

## Trefftz in translation

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This paper reviews the important concepts presented by Trefftz in 1926 regarding bounds to solutions, error estimation, and hybrid fields for use with domain decomposition. Observations are offered from the perspective of today's relatively mature state of the art in finite element methods. The numerical examples presented by Trefftz are also reviewed with the benefit of 'exact' solutions made available from current commercial finite element methods. The accuracies of the solutions given by Trefftz are quantified and compared, and the effectivity indices of Trefftz's proposed error estimates are also quantified. An English translation of the original German version of Trefftz's paper is included for reference in an Appendix.

### 1. INTRODUCTION

The seminal work presented by Trefftz in 1926 [1] has recently been translated into the English language. The translation was undertaken for reference at the 3rd International Workshop/EuroConference on Trefftz Methods held in Exeter in September 2002, and to enable the work of Trefftz to reach a much wider audience.

Whilst Trefftz made many contributions to applied mathematics and mechanics [2], he is mainly remembered in the finite element community for proposing the new concept of using trial functions which satisfy the governing differential equations of a boundary value problem but not the boundary conditions. However the driving force behind these concepts was the need to obtain both upper and lower bounds to exact solutions of boundary value problems so that the global error of an approximate solution could be estimated. The Trefftz method was proposed as a means to complement the Ritz method in order to achieve such bounds. Similar aims were behind the development of dual finite element analyses some 50 years later [3], and it remains a topic of current research [4, 5].

The present paper is intended to accompany the translation which appears in the Appendix. In order to identify equations and/or figures in the translation, these carry the original reference numbers but with a prefix "A". Observations are included from the current viewpoint which benefits from the relative maturity of finite element techniques and computational mechanics.

Section 2 interprets the Dirichlet integral of the error as defined by Trefftz in terms of energy norms as currently defined. In Section 3 the bounds on solutions of the Laplace problem are reviewed, and Section 4 extends the review to generalised methods with domain decomposition. Trefftz presented two applications to torsion problems as solutions to Poisson's equation with a single domain and with domain decomposition. Section 5 presents further aspects of these solutions together with exact values of energy and stress which enable the error estimations made by Trefftz to be compared with actual errors. Conclusions are presented in Section 6.



## 2. ERROR MEASURE

The problems considered by Trefftz were the classical Dirichlet boundary value problems as exemplified by the equation governing steady state heat conduction in 2D. Equation (1) gives the typical form as given in [6, 7].

$$\frac{\partial}{\partial x} \left( k \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial u}{\partial y} \right) + Q = 0, \quad (1)$$

in domain  $G$ , with temperature (potential)  $u$  specified by the function  $g(s)$  on the entire boundary  $K$ .  $k$  denotes the thermal conductivity, although this is not explicitly mentioned by Trefftz. It may be considered as a constant equal to unity.  $Q$  denotes a scalar heat source function. The (heat) flux vector  $\mathbf{e}$  within the domain satisfies Eq. (2).

$$\mathbf{e} = -k \cdot \text{grad } u \quad \text{and} \quad -\text{div } \mathbf{e} + Q = 0. \quad (2)$$

With  $v$  denoting an approximate solution for temperature (potential), the error  $f$  is defined by Eq. (3).

$$f = v - u \quad (3)$$

and an integral global measure  $F$  is defined by the Dirichlet integral dot product in Eq. (4).

$$F = \iint_G \text{grad } f \cdot \text{grad } f \, dx \, dy, \quad (4)$$

$F$  may be interpreted as an energy measure of the error since  $\text{grad } f = -\text{error in flux } \mathbf{e}$  for unit conductivity. However  $F^{0.5}$  is the  $L_2$  norm of the flux error, which is also termed the energy norm for the case of unit conductivity [7].

## 3. ERROR BOUNDS FOR A SINGLE DOMAIN

### 3.1. Ritz method

The Ritz method is recalled for the Laplace problem with essential boundary conditions. Trial potential functions are used which satisfy the boundary conditions. By invoking Gauss's divergence theorem it is shown that: the error in flux is orthogonal to the true flux; the energy of the error equals the error of the energy; and consequently the trial solution which minimises the energy of the solution provides an upper bound to this energy. It also minimises the energy of the error and is thereby an optimum one.

