

Solution representations for Trefftz-type finite elements

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Solution representations are available for several differential equations. For elasticity problems some of the solution representations are considered in this paper. The solution representations can be used for a systematic construction of Trefftz functions for the derivation of Trefftz-type finite elements. For the example of a thick plate a set of Trefftz functions is presented.

Keywords: Trefftz functions, Trefftz-type finite elements

1. INTRODUCTION

The Trefftz method had been introduced by Erich Trefftz in 1926 [1]. Numerical applications of the method appeared in the seventies e.g. by Stein [2], Ruoff [3], and Zienkiewicz et al. [4]. Numerous contributions to different aspects of the Trefftz method have been made. Among those contributions are the papers of Almeida/Pereira [5], Dumont [6], Freitas/Ji [7], Freitas/Leitão [8], Herrera [9], Jirousek et al. [10–14], Kita/Kamiya [15], Kompis/Konkol/Vasko [16], Leitão [17, 18], Maunder/Almeida [19, 20], Moorthy/Ghosh [21], Petrolito [22, 23], Piltner [24–31], Qin [32], Reutsky/Tirozzi [33], Stein [34], Szabo/Babuska [35], Zhang/Katsube [36], Zielinski/Zienkiewicz [37], Zielinski [38], Zienkiewicz/Taylor [39].

2. SELECTED ELASTICITY SOLUTION REPRESENTATIONS

For two and three-dimensional elasticity problems solution representations useful for the Trefftz method are listed below.

2.1. Plane strain and plane stress

The displacement components for plane strain/stress can be written in terms of two arbitrary functions $\Phi(z)$ and $\Psi(z)$ in the following form:

$$2\mu u = \operatorname{Re} \left[\kappa \Phi(z) - z \overline{\Phi'(z)} - \overline{\Psi(z)} \right], \quad (1)$$

$$2\mu v = \operatorname{Im} \left[\kappa \Phi(z) - z \overline{\Phi'(z)} - \overline{\Psi(z)} \right], \quad (2)$$

where

$$\begin{aligned} z &= x + iy, \\ 2\mu &= E/(1 + \nu), \\ \kappa &= \begin{cases} (3 - \nu)/(1 + \nu) & \text{for plane stress,} \\ (3 - 4\nu) & \text{for plane strain.} \end{cases} \end{aligned} \quad (3)$$

The Muskhelishvili-Kolosov complex solution representation (1)–(2) [40] can be used to construct a set of linearly independent trial functions for the displacement components, the strains and stresses. For the complex functions we can assume, for example, complex power series:

$$\Phi = \sum_j a_j z^j, \quad (4)$$

$$\Psi = \sum_j b_j z^j. \quad (5)$$

2.2. Plane strain and plane stress in anisotropic bodies

For anisotropic bodies the displacement field can be expressed as

$$u = 2\text{Re} [p_1\Phi_1(z_1) + p_2\Phi_2(z_2)], \quad (6)$$

$$v = 2\text{Re} [q_1\Phi_1(z_1) + q_2\Phi_2(z_2)], \quad (7)$$

where the two complex variables z_1 and z_2 are defined as

$$z_1 = x + \mu_1 y = (x + \alpha y) + i\beta y, \quad (8)$$

$$z_2 = x + \mu_2 y = (x + \gamma y) + i\delta y. \quad (9)$$

The values p_1 , p_2 , q_1 and q_2 depend on the elastic compliance coefficients. The complex values μ_1 and μ_2 are obtained from a characteristic equation with the coefficients depending on the elastic compliance coefficients.

2.3. Bending of isotropic thin plates (Kirchhoff-plate theory)

Using the Kirchhoff theory, the deflection of a plate can be written as

$$w(x, y) = \text{Re} \left[\bar{z}\Phi(z) + \overline{\Psi(z)} \right]. \quad (10)$$

2.4. Bending of anisotropic thin plates

For anisotropic thin plates the complex representation of the plate deflection can be written as

$$w(x, y) = 2\text{Re} [\Phi_1(z_1) + \Phi_2(z_2)], \quad (11)$$

where $z_1 = x + \mu_1 y$ and $z_2 = x + \mu_2 y$, and μ_1 and μ_2 depend on the material properties of the plate.

