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Trefftz methods for plane piezoelectricity

K.Y. Sze[1], W.G. Jin[2], N. Sheng[1], J. Li[2]
[1] *Department of Mechanical Engineering*
The University of Hong Kong, Pokfulam, Hong Kong, P. R. China
[2] *Department of Mechanics & Engineering Science*
Fudan University, Shanghai 200433, P. R. China

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Starting from the governing equations, the general solution and the complete solution set for plane piezoelectricity are derived in this paper. Subsequently, the Trefftz collocation method (TCM) is formulated. TCM falls into the category of Trefftz indirect methods which adopt the truncated complete solution set as the trial functions. Similar to the boundary element method, the solution procedure of TCM requires only boundary discretization. Numerical examples are presented to illustrate the efficacy of the formulation.

Keywords: Trefftz, piezoelectricity, boundary element, collocation.

1. Introduction

Piezoelectric materials have been extensively used in sensors, actuators, resonators and intelligent structures. Owing to the material anisotropy and the electromechanical coupling, analytical methods are limited to relatively simple problems [1–4]. In real engineering analysis, numerical methods are often resorted to. In recent years, the boundary element method has been applied to piezoelectric problems [5–10]. Most, if not all, of the existing boundary element formulations for piezoelectricity used the fundamental solutions, i.e. singular Green's functions, as the weight functions. This leads to the difficulty in evaluating the singular boundary integrals. To this end, Trefftz-type boundary element methods, which adopt non-singular Trefftz functions as the weight functions and/or trial functions, are advantageous alternatives. Trefftz methods as boundary solution techniques have long been recognized and are applicable to a wide variety of engineering problems [11–18]. According to the different choice of trial functions, Trefftz methods can be classified into indirect and direct methods. In Trefftz indirect methods, the trial functions are taken from the complete solution set. Depending on the choice of the weight functions, they may lead to the Trefftz collocation method (TCM) and the Trefftz Galerkin method (TGM). In Trefftz direct methods [13–16], the weight functions are taken from the complete solution set whereas the trial functions can be conventional interpolation functions – an example being the piecewise Lagrange polynomials.

In this paper, the general solution and the complete solution set for plane piezoelectricity are derived. With the trial functions chosen from the complete solution set, the Trefftz collocation method is formulated by enforcing the boundary conditions at discrete boundary points. Numerical examples are presented to show the efficacy of the formulation.

2. General solutions

In this section, the governing equations for piezoelectrics are first summarized. The procedure leading to the general solution is then outlined.

2.1. Balance equations

$$\begin{aligned} \sigma_{ij,j} + f_i &= 0, \\ D_{i,i} - q &= 0, \end{aligned} \tag{1}$$

where σ_{ij} and D_i are, respectively, the stress tensor and the electric displacement vector; f_i and q are respectively the body force and the free charge densities. Subscript commas denote the partial differentiation with respect to the coordinates.

2.2. Constitutive relations

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}\epsilon_{kl} - e_{kij}E_k, \\ D_i &= e_{ikl}\epsilon_{kl} + \epsilon_{ik}E_k, \end{aligned} \tag{2}$$

where ϵ_{kl}, E_k, C_{ijkl}, e_{ikl} and ϵ_{ik} are the strain tensor, the electric field, the elasticity tensor measured under constant electric field, the piezoelectric tensor and dielectric tensor measured under constant strain, respectively.

2.3. Strain-displacement and electric field-electric potential relations

$$\begin{aligned} \epsilon_{ij} &= \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \\ E_i &= -\phi_{,i}, \end{aligned} \tag{3}$$

where u_i and ϕ are the displacement and the electric potential, respectively.

2.4. Boundary conditions

$$\begin{aligned} \sigma_{ij}n_j &= \bar{t}_i \quad \text{on} \quad \Gamma_t, \\ u_i &= \bar{u}_i \quad \text{on} \quad \Gamma_u, \\ D_i n_i &= -\bar{\omega} \quad \text{on} \quad \Gamma_\omega, \\ \phi &= \bar{\phi} \quad \text{on} \quad \Gamma_\phi, \end{aligned} \tag{4}$$

where t_i is the traction, ω is the surface charge and n_i is the unit outward normal vector. The barred quantities indicate that their values are prescribed. It is assumed as usual that $\Gamma_t \cup \Gamma_u = \Gamma_\omega \cup \Gamma_\phi = \Gamma$ in which Γ denotes the entire boundary of the problem domain.