### 3.2. Trefftz method

Trefftz introduces his new method for the problem using trial potential functions which are harmonic, i.e. they each satisfy the homogeneous form of the governing differential equation (1), but they do not satisfy any boundary conditions. In this case it is shown that the energy of the error equals minus the error of the energy. The aim remains to minimise the error but the result of this is to provide a lower bound to the energy of the solution. Consequently both upper and lower bounds can be obtained to bracket the exact, and generally unknown, solution. From the pedagogical point of view it appears to be more logical to start with the aim of minimising an error measure and then conclude that this is achieved by minimising the energy of the approximate solution. Indeed this is exactly the approach taken by Trefftz in his "New Method".



Two important characteristics of his new method are mentioned: greater simplicity and freedom to use trial functions which do not satisfy the boundary conditions (at least for the first type of problem considered, however he appears to lose this freedom in generalising the method for use with subdomains); and local errors in quantities of interest are expected to be largest on the boundary. The latter point has been demonstrated by Jirousek and Wroblewski [8]. The latter characteristic may be bad news for those who need to quantify potentially harmful stress concentrations on the surface of a structure – unless special functions are used.

The initial problems considered by Treffitz are 2D Laplace type problems with boundary conditions which are entirely of the Dirichlet (or displacement) type. We may consider in this case that the lower bound nature of the solutions is also guaranteed in another variational problem by varying flux fields which *a priori* satisfy continuity conditions (cf. statically admissible stress fields in structural problems), and minimising the appropriate functional. Of course by selecting trial harmonic potential fields, the corresponding flux fields do satisfy the continuity conditions automatically in the absence of any “flux” type of boundary conditions.

It could be said that dual analysis offers even more freedom in seeking a lower bound solution since the trial flux fields need only satisfy continuity [9], and need not be derived from potential functions (harmonic or otherwise).

#### 4. GENERALISATION OF THE METHODS WITH DOMAIN DECOMPOSITION

##### 4.1. Ritz method

This method is generalised by introducing another set of trial potential functions  $h_p$  which are defined along the interfaces between subdomains. Two types of solution are then to be derived independently for each subdomain: particular solutions  $q_0$  which conform with the specified essential boundary conditions on external boundaries and with homogeneous essential boundary conditions on potential interfaces; and complementary solutions  $q_p$  which conform with homogeneous boundary conditions on external boundaries and with each trial interface function.

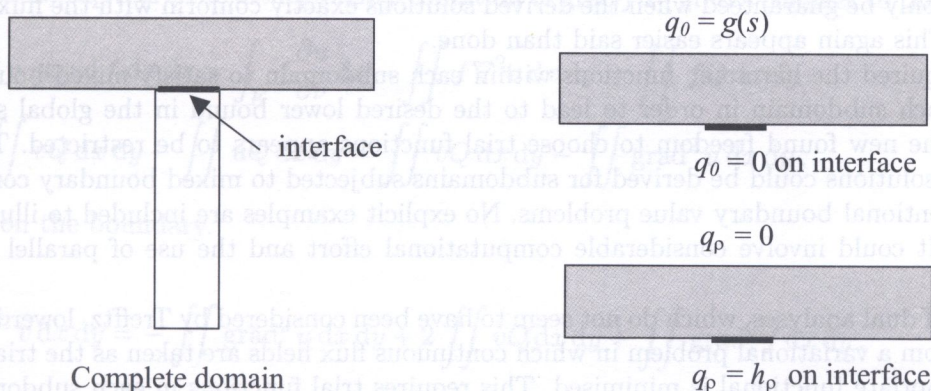


Fig. 1. Essential boundary conditions for a subdomain in the Ritz method

These derived solutions are then combined for the whole domain so as to enforce continuity of potential and minimum energy of the error. It should be noted that continuity of flux is not enforced, and the upper bound nature of the solution can only be guaranteed when the derived solutions exactly conform with the assumed boundary conditions. The latter conformity may be easier said than done!

Obtaining solutions by analysing each subdomain independently appears to be an expensive way to obtain global trial solutions. For the Ritz method it is only strictly necessary to use piecewise



continuous global trial functions which satisfy the essential boundary conditions in order to obtain an upper bound solution.

#### 4.2. Trefftz method

This method is generalised by introducing another set of trial functions  $k_\rho$  representing normal flux distributions along interfaces between subdomains. Now each subdomain is analysed to derive harmonic solutions  $p_0, p_\rho$  satisfying mixed boundary conditions. These boundary conditions include homogeneous and/or non-homogeneous essential boundary conditions on external boundaries, and homogeneous and/or non-homogeneous natural boundary conditions at interfaces.