2.5. Bending and stretching of isotropic thick plates

2.5.1. Stretching solution involving powers of z

In terms of the complex functions $\Phi(\zeta)$ and $\chi(\zeta)$ the displacement components for the stretching case of a thick plate can be given as

$$2\mu u = \operatorname{Re} \left[\frac{3-\nu}{1+\nu} \Phi - \zeta \bar{\Phi}' - \bar{\chi}' \right] - 2 \frac{\nu}{1+\nu} \left[\frac{h^2}{12} - z^2 \right] \operatorname{Re} [\Phi''], \quad (12)$$

$$2\mu v = \operatorname{Im} \left[\frac{3-\nu}{1+\nu} \Phi - \zeta \bar{\Phi}' - \bar{\chi}' \right] + 2 \frac{\nu}{1+\nu} \left[\frac{h^2}{12} - z^2 \right] \operatorname{Im} [\Phi''], \quad (13)$$

$$2\mu w = -4z \frac{\nu}{1+\nu} \operatorname{Re} [\Phi'], \quad (14)$$

where $\zeta = x + iy$.

2.5.2. Bending solution involving powers of z

The bending solution contribution to the three-dimensional displacement field involves warping and the change of thickness of the plate. The bending solution contribution with powers of the thickness coordinate z is given as

$$2\mu u = -z \operatorname{Re} [\Phi + \zeta \bar{\Phi}' + \bar{\chi}'] - \frac{1}{1-\nu} \left[h^2 z - 2(2-\nu) \frac{z^3}{3} \right] \operatorname{Re} [\Phi''], \quad (15)$$

$$2\mu v = -z \operatorname{Im} [\Phi + \zeta \bar{\Phi}' + \bar{\chi}'] + \frac{1}{1-\nu} \left[h^2 z - 2(2-\nu) \frac{z^3}{3} \right] \operatorname{Im} [\Phi''], \quad (16)$$

$$2\mu w = \operatorname{Re} [\zeta \Phi + \chi] + \frac{2\nu}{1-\nu} z^2 \operatorname{Re} [\Phi']. \quad (17)$$

Remark 1. If we want to look at the complete bending solution split into a part depending on powers of the thickness coordinate z , on trigonometric functions of z and on hyperbolic functions of z , we can write the displacement components in terms of derivatives of the functions G , g_n and \hat{G} :

- Contributions involving powers of z :

$$2\mu u = -z \frac{\partial}{\partial x} G - \frac{1}{4(1-\nu)} \left[h^2 z - 2(2-\nu) \frac{z^3}{3} \right] \frac{\partial}{\partial x} \Delta G,$$

$$2\mu v = -z \frac{\partial}{\partial y} G - \frac{1}{4(1-\nu)} \left[h^2 z - 2(2-\nu) \frac{z^3}{3} \right] \frac{\partial}{\partial y} \Delta G,$$

$$2\mu w = G + \frac{\nu}{2(1-\nu)} z^2 \Delta G.$$

- Contributions involving trigonometric functions in z :

$$2\mu u = \frac{\partial g_n}{\partial y} \sin \omega_n z,$$

$$2\mu v = -\frac{\partial g_n}{\partial x} \sin \omega_n z,$$

$$2\mu w = 0.$$

- Contributions involving hyperbolic functions in z :

$$2\mu u = -\frac{\partial \widehat{G}}{\partial x} [a\hat{z} \cosh q\hat{z} + b\hat{z} \sinh q\hat{z} + c \cosh q\hat{z} + d \sinh q\hat{z}],$$

$$2\mu v = -\frac{\partial \widehat{G}}{\partial y} [a\hat{z} \cosh q\hat{z} + b\hat{z} \sinh q\hat{z} + c \cosh q\hat{z} + d \sinh q\hat{z}],$$

$$2\mu w = \widehat{G} \{ a\{(3 - 4\nu) \cosh q\hat{z} - q\hat{z} \sinh q\hat{z}\} + b\{(3 - 4\nu) \sinh q\hat{z} - q\hat{z} \cosh q\hat{z}\} - cq \sinh q\hat{z} - dq \cosh q\hat{z} \},$$

where $\hat{z} = z + h/2$.