As most of the commonly used piezoelectrics are transverse isotropic, only this class of material symmetry will be considered. With direction 3 taken to be the poling direction and under the electromechanical plane strain conditions, the constitutive relations in (2) reduce to:

$$\begin{aligned} \sigma_x &= c_{11}\frac{\partial u}{\partial x} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \phi}{\partial z}, \\ \sigma_z &= c_{13}\frac{\partial u}{\partial x} + c_{33}\frac{\partial w}{\partial z} + e_{33}\frac{\partial \phi}{\partial z}, \\ \tau_{xz} &= c_{44}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + e_{15}\frac{\partial \phi}{\partial x}, \\ D_x &= e_{15}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - \epsilon_{11}\frac{\partial \phi}{\partial x}, \\ D_z &= e_{31}\frac{\partial u}{\partial x} + e_{33}\frac{\partial w}{\partial z} - \epsilon_{33}\frac{\partial \phi}{\partial z}, \end{aligned} \tag{5}$$

in which $x = x_1$, $z = x_3$, $u = u_1$ and $w = u_3$. Substituting (5) into (1) and neglecting the body force and free charge densities, the governing equations can be expressed as:

$$\mathcal{A}\mathbf{u} = \mathbf{0}, \tag{6}$$

where

$$\mathbf{u} = \left\{ \begin{array}{c} u \\ w \\ \phi \end{array} \right\}, \quad \mathcal{A} = \left[\begin{array}{ccc} c_{11}\frac{\partial^2}{\partial x^2} + c_{44}\frac{\partial^2}{\partial z^2} & (c_{13}+c_{44})\frac{\partial^2}{\partial x \partial z} & (e_{15}+e_{31})\frac{\partial^2}{\partial x \partial z} \\ (c_{13}+c_{44})\frac{\partial^2}{\partial x \partial z} & c_{44}\frac{\partial^2}{\partial x^2} + c_{33}\frac{\partial^2}{\partial z^2} & e_{15}\frac{\partial^2}{\partial x^2} + e_{33}\frac{\partial^2}{\partial z^2} \\ (e_{15}+e_{31})\frac{\partial^2}{\partial x \partial z} & e_{15}\frac{\partial^2}{\partial x^2} + e_{33}\frac{\partial^2}{\partial z^2} & -\left(\epsilon_{11}\frac{\partial^2}{\partial x^2} + \epsilon_{33}\frac{\partial^2}{\partial z^2}\right) \end{array} \right].$$

Introducing a displacement function Ψ and solving (6) yield [19]:

$$\begin{aligned} u &= \left(B_1^{31}\frac{\partial^4}{\partial x^3 \partial z} + B_2^{31}\frac{\partial^4}{\partial x \partial z^3} \right)\Psi, \\ w &= \left(B_1^{32}\frac{\partial^4}{\partial x^4} + B_2^{32}\frac{\partial^4}{\partial x^2 \partial z^2} + B_3^{32}\frac{\partial^4}{\partial z^4} \right)\Psi, \\ \phi &= \left(B_1^{33}\frac{\partial^4}{\partial x^4} + B_2^{33}\frac{\partial^4}{\partial x^2 \partial z^2} + B_3^{33}\frac{\partial^4}{\partial z^4} \right)\Psi, \end{aligned} \tag{7}$$

where

$$\begin{aligned} B_1^{31} &= c_{13}e_{15} - e_{31}c_{44}, \quad B_2^{31} = (c_{13}+c_{44})\, e_{33} - (e_{31}+e_{15})\, c_{33}, \\ B_1^{32} &= -c_{11}e_{15}, \quad B_2^{32} = -\left[c_{11}e_{33} - c_{13}(e_{31}+e_{15}) - c_{44}e_{31}\right], \quad B_3^{32} = -c_{44}e_{33}, \\ B_1^{33} &= c_{11}c_{44}, \quad B_2^{33} = c_{11}c_{33} - c_{13}(c_{13}+2c_{44}), \quad B_3^{33} = c_{33}c_{44}. \end{aligned}$$