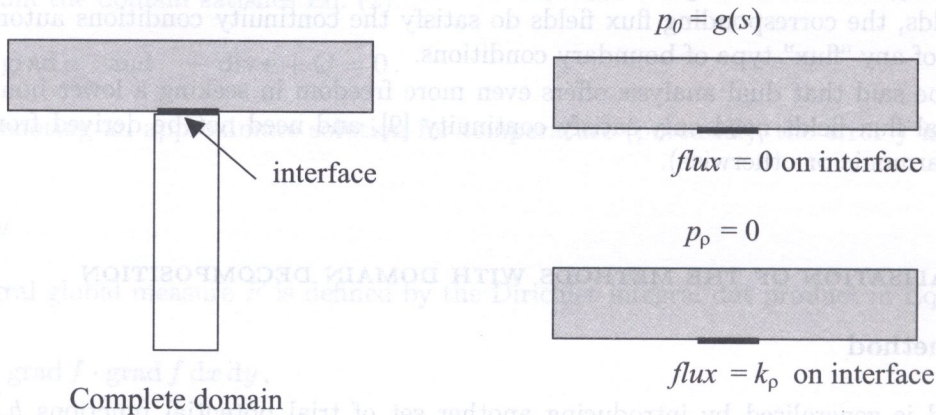


Fig. 2. Mixed boundary conditions for a subdomain in the Trefftz method

The solutions are then combined for the whole domain so as to enforce flux continuity and minimum energy of the error. It should be noted that continuity of potential is not enforced exactly but only in a weighted integral sense as defined in Eq. (A21), and the lower bound nature of the solution can only be guaranteed when the derived solutions exactly conform with the flux continuity conditions. This again appears easier said than done.

Trefftz required the harmonic functions within each subdomain to satisfy mixed boundary conditions for each subdomain in order to lead to the desired lower bound in the global solution! In which case the new found freedom to choose trial functions appears to be restricted. Trefftz considered that solutions could be derived for subdomains subjected to mixed boundary conditions by solving conventional boundary value problems. No explicit examples are included to illustrate such derivations. It could involve considerable computational effort and the use of parallel computing techniques!

In terms of dual analyses, which do not seem to have been considered by Trefftz, lower bounds can be derived from a variational problem in which continuous flux fields are taken as the trial functions and an appropriate functional is minimised. This requires trial functions in each subdomain which also satisfy continuity of normal flux across the interface, but do not need to satisfy the global essential type of boundary conditions. The interface continuity condition can be enforced through the use of the interface potential functions  $h_\rho$  which can be viewed as weighting functions in order to zero the residual flux discontinuities, i.e. functions  $h_\rho$  are then common hybrid potential functions for the subdomains. Functions  $k_\rho$  could also be viewed as dual hybrid tractions on the sides of the subdomains where the interfaces occur.

It is of interest to compare this generalised method to recent hybrid finite element methods which utilise hybrid functions along interfaces, or element edges, in the assembly of elements [8–12]. For example the HT-D and HT-T elements [8, 12] utilise internal Trefftz type displacement fields and displacement frame functions or edge traction functions respectively; hybrid equilibrium



elements [10] exploit the weighted residual concept for element edges [9] to achieve stress continuity sufficient for strong equilibrium; and a hybrid stress based element with internal Trefftz type stress fields and edge displacement functions is also considered [11].

In these hybrid elements the two fields are generally defined independently of each other, and bounded solutions are not guaranteed except for the equilibrium model. So one can have more freedom of choice for hybrid functions, but generally this freedom incurs penalties: e.g. a loss of bounds, and instability when spurious kinematic modes are present!

## 5. APPLICATIONS TO TORSION PROBLEMS

### 5.1. Torsion of a square section – a single domain

This problem is formulated in terms of the Prandtl stress function  $\varphi$  as a solution to Poisson's Eq. (5), with essential boundary condition  $\varphi = 0$ .

$$\nabla^2 \varphi = 2G\omega. \quad (5)$$

It should be noted here that this application involves upper and lower bound properties of solutions but the bounds are opposite to those for the Laplace problem discussed in Section 3! The nature of the bounds are considered in the following sections.

