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The Lyapunov exponents for the partitioned-pipe mixer

Tomasz Stręk
Institute of Applied Mechanics, Poznań University of Technology,
ul. Piotrowo 3, 60-965 Poznań, Poland

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This paper presents a mechanical model of the partitioned-pipe mixer (PPM) in case where pipe of the static mixer rotates with angular periodic velocity. Mixing becomes more efficient if the forcing of fluid mixing process is time periodic. Chaos in duct flows can be achieved by time modulation or by spatial changes along the duct axis. The values of Lyapunov exponents for flow in PPM are calculated.

Keywords: partitioned-pipe mixer, chaos, Lyapunov exponents

1. Introduction

During the past two decades there has been considerable interest in dynamical chaos and examples have been documented among physical, chemical, and biological systems. A review of the application of chaos theory in mechanical, civil, electrical and chemical engineering one can find in [3]. In most practical cases, however, chaos is to be minimised or avoided altogether. The application of chaos to fluid mixing, however, provides a counter example. Fluid mixing is an instance in which chaos is beneficial.

One industrial mixing device that is used in a variety of applications from polymer processing to biotechnology is the static mixer. Frequently industrial application for static mixers is blending a small quantity of a minor component into a larger stream of a major component.

The partitioned-pipe mixer (PPM) consists of a pipe partitioned into a sequence of semi-circular ducts by means of rectangular plates placed orthogonally to each other (Fig. 1). The fluid is forced through the pipe by means of an axial pressure gradient while the pipe is rotated around its axis relative to the assembly of plates, thus resulting in a cross-sectional flow in the (r, θ) plane in each semi-circular element. In paper [12] the theoretical model for the mechanical mixing of a Newtonian fluid in such a mixer when it operates under creeping flow conditions has been presented. These authors considered a case when the pipe rotates with constant angular velocity.

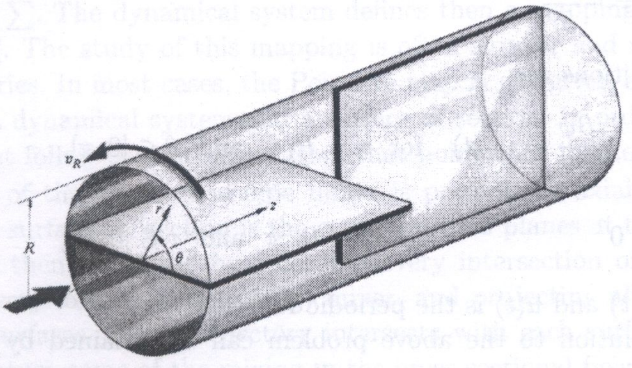


Fig. 1. Schematic view of the partitioned-pipe mixer [13]

The purpose of the presented paper is a mechanical model of the mentioned mixer in a case where the pipe rotates with angular periodic velocity. The expectation that the flow in the partitioned-pipe mixer is chaotic is based on its similarity to the Kenics static mixer, which resembles the Baker's transform in terms of its cross-sectional mixing. Under ideal conditions, each stream is divided into two in each element. The main difference between the two mixers is that the idea of rotation of the cross-sectional flow is the same in adjacent elements of the partitioned-pipe mixer while in the static mixer it is opposite.

2. Velocity field of flow in the partitioned-pipe mixer

We consider an approximate solution to the fully developed Stokes flow of a Newtonian fluid in the semi-circular compartment of an element of the mixer under creeping flow. Similarly to the idea given in paper [12] the flow field in the mixer is decomposed to axial and cross-sectional components. Moreover, it is assumed that these components are independent of each other and can be considered separately.