Substituting (7) into (5) yields

$$\begin{aligned} \sigma_x &= \left(m_{11}\frac{\partial^5}{\partial x^2 \partial z^3} + m_{12}\frac{\partial^5}{\partial z^5} \right)\Psi, \\ \sigma_z &= \left(m_{21}\frac{\partial^5}{\partial x^4 \partial z} + m_{22}\frac{\partial^5}{\partial x^2 \partial z^3} \right)\Psi, \\ \tau_{xz} &= \left(m_{31}\frac{\partial^5}{\partial x^3 \partial z^2} + m_{32}\frac{\partial^5}{\partial x \partial z^4} \right)\Psi, \\ D_x &= \left(m_{41}\frac{\partial^5}{\partial x^5} + m_{42}\frac{\partial^5}{\partial x^3 \partial z^2} + m_{43}\frac{\partial^5}{\partial x \partial z^4} \right)\Psi, \\ D_z &= \left(m_{51}\frac{\partial^5}{\partial x^4 \partial z} + m_{52}\frac{\partial^5}{\partial x^2 \partial z^3} + m_{53}\frac{\partial^5}{\partial z^5} \right)\Psi, \end{aligned} \tag{8}$$

where

$$\begin{aligned} m_{11} &= c_{11}(e_{33}c_{44} - c_{33}e_{15}) + c_{13}(c_{13}e_{15} - c_{44}e_{31}), \quad m_{12} = c_{44}(c_{33}e_{31} - c_{13}e_{33}), \\ m_{21} &= c_{11}(e_{33}c_{44} - c_{33}e_{15}) + c_{13}(c_{13}e_{15} - c_{44}e_{31}), \quad m_{22} = c_{44}(c_{33}e_{31} - c_{13}e_{33}), \\ m_{31} &= -c_{11}(e_{33}c_{44} - c_{33}e_{15}) - c_{13}(c_{13}e_{15} - c_{44}e_{31}), \quad m_{32} = -c_{44}(c_{33}e_{31} - c_{13}e_{33}), \\ m_{41} &= -c_{11}(e_{15}e_{15} + \epsilon_{11}c_{44}), \\ m_{42} &= e_{15}(2e_{15}c_{13} - c_{11}e_{33} + c_{13}e_{31}) - \epsilon_{11}(c_{11}c_{33} - 2c_{13}c_{44} - c_{13}c_{13}), \\ m_{43} &= e_{15}(e_{33}c_{13} - c_{33}e_{31} - e_{15}c_{33}) - \epsilon_{11}c_{44}c_{33}, \\ m_{51} &= e_{31}(e_{15}c_{13} - e_{31}c_{44}) - c_{11}(e_{33}e_{15} + \epsilon_{33}c_{44}), \\ m_{52} &= 2e_{31}e_{33}(c_{13}+c_{44}) - e_{31}c_{33}(e_{31}+e_{15}) - e_{33}(e_{33}c_{11} - e_{15}c_{13}) \\ &\quad + \epsilon_{33}(c_{13}c_{13} + 2c_{13}c_{44} - c_{11}c_{33}), \\ m_{53} &= c_{44}(-e_{33}e_{33} - \epsilon_{33}c_{33}). \end{aligned}$$

The displacement function Ψ satisfies the following sixth-order linear partial differential equation:

$$\left(a\frac{\partial^6}{\partial z^6}+b\frac{\partial^6}{\partial z^4\partial x^2}+c\frac{\partial^6}{\partial z^2\partial x^4}+d\frac{\partial^6}{\partial x^6}\right)\Psi=0, \tag{9}$$

where

$$\begin{aligned}
a &= -c_{44}\left(e_{33}e_{33}+c_{33}\epsilon_{33}\right),\\
b &= f\epsilon_{33}-c_{33}c_{44}\epsilon_{11}+ge_{33}-\left(e_{15}+e_{31}\right)^2c_{33},\\
c &= f\epsilon_{11}-c_{11}c_{44}\epsilon_{33}+\left(g-c_{11}e_{33}\right)e_{15}-c_{44}\left(e_{31}^2+2e_{15}e_{31}\right),\\
d &= -\left(c_{11}c_{44}\epsilon_{11}+c_{11}e_{15}e_{15}\right).
\end{aligned}$$