The function G has to satisfy $\Delta\Delta G = 0$ whereas g_n and \widehat{G} have to satisfy $\Delta g_n - \omega_n^2 g = 0$ and $\Delta \widehat{G} + q^2 \widehat{G} = 0$.

2.6. A general 3-dimensional elasticity solution representation

Defining complex variables ζ_1, ζ_2 , and ζ_3 with the space coordinates x, y, z it becomes possible to derive a three-dimensional elasticity solution representation in terms of the six arbitrary functions $\Phi_1(\zeta_1), \Phi_2(\zeta_2), \Phi_3(\zeta_3), \chi_1(\zeta_1), \chi_2(\zeta_2)$, and $\chi_3(\zeta_3)$. The three complex variables

$$\begin{aligned} \zeta_1 &= ix + b_1(t)y + c_1(t)z, \\ \zeta_2 &= a_2(t)x + iy + c_2(t)z, \\ \zeta_3 &= a_3(t)x + b_3(t)y + iz, \end{aligned} \tag{18}$$

contain parameter functions $b_1(t), c_1(t), a_2(t), c_2(t), a_3(t), b_3(t)$, which have to satisfy the following characteristic equations:

$$\begin{aligned} i^2 + b_1^2 + c_1^2 &= 0, \\ i^2 + a_2^2 + c_2^2 &= 0, \\ i^2 + a_3^2 + b_3^2 &= 0. \end{aligned} \tag{19}$$

A simple choice for the parameter functions would be, for example, $b_1 = \cos t$ and $c_1 = \sin t$. Using the complex functions $\Phi_1(\zeta_1), \Phi_2(\zeta_2), \Phi_3(\zeta_3), \chi_1(\zeta_1), \chi_2(\zeta_2)$, and $\chi_3(\zeta_3)$ the following 3D elasticity solution representation can be derived:

$$\begin{aligned} 2\mu u &= \int \{ \text{Im} [(3 - 4\nu)\Phi_1 + \bar{\zeta}_1\Phi'_1 + \chi'_1] \\ &\quad + a_2 \text{Re} [(3 - 4\nu)\Phi_2 - \bar{\zeta}_2\Phi'_2 - \chi'_2] \\ &\quad + a_3 \text{Re} [(3 - 4\nu)\Phi_3 - \bar{\zeta}_3\Phi'_3 - \chi'_3] \} dt, \\ 2\mu v &= \int \{ b_1 \text{Re} [(3 - 4\nu)\Phi_1 - \bar{\zeta}_1\Phi'_1 - \chi'_1] \\ &\quad + \text{Im} [(3 - 4\nu)\Phi_2 + \bar{\zeta}_2\Phi'_2 + \chi'_2] \\ &\quad + b_3 \text{Re} [(3 - 4\nu)\Phi_3 - \bar{\zeta}_3\Phi'_3 - \chi'_3] \} dt, \\ 2\mu w &= \int \{ c_1 \text{Re} [(3 - 4\nu)\Phi_1 - \bar{\zeta}_1\Phi'_1 - \chi'_1] \\ &\quad + c_2 \text{Re} [(3 - 4\nu)\Phi_2 - \bar{\zeta}_2\Phi'_2 - \chi'_2] \\ &\quad + \text{Im} [(3 - 4\nu)\Phi_3 + \bar{\zeta}_2\Phi'_3 + \chi'_3] \} dt. \end{aligned} \tag{20}$$

Depending on the choice of parameter functions, the limits of integration can be specified. For $\sin t$ and $\cos t$ as parameter functions, the limits of integration will be $-\pi$ and π . For example, a displacement component could contain the following integral term:

$$\int_{-\pi}^{\pi} (z + ix \cos t + iy \sin t)^n \frac{\cos}{\sin} mtdt = \frac{2\pi i^m n!}{(n+m)!} r^n P_n^m(\cos \theta) \frac{\cos}{\sin} m\phi, \tag{21}$$

where P_n^m are Legendre functions.