#### 5.1.1. Ritz method

Before discussing the specific solution to this problem, the nature of the bound of the approximate solution  $v$  to the exact solution  $u$  of Eq. (6) is proved in a similar way to Trefftz in Eqs. (A1) to (A4).

$$\nabla^2 u + Q = 0 \quad \text{with} \quad u = 0 \quad \text{on the boundary.} \quad (6)$$

Starting from Eq. (A4), the orthogonality condition of  $\text{grad } f$  to  $\text{grad } u$  changes to that in Eq. (7).

$$\begin{aligned} \iint \text{grad } u \cdot \text{grad } f \, dx \, dy &= \int_K f \frac{\partial u}{\partial \nu} \, ds - \iint f \nabla^2 u \, dx \, dy = \iint f Q \, dx \, dy \\ &= \iint v Q \, dx \, dy - \iint u Q \, dx \, dy = \iint v Q \, dx \, dy - \iint \text{grad}^2 u \, dx \, dy, \end{aligned} \quad (7)$$

since  $u = 0$  on the boundary.

So now

$$\iint \text{grad}^2 v \, dx \, dy = - \iint \text{grad}^2 u \, dx \, dy + 2 \iint v Q \, dx \, dy + \iint \text{grad}^2 f \, dx \, dy,$$

or

$$\iint \text{grad}^2 f \, dx \, dy = \iint \text{grad}^2 v \, dx \, dy - 2 \iint v Q \, dx \, dy + \iint \text{grad}^2 u \, dx \, dy.$$

Minimisation of the global error leads to  $\iint \text{grad}^2 v \, dx \, dy = \iint v Q \, dx \, dy$ , so that the energy of the error is minus the error of the energies and finally the bounded nature of the solution appears in Eq. (8) as a *lower* bound.

$$\iint \text{grad}^2 v \, dx \, dy = \iint \text{grad}^2 u \, dx \, dy - \iint \text{grad}^2 f \, dx \, dy \leq \iint \text{grad}^2 u \, dx \, dy. \quad (8)$$



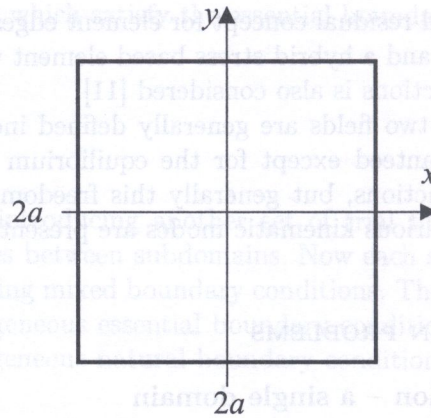


Fig. 3. Square cross-section for the torsion problem

For the torsion problem governed by Eq. (5), and illustrated in Fig. 3, a quartic polynomial is used with two variable parameters to approximate  $\varphi$  (Eq. (A29)) and satisfy the boundary conditions. Minimising the error leads to the solution for function  $\varphi$  in Eq. (9),

$$\varphi = (x^2 - a^2)(y^2 - a^2) [74a^2 + 15(x^2 + y^2)] \frac{105G\omega}{13296a^4} \quad (9)$$

and a lower bound for the torsional stiffness in Eq. (10)

$$2.2463Ga^4 \quad [\text{torque per unit twist}], \quad (10)$$

cf.  $2.2496Ga^4$  for the exact stiffness [13].

Trefftz gives the result as an upper bound for the torsional flexibility in Eq. (A36). It is noted that Trefftz moves rather swiftly from detailed consideration of the Laplace problem to the Poisson problem for torsion, and casts the Ritz method as minimising the strain energy of the solution for a given torsional moment. This step can be analysed with a little more depth to consolidate understanding! Integration by parts of the expression for torque in Eq. (A26) leads to Eq. (11).

$$M = \iint \left( x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} \right) dx dy = -2 \iint v dx dy \quad \text{when } v = 0 \quad \text{on the boundary.} \quad (11)$$

But after minimisation of the error  $f$ ,  $\iint \text{grad}^2 v dx dy = \iint vQ dx dy$  where  $Q = -2G\omega$  and the relationships for torque and strain energy become as in Eq. (12).

$$M = \frac{1}{G\omega} \iint \text{grad}^2 v dx dy; \quad (12)$$

$$\text{strain energy} = \frac{1}{2G} \iint \text{grad}^2 v dx dy = \frac{1}{2} M\omega.$$

Thus the approximate solution  $v$  actually provides a lower bound solution in terms of strain energy for a specified twist  $\omega$  on the right hand side of Eq. (5), but Trefftz considers the Ritz solution as an upper bound for strain energy for a specified torque.