The axial flow is simply a pressure driven flow in a semi-circular duct and in cylindrical coordinates is given by

$$v_z = \langle v_z \rangle \sum_{k=1}^{\infty} \left(\left(\left(\frac{r}{R}\right)^{2k-1} - \left(\frac{r}{R}\right)^2 \right) \frac{\sin((2k-1)\theta)}{(2k-1)(4-(2k-1)^2)} \right), \tag{1}$$

where R is the radius of the pipe, and the average axial velocity $\langle v_z \rangle$ is

$$\langle v_z \rangle = \frac{8-\pi^2}{4\pi\mu} \frac{\partial p}{\partial z} R^2, \tag{2}$$

where μ is the viscosity, and p is the pressure. Due to periodic rotation of the pipe the cross-sectional component is really unsteady. Here the idea of time modulated flow given in fundamental paper [1] is adapted for the description of a cross-sectional component. According to this method the cross-sectional component is a two-dimensional flow in a semi-circular cavity, and is given as a solution of

$$\nabla^4 \psi = 0, \tag{3}$$

where

$$\nabla^2 = \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2} \tag{4}$$

with the velocity field given by

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad v_\theta = -\frac{\partial \psi}{\partial r}. \tag{5}$$

The boundary conditions are

$$\psi = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0, \qquad -\frac{\partial \psi}{\partial r} = v_R(t) \quad \text{for } r = R, \quad \text{and } \theta \in [0, \pi], \tag{6}$$

$$\psi = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \psi}{\partial r} = 0 \qquad \text{for } \theta = 0, \pi \text{ and } r \in [0, R], \tag{7}$$

where $v_R(t) = v_\theta(R)u(t)$ and $u(t)$ is the periodic function.

An approximate solution to the above problem can be obtained by the method of weighted residuals [9].

The motion is obtained by integrating the following set of equations, which represents the approximate velocity field

$$
\begin{aligned}
v_r &= \frac{d\widetilde{r}}{d\widetilde{t}} = \beta\left(\widetilde{t}\right)\widetilde{r}\left(1-\widetilde{r}^n\right)\sin 2\theta, \\
\frac{v_\theta}{\widetilde{r}} &= \frac{d\theta}{d\widetilde{t}} = \beta\left(\widetilde{t}\right)\left[2-(2+n)\widetilde{r}^n\right]\sin^2\theta, \\
v_z &= \frac{d\widetilde{z}}{d\widetilde{t}} = \frac{16\pi}{\pi^2-8}\sum_{k=1}^{3}\left[\left(\widetilde{r}^{2k-1}-\widetilde{r}^2\right)\frac{\sin((2k-1)\theta)}{(2k-1)(4-(2k-1)^2)}\right].
\end{aligned}
\tag{8}
$$

The above equations are dimensionless, with radial distances made dimensionless with respect to R, axial distances with respect to the length of an element L, and time with respect to $L/\langle v_z\rangle$. The dimensionless function (parameter) β, which is referred to as the mixing strength, is defined as

$$
\beta\left(\widetilde{t}\right) = \frac{4v_R\left(\widetilde{t}\right)L}{3n\langle v_z\rangle R} \quad \text{or as} \quad \beta\left(\widetilde{t}\right) = Bu\left(\widetilde{t}\right) \quad \text{where} \quad B = \frac{4v_\theta L}{3n\langle v_z\rangle R}, \tag{9}
$$

where $n = (11/3)^{1/2} - 1$ and is essentially a measure of the cross-sectional stretching per element (as opposed to axial stretching).

The trajectory (path of fluid element) can be obtained by carrying out repeatedly the following steps:

- integration from the beginning to the end of an element,
- turning the co-ordinate system by 90°.

It is clear from this procedure that though the particle path is continuos, its derivatives are not, and this results in infinite stresses.

3. Dynamical system

3.1. Poincaré map

The system of differential equations for a path of fluid elements can be considered as a dynamical system. A solution can be represented by a curve, or trajectory, in a three-dimensional phase space.

A frequently used technique consists in considering the successive intersections of the trajectory with a surface of section $\sum$. The dynamical system defines then a mapping of $\sum$ on itself, known as the Poincaré map [10]. The study of this mapping is often simpler and more illuminating than the study of the trajectories. In most cases, the Poincaré map is not given by explicit equations. It is defined implicitly by a dynamical system and a surface of section. In order to find the image of a point P of $\sum$, one must follow the trajectory from that point until it intersects $\sum$ again [4].

The flow in the case of the partitioned-pipe mixer is periodic in axial distance, so the most convenient choice for the surfaces of section is the cross-sectional planes at the end of each periodic unit. Poincaré maps are then generated by recording every intersection of a trajectory with the surfaces of section in a very long (infinitely long) mixer, and projecting all the intersections onto a plane parallel to the surfaces. Every trajectory intersects with each surface of section, and the Poincaré map should capture some of the mixing in the cross-sectional flow.