2.6.1. Recovery of the 2D representation from the 3D representation

If we choose the parameters $a_2 = 1, c_2 = 0$ and the functions $\Phi_1 = \chi'_1 = \Phi_3 = \chi'_3 = 0$ we obtain the Muskhelishvili-Kolosov representation for the plane strain case:

$$\begin{aligned} 2\mu u &= \text{Re} [(3 - 4\nu)\Phi_2 - \bar{\zeta}_2\Phi'_2 - \chi'_2], \\ 2\mu v &= \text{Im} [(3 - 4\nu)\Phi_2 + \bar{\zeta}_2\Phi'_2 + \chi'_2]. \end{aligned}$$

3. EXAMPLE: CONSTRUCTION OF TREFFTZ FUNCTIONS FOR THICK PLATES

A simple Trefftz-approximation can be constructed by considering the solution part which involves powers of the thickness coordinate z . In terms of a real function G and its partial derivatives, all displacements and stresses are listed as follows:

$$\begin{aligned} 2\mu u &= -z \frac{\partial}{\partial x} G - \frac{1}{4(1-\nu)} \left[h^2 z - 2(2-\nu) \frac{z^3}{3} \right] \frac{\partial}{\partial x} \Delta G, \\ 2\mu v &= -z \frac{\partial}{\partial y} G - \frac{1}{4(1-\nu)} \left[h^2 z - 2(2-\nu) \frac{z^3}{3} \right] \frac{\partial}{\partial y} \Delta G, \\ 2\mu w &= G + \frac{\nu}{2(1-\nu)} z^2 \Delta G, \\ \sigma_{zz} &= 0 \end{aligned} \tag{22}$$

$$\begin{aligned} \sigma_{xx} &= -\frac{1}{(1-\nu)} z [G_{xx} + \nu G_{yy}] - \frac{1}{4(1-\nu)} \left[h^2 z - 2(2-\nu) \frac{z^3}{3} \right] \frac{\partial^2}{\partial x^2} \Delta G \\ \sigma_{yy} &= -\frac{1}{(1-\nu)} z [G_{yy} + \nu G_{xx}] - \frac{1}{4(1-\nu)} \left[h^2 z - 2(2-\nu) \frac{z^3}{3} \right] \frac{\partial^2}{\partial y^2} \Delta G \\ \tau_{xy} &= -z G_{xy} - \frac{1}{4(1-\nu)} \left[h^2 z - 2(2-\nu) \frac{z^3}{3} \right] \frac{\partial^2}{\partial x \partial y} \Delta G \\ \tau_{xz} &= \frac{1}{2(1-\nu)} \left[z^2 - \frac{h^2}{4} \right] \frac{\partial}{\partial x} \Delta G \\ \tau_{yz} &= \frac{1}{2(1-\nu)} \left[z^2 - \frac{h^2}{4} \right] \frac{\partial}{\partial y} \Delta G \end{aligned} \tag{23}$$

The stresses listed satisfy the homogeneous 3D equilibrium equations. The displacement field contains warping and change of thickness terms. Expressing G in complex form as in (15)–(17) and

using four terms in the complex power series, we obtain a set of linearly independent terms for the displacements u , v and w . The complex representation can be used in a symbolic manipulation program such as MATHEMATICA or MAPLE. Using the representations (15)–(17) and (22) in MATLAB, the following first 16 terms for the displacements u , v and w have been automatically produced:

$$\begin{aligned}
 u(1) &= -2zx, \\
 u(2) &= -z(3x^2 + y^2) - 8f_3, \\
 u(3) &= -4zx^3 - 24f_3x, \\
 u(4) &= -z(-6x^2y^2 - 3y^4 + 5x^4) - f_3(-48y^2 + 48x^2), \\
 u(5) &= 0, \\
 u(6) &= 2zxy, \\
 u(7) &= -z(-6x^2y - 2y^3) + 24f_3y, \\
 u(8) &= -z(-12x^3y - 4xy^3) + 96f_3xy, \\
 u(9) &= -z, \\
 u(10) &= -2zx, \\
 u(11) &= -z(-3y^2 + 3x^2), \\
 u(12) &= -z(-12xy^2 + 4x^3), \\
 u(13) &= 0, \\
 u(14) &= 2zy, \\
 u(15) &= 6zxy, \\
 u(16) &= -z(4y^3 - 12x^2y),
 \end{aligned} \tag{24}$$