In b and c, $f=c_{13}\left(c_{13}+2c_{44}\right)-c_{11}c_{33}$ and $g=2c_{13}\left(e_{31}+e_{15}\right)+2c_{44}e_{31}-c_{11}e_{33}$. Factorization of (9) gives:

$$\prod_{k=1}^{3}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z_k^2}\right)\Psi=0, \tag{10}$$

where $z_k=\lambda_k z$ $(k=1,2,3)$ and λ_k^2 are the three roots of the cubic equation

$$a\lambda^3-b\lambda^2+c\lambda-d=0. \tag{11}$$

By applying the generalized Almansi's theorem [20], the solution of (10) can be expressed as:

$$\Psi=\Psi_1+\Psi_2+\Psi_3 \quad \text{for} \quad \lambda_1^2\neq\lambda_2^2\neq\lambda_3^2, \tag{12}$$

where Ψ_k $(k=1,2,3)$ satisfy

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z_k^2}\right)\Psi_k=0. \tag{13}$$

By introducing the new functions

$$\psi_k=\frac{\partial^3\Psi_k}{\partial z_k^3}, \tag{14}$$

the order of the derivatives in (7) can be reduced and the general solution for the plane strain piezoelectric problem can be simplified to

$$\begin{Bmatrix}u\\ w\\ \phi\end{Bmatrix}=\sum_{k=1}^{3}\begin{bmatrix}\left(-\lambda_kB_1^{31}+\lambda_k^3B_2^{31}\right)\partial/\partial x\\ \left(B_1^{32}-\lambda_k^2B_2^{32}+\lambda_k^4B_3^{32}\right)\partial/\partial z_k\\ \left(B_1^{33}-\lambda_k^2B_2^{33}+\lambda_k^4B_3^{33}\right)\partial/\partial z_k\end{bmatrix}\psi_k, \tag{15}$$

which leads to

$$\begin{aligned}
\sigma_x &= \sum_{k=1}^{3}\left(-\lambda_k^3m_{11}+\lambda_k^5m_{12}\right)\frac{\partial^2\psi_k}{\partial z_k^2},\\
\sigma_z &= \sum_{k=1}^{3}\left(\lambda_k^3m_{21}-\lambda_k^3m_{22}\right)\frac{\partial^2\psi_k}{\partial z_k^2},\\
\tau_{xz} &= \sum_{k=1}^{3}\left(-\lambda_k^2m_{31}+\lambda_k^4m_{32}\right)\frac{\partial^2\psi_k}{\partial x\partial z_k},\\
D_x &= \sum_{k=1}^{3}\left(m_{41}-\lambda_k^2m_{42}+\lambda_k^4m_{43}\right)\frac{\partial^2\psi_k}{\partial x\partial z_k},\\
D_z &= \sum_{k=1}^{3}\left(\lambda_km_{51}-\lambda_k^3m_{52}+\lambda_k^5m_{53}\right)\frac{\partial^2\psi_k}{\partial z_k^2}.
\end{aligned}$$

To recapitulate, the plain strain piezoelectric problem has been reduced to one of finding the complex potentials ψ_k $(k = 1, 2, 3)$ which satisfy the following Laplace equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_k^2}\right)\psi_k = 0. \tag{16}$$

3. Complete solution sets

It is well known that the complete solution sets of the standard Laplacian equation for interior and external domain problems are respectively [21]:

$$\begin{aligned} \mathbf{B}_1 &= \{1,\ \mathrm{Re}(Z^n),\ \mathrm{Im}(Z^n);\quad n = 1, 2, 3, \ldots\}, \\ \mathbf{B}_2 &= \left\{1,\ \ln\sqrt{x^2 + z^2},\ \mathrm{Re}(Z^{-n}),\ \mathrm{Im}(Z^{-n});\quad n = 1, 2, 3, \ldots\right\}, \end{aligned} \tag{17}$$

in which $Z = (x + \mathrm{i}z)$; Re and Im denote the real and the imaginary parts of a given complex expression, respectively. Thus, the complete Laplacian solution set of (16) for interior domain problems is:

$$\mathbf{B}_{1\psi_k} = \{1,\ \mathrm{Re}(Z_k^n),\ \mathrm{Im}(Z_k^n);\quad n = 1, 2, 3, \ldots\}, \tag{18}$$

which leads to

$$\psi_k = A_0 + \sum_{i=1}^{n} \left[\alpha_{ik}\mathrm{Re}(Z_k^i) + \beta_{ik}\mathrm{Im}(Z_k^i)\right]. \tag{19}$$

