,\ j) - \sum_{\nu=1}^{n} \widetilde{x}_\nu \cdot \omega(j,\ \nu)\right)\right\}\right)^{\mathrm{T}} [p], \\ [C] &= I - R \cdot A([p]).\end{aligned} \tag{9}$$

Define $[V] \in \mathbb{IR}^n$ *by*

$$[V]_i = \left\{\Diamond\Big([Z] + [C]\cdot[U]\Big)\right\}_i, \tag{10}$$

where $[U] = ([V]_1, \ldots, [V]_{i-1}, [Y]_i, \ldots, [Y]_n)^{\mathrm{T}}, \quad i = 1, \ldots, n.$
If

$$[V] \subsetneq [Y],$$

then R *and every matrix* $A(p)$, $p \in [p]$ *is regular, and for every* $A(p)$, $b(p)$ *with* $p \in [p]$ *the unique solution* $\tilde{x} = A(p)^{-1}b(p) \in \widetilde{x} + [V]$.

The following inner and outer estimations hold true for $1 \le i \le n$:

$$\widetilde{x}_i + \inf([Z]_i) + \sup([\Delta]_i) \ge \inf_{\sigma_i \in \Sigma} \sigma_i,$$

$$\widetilde{x}_i + \sup([Z]_i) + \inf([\Delta]_i) \le \sup_{\sigma_i \in \Sigma} \sigma_i,$$

where $[\Delta] = \Diamond([C]\cdot[V])$ *and* $\sum$ *denotes the solution set defined by Eq. (8).*

□

Theorem 1 gives inner and outer bounds on $\sum(A(p),\, b(p),\, [p])$

$$\widetilde{x} + [V'] \subseteq \sum \subseteq \widetilde{x} + [V], \tag{11}$$

where $[V']_i = [\inf([Z]_i) + \sup([\Delta]_i),\ \sup([Z]_i) + \inf([\Delta]_i)]$, and $[V]$ is given by Eq. (10).

In Sec. 4 we present algorithms based on Rump's theorem. We use interval iteration with the so-called ε-inflation, introduced in [15]. The following theorem gives conditions of convergence of the iteration.

Theorem 2 (Rump 1994) *Let* $[C] \in \mathbb{IR}^{n\times n}$, $[Z] \in \mathbb{IR}^n$ *and for* $[X^0] \in \mathbb{IR}^n$ *define the iteration*

$$[X^{k+1}] = [Z] + [C]\cdot[X^k] + [E^k], \quad k \in N,$$

where $[E^k] \in \mathbb{IR}^n$, $[E^k] \to [E] \in \mathbb{IR}^n$ *with* $0 \in \mathrm{int}([E])$. *Then the following two conditions are equivalent*

$$\forall [X^0] \in \mathbb{IR}^n\ \exists k \in N:\ [Z] + [C][X^k] \subseteq \mathrm{int}([X^k]),$$

$$\rho(|[C]|) < 1.$$

□

The quality of the bounds is determined by the width of $[\Delta]$, which is a product of matrix $[C]$ and vector $[V]$. Matrix $[C]$ is computed here by straightforward evaluation of interval expressions. Such straightforward evaluation usually produces overestimation (see Sec. 2).

Now we use linear dependencies (7) to eliminate multiple occurrence of parameter $[p]$ in formula $[C]_{ij} = I_{ij} - \{R\cdot A([p])\}_{ij}$.

$$\{R\cdot A([p])\}_{ij} = \left\{\Diamond\{R\cdot A(p)\ \mid\ p \in [p]\}\right\}_{ij}$$

$$= \Diamond\left\{\sum_{\nu=1}^{n} R_{i\nu}A_{\nu j}(p)\ \mid\ p \in [p]\right\} = \Diamond\left\{\sum_{\nu=1}^{n} R_{i\nu}\omega(\nu, j)^{\mathrm{T}}p\ \mid\ p \in [p]\right\}$$

$$= \Diamond\left\{\left(\sum_{\nu=1}^{n} R_{i\nu}\omega(\nu, j)\right)^{\mathrm{T}} p\ \mid\ p \in [p]\right\} = \left(\sum_{\nu=1}^{n} R_{i\nu}\omega(\nu, j)\right)^{\mathrm{T}}[p].$$