The tangential shear stress at the midpoint of a side can be derived as in Eq. (13).

$$\tau = \frac{801}{160} \frac{M}{(2a)^3} = 5.0062 \frac{M}{(2a)^3}, \quad (13)$$

cf.  $4.8077 \frac{M}{(2a)^3}$  for the exact maximum stress [13].



5.1.2. Trefftz method

In the new method Trefftz uses a trial function for  $\varphi$  which combines a particular solution ( $p_0 = r^2 = x^2 + y^2$  so that  $\nabla^2 p_0 = 4$ ) and an harmonic function ( $p_1 = x^4 - 6x^2y^2 + y^4$  so that  $\nabla^2 p_1 = 0$ ). The solution quoted by Trefftz is recalled in Eq. (14),

$$\varphi = \frac{G\omega}{2a^2} \left\{ a^2 (x^2 + y^2) + \frac{7}{36} (x^4 - 6x^2y^2 + y^4) \right\} \quad \text{and} \tag{14}$$

$$\varphi = \frac{G\omega a^2}{72} \left\{ 43 - 6 \left(\frac{y}{a}\right)^2 + 7 \left(\frac{y}{a}\right)^4 \right\} \quad \text{when } x = a$$

and this provides an upper bound for the torsional stiffness in Eq. (15).

$$2.2518Ga^4 \text{ [torque per unit twist];} \tag{15}$$

cf. Eq. (10).

This solution satisfies the governing differential equation within the square domain, but as expected it leads to error on the boundary. The stress function itself is not constant but varies as shown in Fig. 4 along the side  $x = a$ . Of more interest to the engineer is the error in shear stresses. The expression for  $\tau_x$  along the side  $x = a$  (which should be zero) is given in Eq. (16).

$$\tau_x = -\frac{\partial\varphi}{\partial y} = \frac{G\omega a}{18} \left\{ 3 \left(\frac{y}{a}\right) - 7 \left(\frac{y}{a}\right)^3 \right\} \tag{16}$$

and this is plotted in Fig. 5. This plot indicates that the largest error in stress occurs at a corner where the residual surface shear stress is about 16% of the exact maximum shear stress  $\tau_y$  at the midpoint of the side. The latter stress is estimated by the Trefftz method in Eq. (17).

$$\tau_y = \frac{\partial\varphi}{\partial x} = 25 \frac{G\omega a}{18} = \frac{375}{76} \frac{M}{(2a)^3} = 4.9342 \frac{M}{(2a)^3}, \tag{17}$$

which exceeds the exact maximum stress in Eq. (13) by 2.66%.

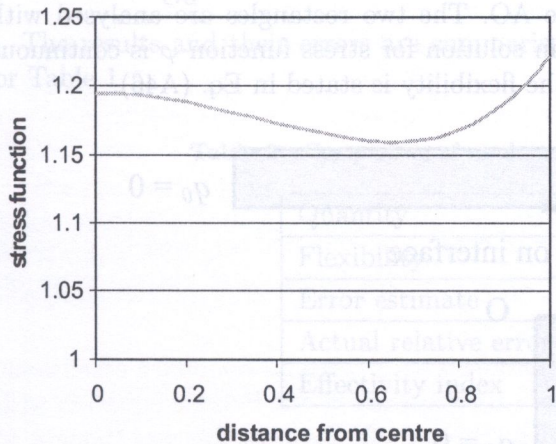


Fig. 4. Variation of stress function  $\varphi$  along the side  $x = a$  in terms of the non-dimensional distance from the centre to the corner

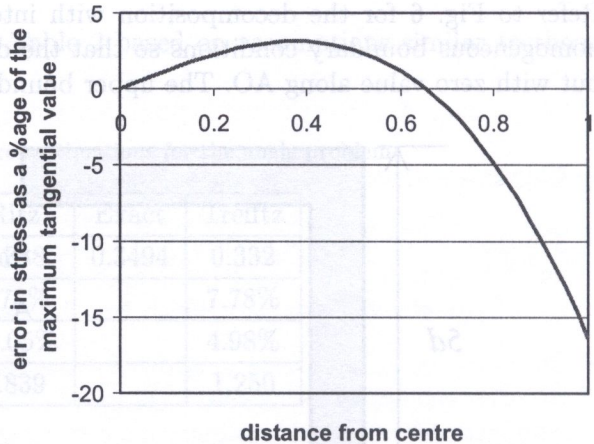


Fig. 5. Error in shear stress  $\tau_x$  compared with the exact maximum stress along the side  $x = a$