Similarly, the complete Laplacian solution set of (16) for exterior domain problems is:

$$\mathbf{B}_{2\psi_k} = \left\{1,\ \ln\sqrt{x^2 + z_k^2},\ \mathrm{Re}(Z_k^{-n}),\ \mathrm{Im}(Z_k^{-n});\quad n = 1, 2, 3, \ldots\right\}, \tag{20}$$

which leads to

$$\psi_k = A_0 + \alpha_{0k}\ln\sqrt{x^2 + z_k^2} + \sum_{i=1}^{n} \left[\alpha_{ik}\mathrm{Re}(Z_k^i) + \beta_{ik}\mathrm{Im}(Z_k^i)\right]. \tag{21}$$

In (19) and (21), $Z_k = (x + \mathrm{i}z_k)$ whereas A_0, α_{0k}, α_{ik} and β_{ik} are generalized coefficients.

By substituting (19) and (21) into (15), the complete solution set of the plane strain piezoelectricity can be obtained for interior and exterior domain problems, respectively.

4. Trefftz indirect formulations

Generalizing the Trefftz indirect method to the plane piezoelectric problems, the trial solution is taken to be

$$\tilde{\mathbf{u}} = \left\{\tilde{u}\quad \tilde{w}\quad \tilde{\phi}\right\}^{\mathrm{T}} = \sum_{i=1}^{n_a} \mathbf{N}_i a_i = \mathbf{N}\mathbf{a}, \tag{22}$$

where $\mathbf{N}_i$'s are the elements from the complete solution set, $\mathbf{N} = [\mathbf{N}_1, \ldots, \mathbf{N}_{n_a}]$ and $\mathbf{a} = \{a_1, \ldots, a_{n_a}\}^{\mathrm{T}}$ is the coefficient vector to be determined. Let $(\tilde{\sigma}_x,\ \tilde{\tau}_{xz},\ \tilde{\sigma}_z)$ be the stress components and $\left(\tilde{D}_x,\ \tilde{D}_z\right)$ be the electric displacement components derived from $\tilde{\mathbf{u}}$ by virtue of (5), then

$\widetilde{t}_x = n_x\widetilde{\sigma}_x + n_z\widetilde{\tau}_{xz}$, $\widetilde{t}_z = n_x\widetilde{\tau}_{xz} + n_z\widetilde{\sigma}_z$, and $\widetilde{\omega} = -n_x\widetilde{D}_x - n_z\widetilde{D}_z$ can be grouped and expressed symbolically as:

$$\widetilde{\mathbf{p}} = \begin{bmatrix}\widetilde{t}_x & \widetilde{t}_z & \widetilde{\omega}\end{bmatrix}^{\mathrm{T}} = \mathbf{Ma}. \tag{23}$$

Since the trial functions satisfy the balance equations in (1) exactly, only the boundary conditions (4) need to be enforced. In the Trefftz indirect method, the boundary conditions can be enforced by minimizing the following residuals:

$$\mathbf{R}_1 = \widetilde{\mathbf{u}} - \overline{\mathbf{u}} \quad \text{on} \quad \Gamma_{\mathbf{u}}, \qquad \mathbf{R}_2 = \widetilde{\mathbf{p}} - \overline{\mathbf{p}} \quad \text{on} \quad \Gamma_{\mathbf{p}}, \tag{24}$$

in which $\overline{\mathbf{u}} = \begin{bmatrix}\overline{u} & \overline{w} & \overline{\phi}\end{bmatrix}^{\mathrm{T}}$ and $\overline{\mathbf{p}} = \begin{bmatrix}\overline{t}_x & \overline{t}_z & \overline{\omega}\end{bmatrix}^{\mathrm{T}}$ denote the vectors of the prescribed values on the boundaries $\Gamma_{\mathbf{u}}$ and $\Gamma_{\mathbf{p}}$, respectively. The two residuals can be minimized by means of the collocation method or the Galerkin method which leads to the Trefftz collocation method (TCM) or the Trefftz Galerkin method (TGM), respectively [11, 12]. In TCM, the residuals are forced to be zero at selected points $\mathbf{X}_i$'s along the domain boundary Γ, i.e.