The last equality holds since the parameter p occurs only once in the expression. Let us denote

$$[C^d]_{ij} = I_{ij} - \left(\sum_{\nu=1}^{n} R_{i\nu}\omega(\nu, j) \right)^{\mathrm{T}} [p]. \tag{12}$$

It is easy to check that $([C^d]_{ij}) = [C^d] \subseteq [C]$, and hence $\rho(|[C^d]|) \leq \rho(|[C]|)$ and $[C^d][V] \subseteq [C][V]$. Thus $[C^d][V] = [\Delta^d] \subseteq [\Delta]$, and hence we obtain a better enclosure.

In the following we present another way to compute an outer enclosure for the solution set (8).

A square martix $[A] \in \mathbb{IR}^{n\times n}$ is an H-matrix iff there is a real vector $u > 0$ with $\langle [A]\rangle u > 0$. Here $\langle [A]\rangle$ is the *comparison matrix* with entries

$$\langle [A]\rangle_{ii} = \langle [A]_{ii}\rangle \quad = \min\{|A_{ii}|,\ A_{ii} \in [A]_{ii}\},$$

$$\langle [A]\rangle_{ik} = -|[A]_{ik}| = -\max\{|A_{ik}|,\ A_{ik} \in [A]_{ik}\}, \quad i \neq k.$$

H-matrices and the comparison matrix $\langle\cdot\rangle$ were introduced by Ostrowski [9], who proved the important inequality (15) for real H-matrices.

A square interval matrix $[A] \in \mathbb{IR}^{n\times n}$ is called *strongly regular* if $\check{A}^{-1}[A]$ is regular. $\check{A} = \mathrm{mid}([A])$ is a midpoint matrix.

H-matrices and strongly regular matrices are of importance in our investigations.

Theorem 3. *Let* $[A] \in \mathbb{IR}$ *be regular,* $R \in \mathbb{R}^{n\times n}$ *and* $\tilde{x} \in \mathbb{R}^n$. *If* $R[A]$ *is an H-matrix then for all* $[B] \in \mathbb{IR}$, *we have*

$$[A]^{\mathrm{H}}[b] \subseteq \tilde{x} + [-1,\ 1]\langle R[A]\rangle^{-1}|R([b] - [A]\tilde{x})|. \tag{13}$$

Proof. If $x \in \sum([A], [b])$ then $Ax = b$ for some $A \in [A]$, $b \in [b]$. Thus $x = A^{-1}b = \tilde{x} + A^{-1}(b - A\tilde{x}) = \tilde{x} + (RA)^{-1}R(b - A\tilde{x}) \in \tilde{x} + (R[A])^{\mathrm{H}}(R([b] - [A]\tilde{x}))$ and we have

$$[A]^{\mathrm{H}}[b] \subseteq \tilde{x} + (R[A])^{\mathrm{H}}(R([b] - [A]\tilde{x})). \tag{14}$$

Among many properties of H-matrices prooved in [8] we have

$$|[A]^{-1}| \leq \langle [A]\rangle^{-1} \tag{15}$$

and

$$|[A]^{\mathrm{H}}[b]| \leq |[A]|^{-1}|[b]|. \tag{16}$$

Thus

$$[A]^{\mathrm{H}}[b] \subseteq [-1,\ 1]\langle [A]\rangle^{-1}|[b]|. \tag{17}$$

Since by assumption matrix $R[A]$ is an H-matrix then by Eq. (17)

$$(R[A])^{\mathrm{H}}(R([b] - [A]\tilde{x})) \subseteq [-1,\ 1]\langle R[A]\rangle^{-1}|R([b] - [A]\tilde{x})|. \tag{18}$$

Equations (14) and (18) imply the thesis of the theorem. □

If $A([p])$ is regular and $R \cdot A([p])$ is an H-matrix, then by Theorem 3 we obtain the following estimation of the solution set (8):

$$\sum(A(p), b(p), [p]) \subseteq [-1,\ 1]\langle R \cdot A([p])\rangle^{-1}|R(b([p]) - A([p])\tilde{x})|$$

and finally (taking into account data dependencies)

$$\sum(A(p), b(p), [p]) \subseteq [-1,\ 1]\langle [D]\rangle^{-1}|[Z]|, \tag{19}$$

where

$$[D]_{ij} = \left(\sum_{\nu=1}^{n} R_{i\nu}\omega(\nu, j) \right)^{\mathrm{T}} [p]$$

and vector $[Z]$ is given by Eq. (9).