$$\begin{aligned}
 v(1) &= -2zy, \\
 v(2) &= -2zxy, \\
 v(3) &= 4zy^3 + 24f_3y, \\
 v(4) &= -z(-4x^3y - 12xy^3) + 96f_3xy, \\
 v(5) &= 0, \\
 v(6) &= -z(-3y^2 - x^2) + 8f_3, \\
 v(7) &= -z(-2x^3 - 6xy^2) + 24f_3x, \\
 v(8) &= -z(5y^4 - 3x^4 - 6x^2y^2) - f_3(-48x^2 + 48y^2), \\
 v(9) &= 0, \\
 v(10) &= 2zy, \\
 v(11) &= 6zxy, \\
 v(12) &= -z(4y^3 - 12x^2y), \\
 v(13) &= z, \\
 v(14) &= 2zx, \\
 v(15) &= -z(-3x^2 + 3y^2), \\
 v(16) &= -z(12xy^2 - 4x^3),
 \end{aligned} \tag{25}$$

$$\begin{aligned}
w(1) &= x^2 + y^2 + 4f_2z^2, \\
w(2) &= x^3 + xy^2 + 8f_2z^2x, \\
w(3) &= x^4 - y^4 + f_2z^2(-12y^2 + 12x^2), \\
w(4) &= -2x^3y^2 - 3xy^4 + x^5 + f_2z^2(-48xy^2 + 16x^3), \\
w(5) &= 0, \\
w(6) &= -y^3 - x^2y - 8f_2z^2y, \\
w(7) &= -2x^3y - 2xy^3 - 24f_2z^2xy, \\
w(8) &= y^5 - 3x^4y - 2x^2y^3 + f_2z^2(-48x^2y + 16y^3), \\
w(9) &= x, \\
w(10) &= x^2 - y^2, \\
w(11) &= -3xy^2 + x^3, \\
w(12) &= -6x^2y^2 + x^4 + y^4, \\
w(13) &= -y, \\
w(14) &= -2xy, \\
w(15) &= -3x^2y + y^3, \\
w(16) &= 4xy^3 - 4x^3y,
\end{aligned} \tag{26}$$

where

$$f_3 = \frac{1}{4(1-\nu)} \left(h^2z - 2(2-\nu)\frac{z^3}{3} \right), \tag{27}$$

$$f_2 = \frac{\nu}{2(1-\nu)} \left(z^2 - \frac{h^2}{4} \right) \tag{28}$$

The terms with the indices 5, 9, and 13 lead to zero stresses and will be omitted for a hybrid Trefftz element. Using the fortran conversion command in MATLAB, the relevant lines for a FORTRAN subroutine can be written automatically.

4. VARIATIONAL FORMULATIONS FOR TREFFTZ-TYPE ELEMENTS

The use of Trefftz functions in a variational formulation leads to a formulation in terms of boundary integrals:

$$\delta\Pi = - \int_S \delta\mathbf{T}^T(\mathbf{u} - \tilde{\mathbf{u}})dS + \int_S \delta\mathbf{u}^T(\mathbf{T} - \bar{\mathbf{T}})dS. \tag{29}$$