$$\begin{aligned} &\widetilde{u}(\mathbf{X}_i) = \overline{u}(\mathbf{X}_i) \quad \text{or} \quad \widetilde{t}_x(\mathbf{X}_i) = \overline{t}_x(\mathbf{X}_i), \\ &\widetilde{w}(\mathbf{X}_i) = \overline{w}(\mathbf{X}_i) \quad \text{or} \quad \widetilde{t}_z(\mathbf{X}_i) = \overline{t}_z(\mathbf{X}_i), \\ &\widetilde{\phi}(\mathbf{X}_i) = \overline{\phi}(\mathbf{X}_i) \quad \text{or} \quad \widetilde{\omega}(\mathbf{X}_i) = \overline{\omega}(\mathbf{X}_i) \quad \text{for} \quad i = 1, \ldots, n_c, \end{aligned} \tag{25}$$

where n_c is the total number of the collocation points. Substituting (22) into (25) yields the matrix equation

$$\mathbf{K}^c\mathbf{a} = \mathbf{f}^c, \tag{26}$$

where $\dim(\mathbf{K}^c) = 3n_c \times \dim\{\mathbf{a}\}$. Generally, the total number of equations ($3n_c$) to be solved exceeds the number of unknown coefficients ($\dim\{\mathbf{a}\}$). Under this circumstance, an approximate solution of the over-constrained equation system can be obtained by the least square method which pre-multiplies both sides of (26) with the transpose of $\mathbf{K}^c$. The implementation of the TCM is simple since the collocation at the boundary points is directly used to build the discrete set of equations governing the problems. However, a large number of collocation points may be required for attaining a reasonable accuracy in practical applications. Moreover, the equation system is over-constrained and matrix $\mathbf{K}^c$ is not symmetric. These shortcomings can be alleviated or overcome by TGM [11, 12, 15] which, however, will not be discussed in this paper due to length limitations.

5. Numerical examples

The numerical tests are conducted in this section to validate the developed TCM. The commercially available PZT-4 ceramic is considered. Its non-zero constitutive coefficients include $c_{11} = 139$, $c_{13} = 74.3$, $c_{33} = 113$, $c_{44} = 25.6$ [GPa]; $e_{15} = 13.44$, $e_{31} = -6.98$, $e_{33} = 13.84$ [Cm^{-2}]; and $\epsilon_{11} = 6.0$, $\epsilon_{33} = 5.47$ [nC(Vm)$^{-1}$] [22].

5.1. Bending of a piezoelectric strip

This problem considers an 1.0×1.0 [mm] clamped piezoelectric strip with electrodes at $z = \pm h/2$, see Fig. 1. The strip is subjected to a linearly varying stress at the right edge and its lower electrode is earthed. The problem is solved by taking the first seven terms in the complete Laplacian solution set in (18). Moreover, three collocation points are located uniformly on each side of the strip. It can be seen in Table 1 that the computed displacements and electrical potential are identical to exact solutions. The predictions are not sensible to the offset of the collocation points from the corners.

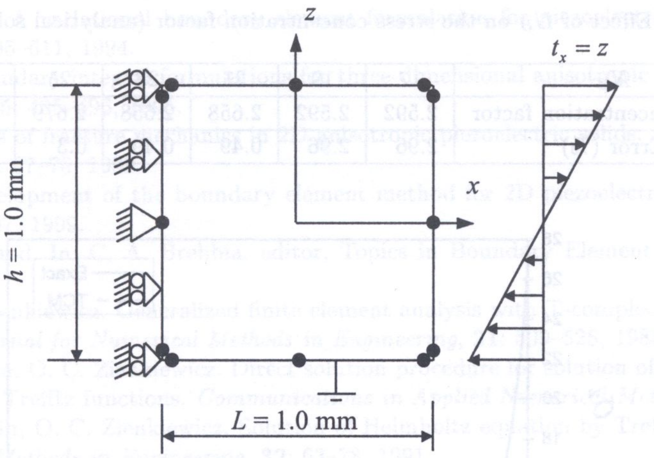

Fig. 1. Bending of a piezoelectric strip (• denotes collocation point)