4. Algorithms

In this section we present algorithms based on the results presented in Sec. 3.

4.1. Rump's Algorithm (RA)

The algorithm presented here allows to compute the real vector $\widetilde{x}$ and interval vectors $[V]$ and $[V']$ satisfying (11). In the case of symmetric matrices it is compatible with Jansson's algorithm [3].

$R := \mathrm{mid}(A([p]))^{-1}$; $[C] := I - R \cdot A([p])$;

$\widetilde{x} := R \cdot \mathrm{mid}(b[p]))$; $[Z]_i := \left(\sum_{j=1}^{n}\left\{R_{ij} \cdot \left(\omega(0, j) - \sum_{\nu=1}^{n} \widetilde{x}_\nu \cdot \omega(j, \nu)\right)\right\}\right)^{\mathrm{T}} [p]$;

$[V] := [Z]$; $k = 0$;

repeat

  $k := k + 1$;

  $[Y] := [V] \cdot [1 - \varepsilon, 1 + \varepsilon] + [-\mu, \mu]$;

  $[\Delta] := [C] \cdot [Y]$;

  $[V] := [Z] + [\Delta]$;

until ($[V] \subset \mathrm{int}[Y]$) or ($k > k_{\max}$);

if $[V] \subset \mathrm{int}[Y]$ then

  begin

    outer $:= \widetilde{x} + [V]$;

    inner $:= \widetilde{x} + [\underline{Z} + \overline{\Delta},\ \overline{Z} + \underline{\Delta}]$

  end

The prescribed range of ε is $(0, 1)$. Practical observations indicate that a good choice of ε is 0.01, $k_{\max} = 15$, and the components of vector μ are equal to the smallest positive real number expressible as floating-point number. It is recommended to choose $R = \mathrm{mid}(A([p]))^{-1}$ and $\widetilde{x} = \mathrm{mid}(A([p]))^{-1} \cdot \mathrm{mid}(b[p]))$, so that $[C]$ and $[Z]$ are of small norms.

Moreover, if $A([p])$ is strongly regular then

$$\|I - \mathrm{mid}(A([p]))^{-1}A([p])\|_u < 1, \qquad 0 < u \in \mathbb{R}^n.$$

One of the properties of the spectral radius, proved in [8], is:

$$\rho(A) \le \|A\|_u, \quad \text{if } A \in \mathbb{R}^{n\times n}.$$

Thus, if $R = \mathrm{mid}(A([p]))^{-1}$ then

$$\rho(|[C]|) < 1,$$

which is a sufficient condition for a finite termination of the algorithm.

4.2. Modified Rump's Algorithm (MRA)

We use formula (12) to modify Rump's algorithm. This modification gives the new method *Modified Rump's Algorithm* (MRA).

$$R := \mathrm{mid}(A([p]))^{-1};$$
$$[C^d]_{ij} := I_{ij} - \left(\sum_{\nu=1}^{n} R_{i\nu}\omega(\nu, j)\right)^{\mathrm{T}} [p];$$
$$\widetilde{x} := R \cdot \mathrm{mid}(b([p])); \; [Z]_i := \left(\sum_{j=1}^{n}\left\{R_{ij} \cdot \left(\omega(0, j) - \sum_{\nu=1}^{n} \widetilde{x}_\nu \cdot \omega(j, \nu)\right)\right\}\right)^{\mathrm{T}} [p];$$
$$[V] := [Z];$$
$$k := 0;$$

repeat

$k := k + 1;$

$[Y] := [V] \cdot [1 - \varepsilon, 1 + \varepsilon] + [-\mu, \mu];$

$[\Delta^d] := [C^d] \cdot [Y];$

$[V] := [Z] + [\Delta^d];$

until $([V] \subset \mathrm{int}[Y])$ or $(k > k_{\max})$;

if $[V] \subset \mathrm{int}[Y]$ then

begin

outer $:= \widetilde{x} + [V];$

inner $:= \widetilde{x} + [\underline{Z} + \overline{\Delta^d},\ \overline{Z} + \underline{\Delta^d}];$

end

4.3. Interval Parametrization Method (IPM)

The IPM method is based on the results from Neumaier [8] and Rump [16]. It computes an outer enclosure for the solution set (8) using Eq. (19).