3.2. Analysis of a system of differential equations

The point $\mathbf{y}^*$ where $\mathbf{f}(\mathbf{y}^*) = \mathbf{0}$ is called a fixed point of differential equations

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{y}). \tag{10}$$

The fixed point $\mathbf{y}^*$ of the differential equations (10) is called an attractor if there exist a neighbourhood $A \subset R^n$ of $\mathbf{y}^*$ such that $\mathbf{y}(t_0) \in A$ implies

$$\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{y}^*. \tag{11}$$

If a fixed point $\mathbf{y}^* = \mathbf{a}$ has this property for $t \to -\infty$, then $\mathbf{y}^*$ is called a repeller.

The analysis of nonlinear dynamical system usually starts with a linear analysis. Let $\mathbf{y}^*$ be a fixed point of the differential equations (10). In analysing fixed points we linearise the differential equations in the neighbourhood of the fixed point. Let us assume that $\mathbf{f}$ is analytic. Thus we have a Taylor series expansion of $\mathbf{f}$ around $\mathbf{y}^*$. Linearising means that we neglect higher-order terms. In case of equations (10) we can write in the neighbourhood of the fixed point

$$\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{y}^*)(\mathbf{y}-\mathbf{y}^*) \tag{12}$$

and study the linear differential equations $\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{y}^*)(\mathbf{y}-\mathbf{y}^*)$. The $n \times n$ matrix $\frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{y}^*)$ is called the Jacobian matrix or functional matrix. To simplify the notation the fixed point $\mathbf{y}^*$ is shifted to the origin of the phase space by $\overline{\mathbf{y}} = \mathbf{y} - \mathbf{a}$, so we have $\frac{d\overline{\mathbf{y}}}{dt} = \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{y}^*)\,\overline{\mathbf{y}}$. The linearised system in the neighbourhood of the fixed point is of the form (we omit the bar) $\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y}$. We assume that $\det \mathbf{A} \neq 0$, and it means we exclude the case of degenerate fixed point.

In analysing the fixed points of the linear system we first determine the eigenvalues of $\mathbf{A}$. The eigenvalues $\overline{\lambda}_i$, $i = 1, \ldots, n$, are the solution of the characteristic equation $\det(\mathbf{A} - \overline{\lambda}\mathbf{I}) = 0$. Any vector satisfying equation $(\mathbf{A} - \overline{\lambda}\mathbf{I})\mathbf{v} = 0$ is called an eigenvector and Lyapunov exponents are equal to the real parts of the eigenvalues of the fixed point.

A solution of autonomous dynamical system (10) can be represented by a curve in n-dimensional phase space. In the case of a harmonic equation the solution can be represented by the closed orbit in the phase space. It means that solution is periodic. Periodic solution of the autonomous system of differential equations correspond to a closed orbit in the phase space and a closed orbit corresponds to a periodic solution [11]. A periodic solution can be also a limit cycle. If a limit cycle is reached asymptotically for $t \to \infty$ a limit cycle is stable (attractor) and if it is for $t \to -\infty$ it is unstable (repeller). To find a fixed point or a periodic orbit of the system governed by the differential equations we can use numerical approaches, in example shooting method [2].

3.3. Fourier analysis of time series

An important practical problem is the interpretation of signals from a nonlinear system. Some methods of diagnostic of the signal (output) from a dynamical system are presented in [5]. The time evolution (output) of a dynamical system is represented by the time variation $g(t)$ (time series) of its dynamical variables.

One of the most popular method of analysis of time series is Fourier analysis of time series [2, 5, 11]. If the time series $g(t)$ and its derivative $dg(t)/dt$ are continuos and $g(t)$ is periodic with T being basic periodicity then $g(t)$ can be expressed as linear combinations whose frequencies are integer multiples of a basic frequency ω_0:

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \tag{13}$$

where c_n is constant and

$$c_n = \frac{\omega_0}{2\pi} \int_{-\pi/\omega_0}^{\pi/\omega_0} g(t) e^{-in\omega_0 t} dt. \tag{14}$$

The functional series (13) is called a Fourier series.