Table 1. TCM results for the bending of a piezoelectric strip

Location (x, z) [m,m]	$u \times 10^{17}$ [m] Exact		$w \times 10^{17}$ [m] Exact		$\phi \times 10^8$ [V] Exact	
	Exact	TCM	Exact	TCM	Exact	TCM
(0, 0.0005)	0.1980	0.1980	–0.1369	–0.1369	0.0	0.0
(0.0005, 0)	0.0	0.0	–0.3961	–0.3961	0.2222	0.2222
(0.00025, 0.0005)	0.2971	0.2971	–0.2607	–0.2607	0.0	0.0
(0.0005, 0.00025)	0.1980	0.1980	–0.4056	–0.4056	0.1667	0.1667

5.2. Infinite piezoelectric plane with a circular hole

An infinite piezoelectric panel with a circular hole of unit radius is subjected to a far field uniform traction in the z direction σ_z^∞ which is taken to be 10 as shown in Fig. 2. Sixty collocation points distributed uniformly along the periphery of the hole are employed. Different number of the terms N_B from the complete Laplacian solution set in (20) is employed and the computed stress concentration factors are listed in Table 2. Comparing with the exact solution [23], the errors of the TCM decrease monotonically from 2.96% to 0.26% when N_B is increased from 17 to 29. Figure 3 compares the exact distribution and the computed distribution ($N_B = 29$) of σ_z along the x axis. The two distributions are practically indistinguishable.

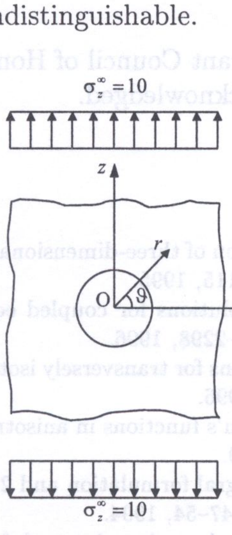

Fig. 2. Infinite piezoelectric medium with a circular hole

Table 2. Effect of N_B on the stress concentration factor (analytical solution is 2.671)

N_B	17	19	21	23	25	27	29
Stress concentration factor	2.592	2.592	2.658	2.658	2.679	2.679	2.678
Error (%)	2.96	2.96	0.49	0.49	0.3	0.3	0.26

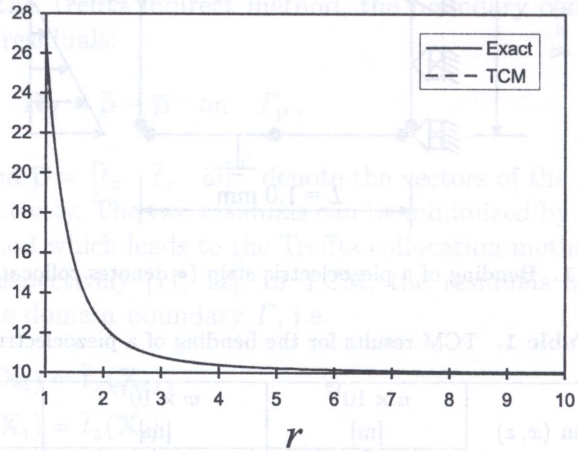

Fig. 3. σ_z versus r on the line $\theta = 0$ ($N_B = 29$)

6. Conclusion

The Trefftz method is developed for the plane piezoelectricity in this paper. The general solution and the complete solution set are first derived. By utilizing truncated complete solution set as the trial solution, the Trefftz collocation method is formulated. As compared to the conventional boundary element method which adopts the fundamental solutions, the Trefftz method does not involve any singular integrals. The numerical examples given in this paper validate the efficacy of the formulation in solving plane piezoelectric problems. Particularly, the present method serves as a good candidate for dealing with the stress concentration problems in piezoelectricity.

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