$$R := \mathrm{mid}(A([p]))^{-1};$$
$$\widetilde{x} := R \cdot \mathrm{mid}(b([p]));$$
$$[Z]_i := \left(\sum_{j,\nu=1}^{n}\left\{R_{ij} \cdot \left(\omega(0, j) - \widetilde{x}_\nu \cdot \omega(j, \nu)\right)\right\}\right)^{\mathrm{T}} [p];$$
$$[D]_{ij} := \left(\sum_{\nu=1}^{n} R_{i\nu}\omega(\nu, j)\right)^{\mathrm{T}} [p];$$
$$\text{outer} := \widetilde{x} + [-1, 1]\langle[D]\rangle^{-1}|[Z]|$$

In [7] Neumaier proved that if $C[A]$ is an H-matrix for some $C \in \mathbb{R}^{n\times n}$ then $\check{A}^{-1}[A]$ is an H-matrix (or $[A]$ is strongly regular) and

$$\langle \check{A}^{-1}[A]\rangle^{-1}|\check{A}^{-1}[b]| \leq \langle C[A]\rangle^{-1}|C[b]|.$$

Hence, if $A([p])$ is strongly regular then the optimal choice of R is $\mathrm{mid}(A([p]))^{-1}$.

The IPM method produces enclosure in one step, hence it is usefull for large systems of equations.

5. Trusses

The standard displacement method of truss analysis leads to a set of linear equations of the form

$$Kd = Q, \tag{20}$$

where K is the global stiffness matrix, Q is the vector of external loads, and d is the displacements vector to be computed. The elements of matrix K depend linearly on stiffnesses (denoted below by s) of the appropriate bars.

The truss structure examples given below were developed and used in the research (on hybrid expert system for analysis of truss structures) supported by grant No. 8T11F00615 from KBN.

5.1. Example 1: 7-bars planar truss structure

Let us consider the planar truss structure depicted in Fig. 1

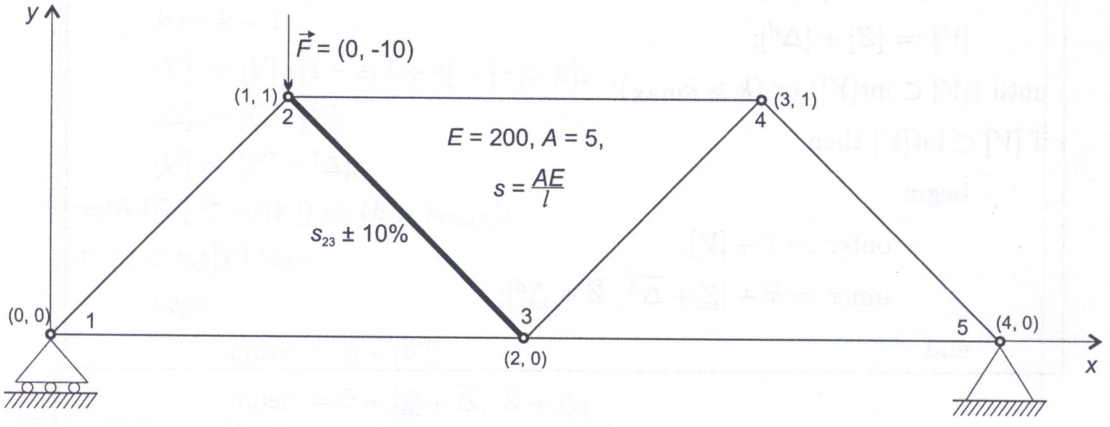

Fig. 1. 7-bar planar truss structure

Parameters of the truss

$A = 5$ is the cross-sectional area,
$E = 200$ is Young modulus,
l length of the bars equals $\sqrt{2}$ for skew bars, and 2 for horizontal bars,
$s = AE/l$,
$F = -10$ is an external load at node No. 2.

The parameters of the truss are given as dimensionless numbers, since the physical values are not relevant to our analysis.