For a Trefftz-type element, frame functions $\tilde{\mathbf{u}}$ on the boundary of the element can be chosen. Not for all Trefftz-type finite element it is the best strategy to derive the element stiffness matrix via a formulation involving boundary integrals along the boundary of the finite element. It can be more efficient to use the formulation with domain integrals. An example where the stiffness matrix can be obtained conveniently and fast using domain integration is the *QE2*-element of Piltner/Taylor [31]. The *QE2*-element is an enhanced quadrilateral plane strain/stress finite element. The enhanced terms \mathbf{e}^i are introduced in the following modified Hu–Washizu variational formulation to make the low-order element less stiff:

$$\Pi(\mathbf{u}, \boldsymbol{\epsilon}, \mathbf{e}^i, \boldsymbol{\sigma}) = \int_V \left[\frac{1}{2} \boldsymbol{\epsilon}^T \mathbf{E} \boldsymbol{\epsilon} - \mathbf{u}^T \bar{\mathbf{f}} \right] dV - \int_S \mathbf{u}^T \bar{\mathbf{T}} dS - \int_V \boldsymbol{\sigma}^T (\boldsymbol{\epsilon} - \mathbf{D}\mathbf{u} - \mathbf{e}^i) dV. \tag{30}$$

The displacement, strain, stress, and enhanced strain fields are chosen in the form

$$\mathbf{u} = \mathbf{N}\mathbf{q}, \quad (31)$$

$$\boldsymbol{\epsilon} = \mathbf{E}^{-1}\mathbf{S}\boldsymbol{\beta}, \quad (32)$$

$$\boldsymbol{\sigma} = \mathbf{S}\boldsymbol{\beta}, \quad (33)$$

$$\boldsymbol{\epsilon}^i = \mathbf{B}^i\boldsymbol{\lambda}, \quad (34)$$

$\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ are obtained from the Trefftz solution terms. The displacement field \mathbf{u} is chosen such that element continuity is achieved. At the element level the parameters $\boldsymbol{\beta}$ and $\boldsymbol{\lambda}$ can be eliminated. For the example of a thick plate the assumed displacement field for an appropriate coupling of elements is described in the following section.

5. EXAMPLE: ASSUMED DISPLACEMENTS FOR A THICK PLATE ELEMENT

In addition to the assumed Trefftz stresses and strains a displacement field is chosen such that continuity of the deflection, rotation, warping parameters and the thickness change are obtained. Denoting the warping parameters as q_x and q_y and the thickness change as H , we can construct elements with the six parameters $(w, \theta_x, \theta_y, q_x, q_y, H)$ or the five parameters $(w, \theta_x, \theta_y, q_x, q_y)$ or the four parameters $(w, \theta_x, \theta_y, H)$. For the case of four parameters the displacement field for the plate is assumed in the following form

$$\mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} z\theta_y \\ -z\theta_x \\ \bar{w} + z^2H \end{bmatrix}, \quad (35)$$

where \bar{w} is the deflection of the middle surface of the plate, θ_x and θ_y are the rotations about the x - and y -axis, and H is the thickness change of the plate. For a quadrilateral plate element the four functions can be chosen in terms of shape functions of the in-plane coordinates ξ and η :

$$\theta_x = \sum_{j=1}^4 N_j(\xi, \eta)\theta_x^j, \quad (36)$$

$$\theta_y = \sum_{j=1}^4 N_j(\xi, \eta)\theta_y^j, \quad (37)$$

$$\bar{w} = \sum_{j=1}^4 N_j(\xi, \eta)w_j, \quad (38)$$

$$H = \sum_{j=1}^4 N_j(\xi, \eta)H_j. \quad (39)$$

In the case of six parameters the continuous functions $\theta_x(x, y)$, $\theta_y(x, y)$, $q_x(x, y)$, $q_y(x, y)$ and $\bar{w}(x, y)$ can be used in the assumed element displacement field:

$$\begin{aligned} u &= z\theta_y + z^3q_x, \\ v &= -z\theta_x + z^3q_y, \\ w &= \bar{w} + z^2H. \end{aligned} \quad (40)$$

All six strains are used in the 3D variational formulation. The constitutive equation for three-dimensional elasticity is used. Depending on the choice of the complex functions, and depending on how many nodes per element and how many degrees of freedom are chosen, a family of Trefftz-type plate elements can be derived.

6. CONCLUSIONS

Solution representations in terms of complex valued functions can be very helpful when used in a symbolic manipulation program environment like MATHEMATICA, MAPLE or MATLAB. For some Trefftz-type elements it can be more efficient to use a formulation with domain integrals than a formulation with boundary integrals.

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