The comparisons between the Ritz, Trefftz and exact solutions are tabulated in Table 1. In this table the numerical coefficients for flexibility are taken as the reciprocals of the coefficients in



Eqs. (10) and (15), the error estimate for flexibility is based on assuming that the exact flexibility is the average value from the two solutions, the effectivity indices are based on comparing energy norms (the square root of the error energy, or in this case the error in flexibility), and the shear stresses are taken from Eqs. (13) and (17).

**Table 1.** Comparison of results and error estimation for the single domain torsion problem

Quantity	Ritz	Exact	Trefftz
Flexibility	0.44518	0.44452	0.44408
Error estimate	0.124%		0.124%
Actual relative error	0.148%		0.099%
Effectivity index	0.913		1.118
Maximum shear stress	5.0062	4.8077	4.9342

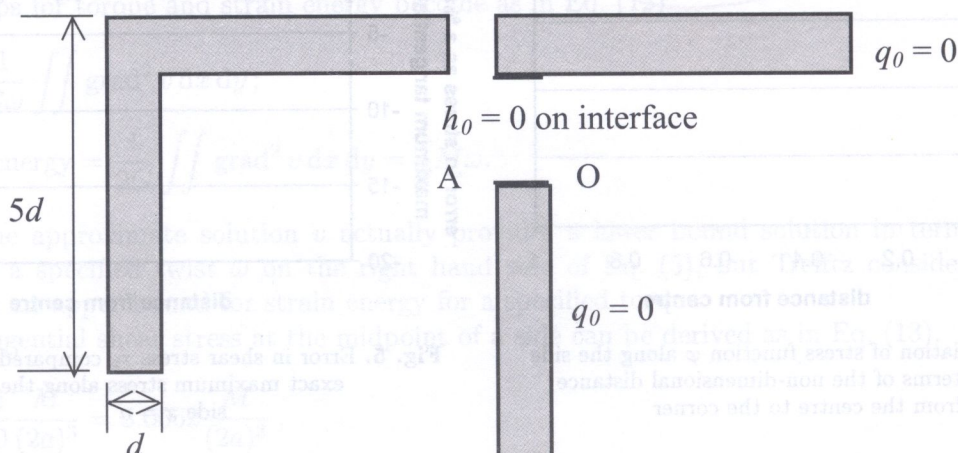
It is to be noted that the relative errors are low and the effectivity indices are well within current normally accepted values, i.e. between 0.8 and 1.2 [6].

## 5.2. Torsion of an angle section – domain decomposition

The last application considered concerns twisting of an angle. The domain is decomposed into two rectangles for the Ritz method and two rectangles and a square for the new method. The trial functions used for the analyses are not explicitly stated, but the interface connections are simply based on enforcing zero values for the stress function or its normal derivative for the Ritz or Trefftz methods respectively. Based on the two solutions, this lead to an estimate of the global twist error of about 8%. No information can be derived about the local error in stress since the solution for  $\varphi$  is not given.

### 5.2.1. Ritz method

Refer to Fig. 6 for the decomposition with interface AO. The two rectangles are analysed with homogeneous boundary conditions so that the domain solution for stress function  $\varphi$  is continuous but with zero value along AO. The upper bound to the flexibility is stated in Eq. (A46).



**Fig. 6.** Equal angle section with subdomains and boundary conditions for the Ritz method



5.2.2. Trefftz method

Refer to Fig. 7 for the decomposition into three subdomains. In this case stress functions are based on larger rectangular or square subdomains and using their axes of symmetry to ensure that the functions give zero normal flux across all internal boundaries. The resulting solution for the complete domain has piecewise but continuous flux fields but with zero flux and discontinuous stress functions at the interfaces. The lower bound to the flexibility is stated in Eq. (A47).