Now we assume the stiffness of the bar (2-3) (the bar is depicted in Fig. 1 by a thick line) to be uncertain, varying around the nominal value by $\pm 10\%$. This gives one interval parameter $[s_{23}]$. The interval stiffness matrix is:

$$[K] = \begin{pmatrix} 853.55 & -353.55 & -353.55 & -500 & 0 & 0 & 0 \\ -353.55 & [1171.75, 1242.46] & [-35.35, 35.35] & [-388.91, -318.19] & [318.19, 388.91] & -500 & 0 \\ -353.55 & [-35.35, 35.35] & [671.75, 742.46] & [318.19, 388.91] & [-388.91, -318.19] & 0 & 0 \\ -500 & [-388.91, -318.19] & [318.19, 388.91] & [1671.75, 1742.46] & [-35.35, 35.35] & -353.55 & -353.55 \\ 0 & [318.19, 388.91] & [-388.91, -318.19] & [-35.35, -35.35] & [671.75, 742.46] & -353.55 & -353.55 \\ 0 & -500 & 0 & -353.55 & -353.55 & 1207.11 & 0 \\ 0 & 0 & 0 & -353.55 & -353.55 & 0 & 707.11 \end{pmatrix}$$

and the interval vector of the forces is $[Q] = (0, 0, [-10, -10], 0, 0, 0, 0)^{\mathrm{T}}$ (real numbers may be considered as intervals consisting of one point). The stiffness matrix is strongly regular.

Below we show how each individual component of $[K]$ depends on the interval parameter $[s_{23}]$. It can be easily derived from the analytical solution of the problem [4].

$$\begin{pmatrix}
\frac{s_{12}}{2}+s_{13} & -\frac{s_{12}}{2} & -\frac{s_{12}}{2} & -s_{13} & 0 & 0 & 0 \\
-\frac{s_{21}}{2} & \frac{s_{21}+[s_{23}]}{2}+s_{24} & \frac{s_{21}-[s_{23}]}{2} & -\frac{[s_{23}]}{2} & \frac{[s_{23}]}{2} & -s_{24} & 0 \\
-\frac{s_{21}}{2} & \frac{s_{21}-[s_{23}]}{2} & \frac{s_{21}+[s_{23}]}{2} & \frac{[s_{23}]}{2} & -\frac{[s_{23}]}{2} & 0 & 0 \\
-s_{31} & -\frac{[s_{32}]}{2} & \frac{[s_{32}]}{2} & s_{31}+\frac{[s_{32}]+s_{34}}{2}+s_{35} & \frac{s_{34}-[s_{32}]}{2} & -\frac{s_{34}}{2} & -\frac{s_{34}}{2} \\
0 & \frac{[s_{32}]}{2} & -\frac{[s_{32}]}{2} & \frac{s_{34}-[s_{32}]}{2} & \frac{[s_{32}]+s_{34}}{2} & -\frac{s_{34}}{2} & -\frac{s_{34}}{2} \\
0 & -s_{42} & 0 & -\frac{s_{43}}{2} & -\frac{s_{43}}{2} & s_{42}+\frac{s_{43}+s_{45}}{2} & 0 \\
0 & 0 & 0 & -\frac{s_{43}}{2} & -\frac{s_{43}}{2} & 0 & \frac{s_{43}+s_{45}}{2}
\end{pmatrix}$$

As can be seen from the table above, the elements of the matrix $[K]$ are of the form:

$$[A]_{ij} = a + b \cdot [s_{23}], \quad a,\ b \in \mathbb{R}.$$

This suggests to use the interval vector $[p] = (1, [s_{23}])^{\mathrm{T}} \in \mathbb{IR}^2$. Then

$$[A]_{ij} = (a,\ b)^{\mathrm{T}} \cdot [p].$$

The elements of the vector of forces can be expressed by the same formula. Then the matrix ω is of the form