When $g(t)$is not periodic it can be expressed in terms of oscillations with continuum frequencies. The Fourier transform $c(\omega)$ of the function $g(t)$ can be written as:

$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) e^{i\omega t} dt. \tag{15}$$

The Fourier transform can be complex so it is useful to define a real-valued function:

$$S(\omega) = |c(\omega)|^2, \tag{16}$$

which is called the power spectrum.

The power spectrum is the quantity, which allows us to determine, for example, the main frequencies of the considered system. For a sinusoidal function there is a single peak of the power spectrum at its frequency f_1, what correspond to a periodic solution. For a well-behaved periodic function there are peaks of exponentially diminishing amplitude at f_1 (fundamental frequency), $2f_1$ (first harmonic), $3f_1$ (second harmonic), etc. Chaotic solution has spectra with broad bands rather than isolated peaks but may have prominent peaks. More details about the characteristic of different types of attractors using the spectral analysis is presented in [11].

3.4. The spectrum of Lyapunov exponents

Lyapunov exponents, named after the Russian mathematician, can be used to obtain a measure of the sensitive dependence of the solution of differential equations $\frac{dy}{dt} = \mathbf{f}$, $\mathbf{y} \in D \subset R^n$, to the initial conditions. Lyapunov exponents are a generalisation of the eigenvalues of a fixed point [11].

Considering a continuous dynamical system in an n-dimensional phase space, we monitor the long-term evolution of an infinitesimal n-sphere of initial conditions. The sphere will become an n-ellipsoid due to the locally deforming nature of the flow. The i-th one-dimensional Lyapunov exponent is then defined in terms of the length of the ellipsoidal principal axis $p_i(t)$:

$$\lambda_i = \lim_{t \to \infty} \log_2 \frac{p_i(t)}{p_i(0)}, \tag{17}$$

where the λ_i are ordered from the largest to the smallest. Thus the Lyapunov exponents are related to the expanding or contracting nature of different directions in a phase space. Since the orientation of the ellipsoid changes continuously as it evolves, the directions associated with a given exponent vary in a complicated way through the attractor. One cannot speak of a well-defined direction associated with a given exponent.

The fact that Lyapunov exponents measure the rate of contraction or expansion they can be used as a simple criterion to distinguish between conservative and dissipative systems. For $\sum_{i=1}^{n} \lambda_i = 0$ the volume of a solution in a phase space is conserved and in this case we have a conservative system. In dissipative systems, a phase space is contracted hence $\sum_{i=1}^{n} \lambda_i < 0$. A dynamical system has an attractor only when $\sum_{i=1}^{n} \lambda_i \leq 0$. For $\sum_{i=1}^{n} \lambda_i > 0$ the system is expanding and may never reach any attractor. Attractors with positive Lyapunov exponents are called strange chaotic attractors. The solution of $\frac{d\mathbf{y}}{dt} = \mathbf{f}$ with $\mathbf{y}(t_0) = \mathbf{y}_0$ is called chaotic if at least one one-dimensional Lyapunov exponent is positive.

The Lyapunov spectrum is closely related to the fractional dimension of the associated strange attractor. There are a number [7] of different fractional-dimension-like quantities, including the fractal dimension, information dimension, and the correlation exponent. The difference between them is often small. It has been conjectured that the information dimension d_f is related to the Lyapunov spectrum by the equation

$$d_f = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|}, \tag{18}$$

where j is defined by the condition that $\sum_{i=1}^{j} \lambda_i > 0$ and $\sum_{i=1}^{j+1} \lambda_i < 0$. The conjectured relation between d_f (a static property of an attracting set) and Lyapunov exponents appears to be satisfied for some systems.

In a system with a few positive exponents we can estimate the Kolmogorov entropy using equation $K \leq \sum_{i=1}^{j} \lambda_i$ where λ_i are ordered from largest to smallest and j is the index of the minimum positive Lyapunov exponent. We can classify dynamic systems [4] using the Kolmogorov entropy. If $K = 0$ motion is regular (periodic, quasi-periodic or steady), if $0 < K < \infty$ motion is chaotic and if $K = \infty$ – accidental.

3.5. Determination of Lyapunov exponents from differential equations

A defining feature of chaotic systems is the sensitive dependence on initial conditions. This implies that after a long time the distance between nearby trajectories grows exponentially as $\exp(\lambda t)$, while such divergence is as t^n for regular systems. In a regular system this divergence can also be exponential if the accessible region of phase space is unbounded. If the region is bounded locally, this exponential separation can only occur during a short period of time.