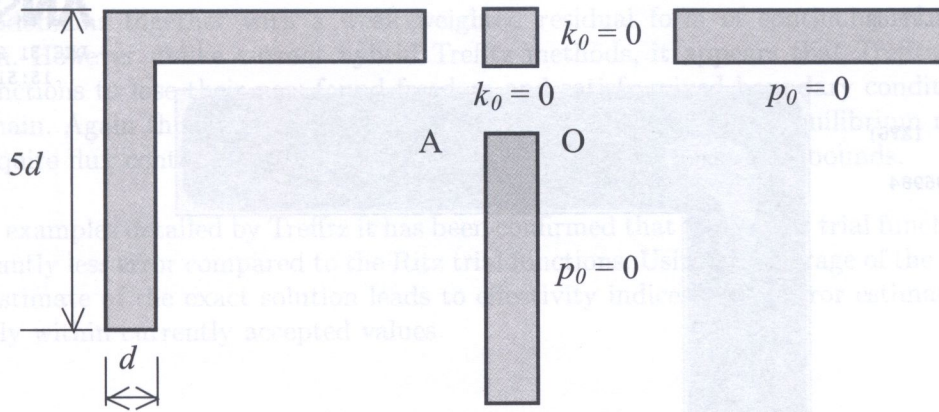


Fig. 7. Equal angle section with subdomains and boundary conditions for the Trefftz method

5.2.3. Further comparisons with a reference solution

In order to compare accuracies and effectivities of error estimates a series of finite element models of the problem were analysed to provide an accurate reference solution. With uniform meshes of square elements, the exact torsional moment was estimated using Richardson’s extrapolation [7] as  $2.8621Gd^4\omega$ , which in turn gives the exact flexibility relation in Eq. (18).

$$\omega = 0.3494 \frac{M}{Gd^4} \tag{18}$$

The results and their errors are summarised in Table 2 based on assumptions similar to those for Table 1.

Table 2. Comparison of results and error estimations for the angle problem

Quantity	Ritz	Exact	Trefftz
Flexibility	0.388	0.3494	0.332
Error estimate	7.78%		7.78%
Actual relative error	11.05%		4.98%
Effectivity index	0.839		1.250

Compared with the previous problem of the square section, the relative errors are much higher, but the effectivity indices are still reasonable. Again the Trefftz solution not only provides error bounds but also is considerably more accurate. Figure 8 provides an accurate contour map of the Prandtl stress function from the final finite element analysis of the series. From this map it can be judged that the interface assumptions for the Trefftz solution (zero normal flux) are more realistic than those for the Ritz solution (zero valued stress function across AO).



The final point made by Trefftz is that the global error should be judged against the uncertainty in knowing the value of the shear modulus  $G$ . Such uncertainty is also a feature of the "error in constitutive relations" concept as quantified by Ladeveze in 1975 and thereafter [14]. This concept also makes use of two solutions which in the context of the Trefftz paper can be considered one as a potential function from a finite element version of the Ritz method, and one as a flux function from a dual solution which could also be a Trefftz solution.