$$\begin{pmatrix}
(0,0) & (0,0) & (0,0) & (0,-10) & (0,0) & (0,0) & (0,0) \\
(\frac{s_{12}}{2}+s_{13},0) & (-\frac{s_{12}}{2},0) & (-\frac{s_{12}}{2},0) & (-s_{13},0) & (0,0) & (0,0) & (0,0) \\
(-\frac{s_{21}}{2},0) & (\frac{s_{21}}{2}+s_{24},\frac{1}{2}) & (\frac{s_{21}}{2},-\frac{1}{2}) & (0,-\frac{1}{2}) & (0,\frac{1}{2}) & (-s_{24},0) & (0,0) \\
(-\frac{s_{21}}{2},0) & (\frac{s_{21}}{2},-\frac{1}{2}) & (\frac{s_{21}}{2},\frac{1}{2}) & (0,\frac{1}{2}) & (0,-\frac{1}{2}) & (0,0) & (0,0) \\
(-s_{31},0) & (0,-\frac{1}{2}) & (0,\frac{1}{2}) & (s_{31}+\frac{s_{34}}{2}+s_{35},\frac{1}{2}) & (\frac{s_{34}}{2},-\frac{1}{2}) & (-\frac{s_{34}}{2},0) & (-\frac{s_{34}}{2},0) \\
(0,0) & (0,\frac{1}{2}) & (0,-\frac{1}{2}) & (\frac{s_{34}}{2},-\frac{1}{2}) & (\frac{s_{34}}{2},\frac{1}{2}) & (-\frac{s_{34}}{2},0) & (-\frac{s_{34}}{2},0) \\
(0,0) & (-s_{42},0) & (0,0) & (-\frac{s_{43}}{2},0) & (-\frac{s_{43}}{2},0) & \left(s_{42}+\frac{s_{43}+s_{45}}{2},0\right) & (0,0) \\
(0,0) & (0,0) & (0,0) & (-\frac{s_{43}}{2},0) & (-\frac{s_{43}}{2},0) & (0,0) & \left(\frac{s_{43}+s_{45}}{2},0\right)
\end{pmatrix}$$

5.2. Example 2: 9-bar planar structure (crane)

The second example is depicted in Fig. 2.

Parameters of the truss

$A = 5$ is the cross-sectional area,
$E = 200$ is Young modulus,
l length of the bars equals $\sqrt{2}$ for skew bars, and 1 for horizontal and vertical bars,
$F = -10$ is the external load at node No. 6.

In this case we assume the stiffness of the bar (3-5) to be uncertain by ±5%. This gives one interval parameter $[s_{35}]$. The interval stiffness matrix is strongly regular.

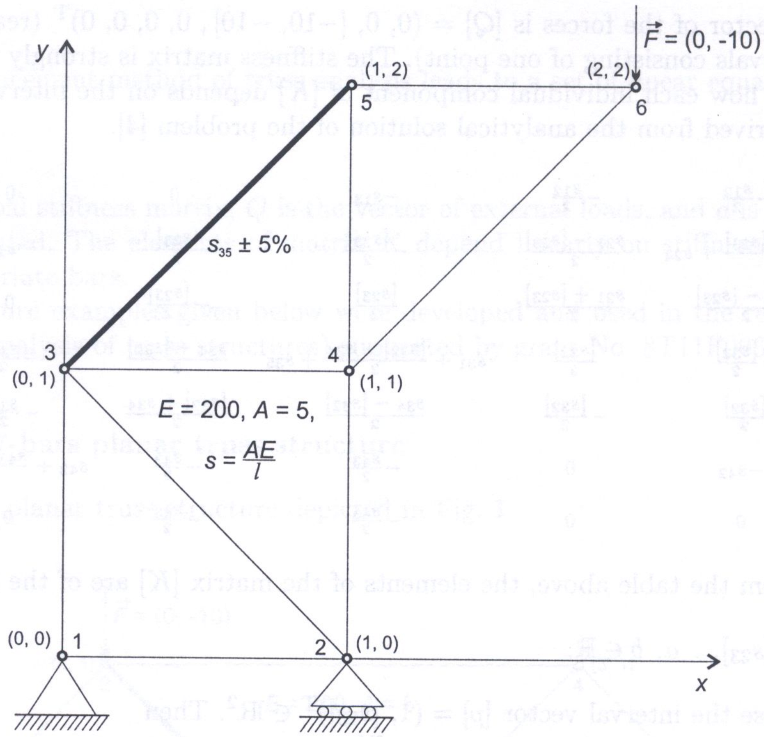

Fig. 2. 9-bar planar truss structure (crane)

5.3. Results

In this section we discuss briefly the results of interval analysis of the example planar truss structures.