To characterize these phenomena one can define the maximum Lyapunov exponent as

$$\lambda_1^n = \frac{\sum_{i=1}^{n-1} \ln \frac{d_{i+1}}{d_i^I}}{t_n - t_1}, \tag{19}$$

$$\lambda_1 = \lim_{t \to \infty} \lambda_1^n, \tag{20}$$

where d_i^I is the distance between two points on nearby trajectories at time t_i and d_{i+1} is the distance between the same two points at time t_{i+1} obtained in a numerical integration of a differential system of equations.

One must be aware that it is not possible to rigorously take a limit by numerical means, and one may find a trajectory for which λ_1^n tends to zero, as far as the computer numerical precision is concerned, while actually $\lambda_1 > 0$.

To determine a complete spectrum from a set of differential equations we recall that Lyapunov exponents are defined by the long-term evolution of the axes of an infinitesimal sphere of states.

A fiducial trajectory, the centre of the sphere, is defined by the action of the nonlinear equations of motion on some initial condition. Trajectories of points on the surface of the sphere are defined by the action of the linearized equations of motion on points infinitesimally separated from the fiducial trajectory. The principal axes are defined by the evolution via the linearized equations of an initially orthonormal vector frame anchored to the fiducial trajectory.

Authors of paper [6] analysed in detail an algorithm for computing Lyapunov exponents from an experimental time series. The complete algorithm for computing complete spectrum from a set of differential equations was presented by Wolf, Swift, Swinney and Vastano [15]. This algorithm uses the Gram-Schmidt reorthonormalization (GSR) procedure on the vector frame.

The complete set of equations for determining Lyapunov spectrum can be written as: $\frac{dy_j}{dt} = f_j$, $j = 1...n$, where n is the number of equations in the system of equations of motion and $f_j = f_j(t, y_1, ..., y_n)$;

$$\frac{dy_{j*n+i}}{dt} = \sum_{s=1}^{n} \frac{\partial f_j}{\partial y_s} y_{s*n+i}, \qquad j = 1, ..., n, \quad i = 1, ..., n. \tag{21}$$

The initial conditions for this problem we can write as: $y_j = y0_j$ for $j = 1, ..., n$, $y_j = 0$ for $j = n+1, ..., nn$ $(nn = n*n)$ and $y_{(n+1)j} = 1$ for $j = 1, ..., n$.

4. Numerical results and conclusions

Flow in the mixer considered here was investigated in order to find chaos regimes. Poincaré sections and Lyapunov spectrum for the partitioned-pipe mixer for various values of B, length of an element L and period of function, which describe periodic rotation of the pipe T were investigated. The results are shown in figures and table. The particle trajectory for the partitioned-pipe mixer are presented in reference [14].

MATLAB environment was used to solve the partitioned-pipe mixer problem. The automatic step size Runge–Kutta–Fehlberg integration method [8] was used in all numerical integrations in this paper. Automatic step size Runge–Kutta algorithms take larger steps where the solution is changing more slowly and uses the 4th and 5th order pair for higher accuracy.

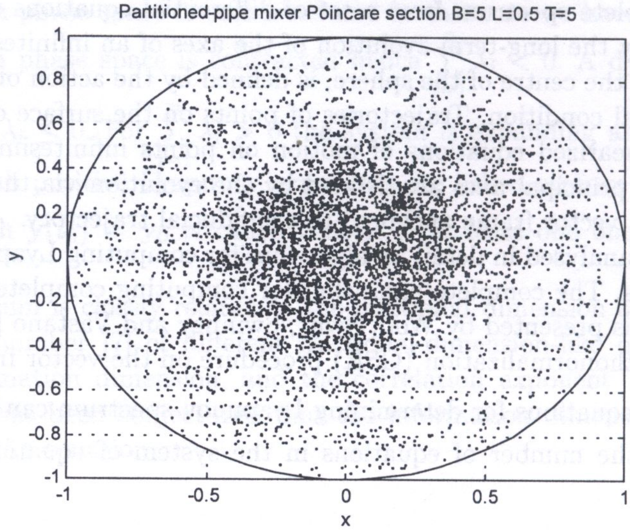
4.1. Poincaré section

Numerical calculations for the presented the Poincaré sections were carried out for total time flow $t_{\max} = 10000$. The initial position of particle was placed in point: $r = 0.5$, $\theta = 0.5$, $z = 0$, periodic rotation of the pipe was described by periodic function.