The results produced by each method are summarized in separate tables. Each table contains the nominal solution $\mathbf{d}_0$, i.e. the solution of the system (20) with the nominal (real) values of the parameters, as well as the widths of the interval estimates relative to the nominal solution ($\mathrm{w}[*]/\mathbf{d}_0$). Table 1, Table 2, Table 4 and Table 5 contain inner $]\cdot[$ and outer $[\cdot]$ estimations produced by RA and MRA methods respectively. Table 1 and Table 4 additionally contain the interval hull, which is the tightest interval enclosure for the solution set (3) we can obtain assuming independence of the input data. Table 3 and Table 6 contain outer estimation of the solution set produced by IPM method. Endpoints of the resulting intervals with the sign different than the sing of the nominal solution are indicated by underlying.

As can be seen from the tables, the parametrization of the system significantly decreased the width of the interval estimation w.r.t. the hull. All of the presented methods produced very sharp

Table 1. RA method results for Example 1

	$\mathbf{d}_0$ $[\times 10^{-4}]$	$\lozenge\sum([A],[b])$ $[\times 10^{-4}]$	$\mathrm{w}[*]/\mathbf{d}_0$ [%]	$]\cdot[$ $[\times 10^{-4}]$	$\mathrm{w}[*]/\mathbf{d}_0$ [%]	$[\cdot]$ $[\times 10^{-4}]$	$\mathrm{w}[*]/\mathbf{d}_0$ [%]
d_1^x	−200	[−8246, −60]	4090	[−272.1, −127.9]	72.1	[−272.2, −127.8]	72.2
d_2^x	−25	[−4256, $\underline{285.7}$]	18200	[−61.2, $\underline{11.2}$]	289.6	[−64.7, $\underline{14.7}$]	317.6
d_2^y	−387.1	[−8468, −246.5]	2120	[−457.8, −316.5]	36.5	[−461.4, −312.9]	38.4
d_3^x	−50	[−5078, $\underline{117.6}$]	10400	[−95.1, −4.9]	180.4	[−95.1, −4.9]	180.4
d_3^y	−341.4	[−10648, −162.2]	3070	[−430.2, −252.6]	52	[−437.4, −245.5]	56.2
d_4^x	−125	[−6367, −16.4]	5080	[−179.2, −70.8]	87.2	[−182.8, −67.2]	92.5
d_4^y	−195.7	[−7860, −62.4]	3980	[−262.7, −128.8]	68.4	[−266.2, −125.2]	72

Table 2. MRA method results for Example 1

	$\mathbf{d}_0$ $[\times 10^{-4}]$	$]\cdot[$ $[\times 10^{-4}]$	$\mathrm{w}[*]/\mathbf{d}_0$ [%]	$[\cdot]$ $[\times 10^{-4}]$	$\mathrm{w}[*]/\mathbf{d}_0$ [%]
d_1^x	−200	−200	0	−200	0
d_2^x	−25	[−26.6, −23.4]	12.8	[−27, −23]	16
d_2^y	−387.1	[−388.7, −385.6]	0.8	[−389.1, −385.2]	1
d_3^x	−50	−50	0	−50	0
d_3^y	−341.4	[−344.6, −338.3]	1.8	[−345.4, −337.5]	2.3
d_4^x	−125	[−126.6, −123.4]	2.6	[−127, −123]	3.2
d_4^y	−195.7	[−197.3, −194.1]	1.6	[−197.7, −193.7]	2

Table 3. IPM method results for Example 1

	$\mathbf{d}_0$ $[\times 10^{-4}]$	$[\cdot]$ $[\times 10^{-4}]$	$\mathrm{w}[*]/\mathbf{d}_0$ [%]
d_1^x	−200	−200	0
d_2^x	−25	[−27, −23]	16
d_2^y	−387.1	[−389.1, −385.2]	1
d_3^x	−50	−50	0
d_3^y	−341.4	[−345.3, −337.5]	2.3
d_4^x	−125	[−127, −123]	3.2
d_4^y	−195.7	[−197.7, −193.7]	2

Table 4. RA method results for Example 2

	$\mathbf{d}_0$ $[\times 10^{-4}]$	$\Diamond\sum([A],[b])$ $[\times 10^{-4}]$	$\mathrm{w}[*]/\mathbf{d}_0$ [%]	$]\cdot[$ $[\times 10^{-4}]$	$\mathrm{w}[*]/\mathbf{d}_0$ [%]	$[\cdot]$ $[\times 10^{-4}]$	$\mathrm{w}[*]/\mathbf{d}_0$ [%]
d_2^x	0	[−34.2, 177.9]	–	[−1.98, 1.98]	–	[−1.98, 1.98]	–
d_3^x	100	[−87.3, 1136.6]	1124	[88.5, 111.6]	23	[88.4, 111.6]	23
d_3^y	100	[35.7, 455.7]	420	[96, 104]	8	[96, 104]	8
d_4^x	0	[−187.3, 1036.6]	–	[−11.6, 11.6]	–	[−11.6, 11.6]	–
d_4^y	−200	[−377.8, −161.9]	108	[−202, −198]	2	[−202, −198]	2
d_5^x	782.8	[459.9, 2782.3]	296.7	[774.7, 791]	2.1	[746.3, 819.4]	9.3
d_5^y	−300	[−655.7, −223.9]	143.9	[−304, 296]	2.7	[−304, 296]	2.7
d_6^x	882.8	[559.9, 2882.3]	263.1	[874.7, 891]	1.8	[846.3, 919.4]	8.3
d_6^y	−1365.7	[−2506.4, −1181.4]	97	[−1367.1, −1364.3]	0.2	[−1392.6, −1338.8]	3.9

bounds for the solution set (8). The new methods (MRA and IPM) produced the best results with minor relative overestimation (in many cases the interval components are thin and equal to the nominal solution). The results produced by MRA an IPM methods are practically the same. However, the MRA method produces both inner and outer enclosures, while the IPM method produces only outer enclosure. Hence, the results produced by the MRA method are much informative. On the other hand, the IPM method is not only easier to understand, but also advantageous for large systems since it computes the enclosure in one step.

Table 5. MRA method results for Example 2

	$\mathbf{d}_0$ [%]	$]\cdot[$ [$\times 10^{-4}$]	$\mathrm{w}[*]/\mathbf{d}_0$ [%]	$[\cdot]$ [$\times 10^{-4}$]	$\mathrm{w}[*]/\mathbf{d}_0$ [%]
d_2^x	0	0	–	0	–
d_3^x	100	100	0	100	0
d_3^y	100	100	0	100	0
d_4^x	0	0	–	0	–
d_4^y	−200	−200	0	−200	0
d_5^x	782.8	[769.5, 796.2]	3.4	[768, 797.7]	3.8
d_5^y	−300	−300	0	−300	0
d_6^x	882.8	[869.4, 896.2]	3	[868, 897.7]	3.4
d_6^y	−1365.7	[−1379.1, −1352.3]	2	[−1380.6, −1350.8]	2.2

Table 6. IPM method results for Example 2

	$\mathbf{d}_0$ [$\times 10^{-4}$]	$[\cdot]$ [$\times 10^{-4}$]	$\mathrm{w}[*]/\mathbf{d}_0$ [%]
d_2^x	0	0	0
d_3^x	100	100	0
d_3^y	100	100	0
d_4^x	0	0	–
d_4^y	−200	−200	0
d_5^x	782.8	[768, 797.7]	3.8
d_5^y	−300	−300	0
d_6^x	882.8	[868, 897.7]	3.4
d_6^y	−1365.7	[−1380.6, −1350.8]	2.2

The methods proved to be powerfull for calculation of the bounds for the solutions of linear mechanical systems with uncertainties.

6. Conclusions

In this paper we have investigated the problem of data dependencies, especially when solving systems of linear equations of structure mechanics. We considered affine-linear dependecies of the matrix and the right hand vector coefficients on the set of interval parameters. We have shown, that proper formulation of the problem in interval form (in this case proper parametrization of the system) and carefull application of interval arithmetic, has led to significant improvements of the estimation of the solution of the problem.

We have presented three methods for solving parametrized systems of linear interval equations, as well as the examples of application of these methods. The proposed methods have proved to be very powerfull for the analysis of problems of structure mechanics with uncertain input data.

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