The figures of the Poincaré section show that for the considered parameters of mixing in the partitioned-pipe mixer the chaotic motion is observed. It means that in this region one has good mixing.

4.2. Lyapunov spectrum

In this part of numerical results Lyapunov spectrum (all Lyapunov exponents – the fourth exponent is equal to zero in all cases of the partitioned-pipe mixer) is presented in Figs. 3. The values of exponents are presented as a function of time. It is clear to see that at the beginning of the mixing process the values of Lyapunov exponents are changing very rapidly and after a long period of time they are stabilized.



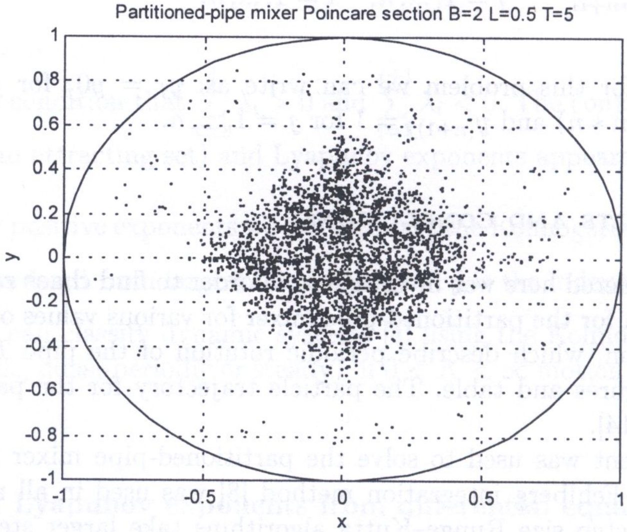


Fig. 2. Poincaré sections for the flow in PPM with the periodic function: $u(t) = \sin^2(2\pi t/T)$; the initial position of the particle was placed: $r = 0.5$, $\theta = 0.5$, $z = 0$ presented for total time flow: $t_{\max} = 10000$

Calculations of the complete Lyapunov spectrum were made from a set of differential equations with algorithm presented in [15]. At least one positive Lyapunov exponent exists in Lyapunov spectrum in three first cases and it means that chaos exists also in the considered examples of PPM. Values of Lyapunov exponents, information dimension and Kolmogorov entropy for different parameters of PPM are presented in Table 1.

Table 1. Values of Lyapunov exponents (λ_i), information dimension (d_f) and Kolmogorov entropy (**K**) of PPM

B	**L**	**T**	$t_{\max}$	λ_1	λ_2	λ_3	d_f	**K**
5	10000	5	1000	0.0187	0.0248	−0.0062	–	0.0435
5	1	5	1000	1.1450	1.8587	−1.0812	4.7781	3.0037
5	1	50	1000	1.0398	1.4028	−1.0361	4.3574	2.4426
2	10	50	1000	0.2373	0.4362	−0.2258	4.9827	0.6735

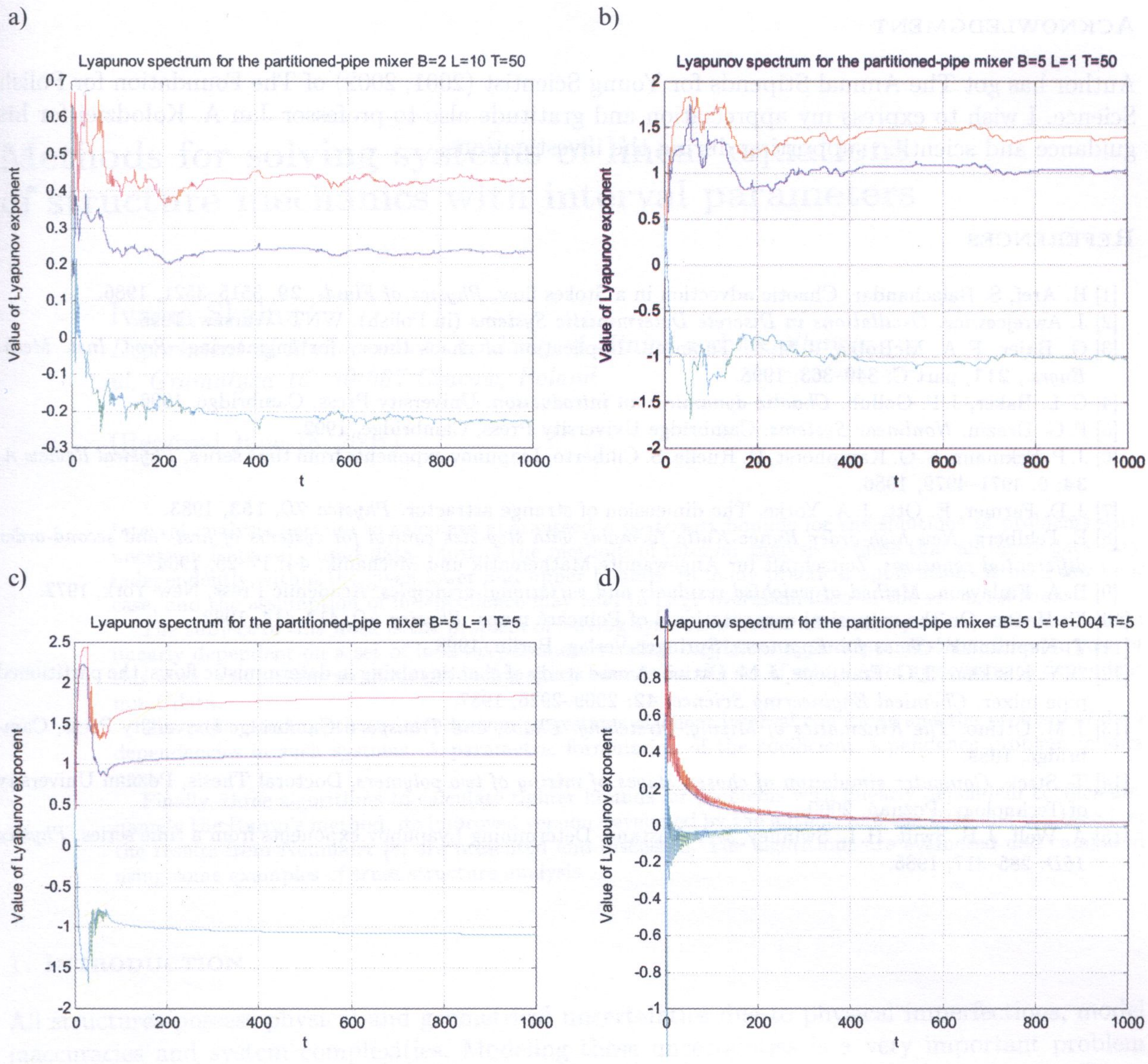


Fig. 3. Lyapunov spectrum for the flow in PPM with the periodic function: $u(t) = \sin^2(2\pi t/T)$; the initial position of the particle was placed: $r = 0.5$, $\theta = 0.5$, $z = 0$ presented for total time flow: $t_{\max} = 1000$

5. Conclusions

Flow in the mixer considered in this paper was investigated in order to find chaos regimes. Poincaré sections, Lyapunov spectrum and Kolmogorov entropy for the partitioned-pipe mixer for various values of parameters of mixer and flow were investigated. The figures of the Poincaré section have shown that for the considered parameters of mixing in the PPM the chaotic motion was observed. It means that in this region one has good mixing. In this paper calculations of the complete Lyapunov spectrum were also made from a set of differential equations. At least one positive Lyapunov exponent exists in Lyapunov spectrum in three first cases presented in paper and it means that chaos exists in the considered examples of PPM. The values of exponents are presented as a function of time. It is clear to see that at the beginning of the mixing process the values of Lyapunov exponents are changing very rapidly and after a long period of time they are stabilized. These examples of the mixing process showed that the value of parameters of partitioned-pipe mixer: B (which is associated with β referred to as the mixing strength), the length of an element L and the period of function (which describe periodic rotation of pipe T) has big influence on the results of the process.

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