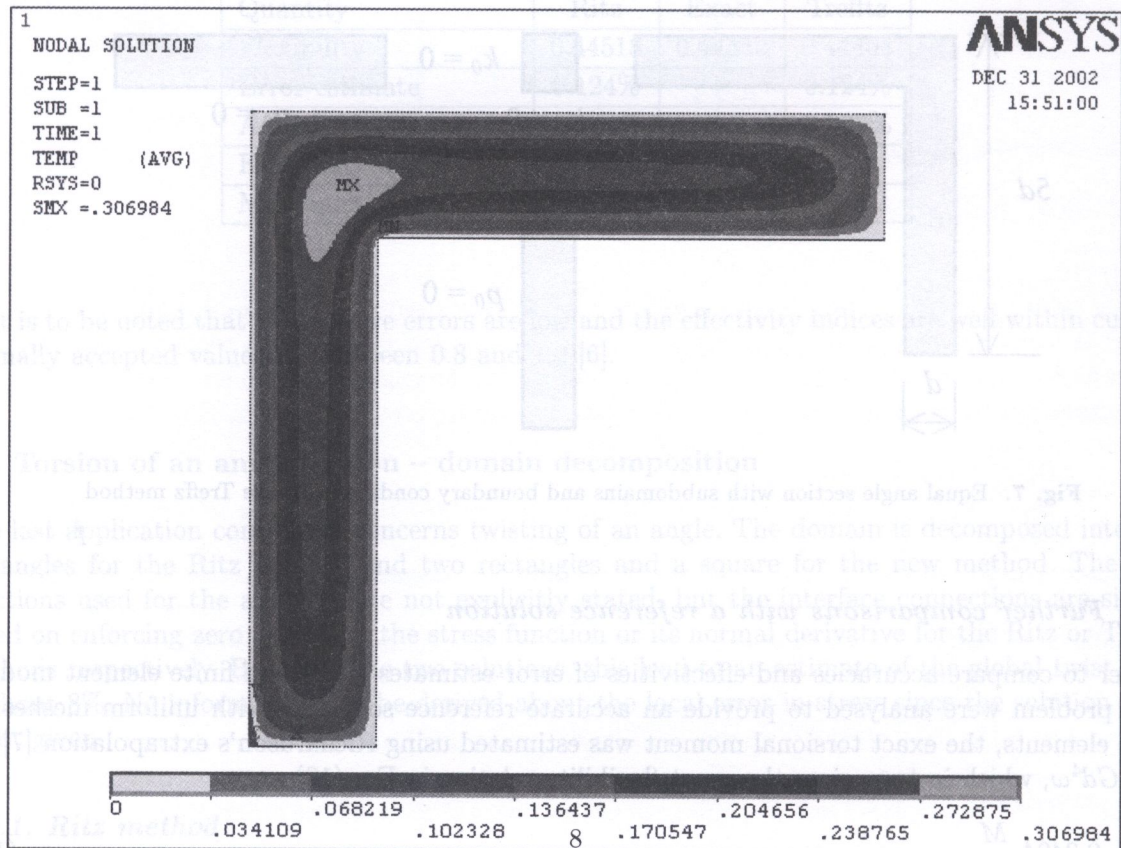


Fig. 8. Contour map of the Prandtl stress function for the angle section

## 6. CONCLUSIONS

- The aim of Trefftz was to obtain bounded estimates of error, and this was achieved by carrying out two analyses to complement each other. Thus he provided a foretaste of current work on dual finite element analyses which has the same aim.
- Proofs of the bounds were provided for the Laplace problem with essential boundary conditions. However the applications used to illustrate the bounds are governed by the Poisson equation with homogeneous essential boundary conditions without further proofs. Perhaps this was thought to be obvious and/or unnecessary for the Poisson problem, although as this paper shows the energy bounds on the solutions are then reversed!
- Trefftz emphasised the freedom of his new method over Ritz in that it may be easier to define trial functions without regard to the boundary conditions. However when viewed from a wider perspective which includes equilibrium models for dual analyses in structural mechanics applications, it is observed that even greater freedom exists for the type of problem considered by



Trefftz in his paper, i.e. trial flux functions can be used, which only need to satisfy “equilibrium” or continuity, to provide similar bounds. Of course the Trefftz particular solutions and harmonic functions do yield flux functions which satisfy the “equilibrium” conditions by default, but with the extra demands of “compatibility”.

- Trefftz proposed domain decomposition for more complex shapes, although only relatively simple cases were considered practical in the pre-digital computer age! This generalisation of the Trefftz method provided another foretaste of recent research in the use of hybrid flux functions to “glue” the subdomains together with a weak weighted residual form of continuity of the potential function. However unlike current hybrid Trefftz methods, it appears that Trefftz required his trial functions to lose their new found freedom and satisfy mixed boundary conditions for each subdomain. Again this freedom can be more than recovered by using equilibrium models which only require flux continuity between subdomains to achieve the desired bounds.
- For the examples detailed by Trefftz it has been confirmed that the Trefftz trial functions incurred significantly less error compared to the Ritz trial functions. Using the average of the two solutions as an estimate of the exact solution leads to effectivity indices of the error estimates which are generally within currently accepted values.

#### APPENDIX: A TRANSLATION OF REFERENCE [1]

Proceedings of the 2<sup>nd</sup> International Congress of Applied Mechanics  
Zurich, 1926  
pp 131 to 137

#### A counterpart to Ritz's Method by E. Trefftz, Dresden

The Ritz method solves partial differential equation boundary value problems using the minimum potential energy theorem. Obviously, the error of the approximate solution is the difference between the strain energy of the exact mathematical model and that obtained using the Ritz trial functions.

It is important to note that the Ritz method does not allow an error estimate or some sort of error bounds since it yields only an upper bound to the true potential energy of the system. The objective of this paper is to present an analogue to the Ritz method that produces a lower bound to the potential energy. A combination of the Ritz method and the novel approach thus yields the desired error bounds.

For the sake of simplicity, we present the new method using the plane potential theory boundary value problem. Generalization to include other differential equations is straightforward.

Given a domain  $G$  with boundary  $K$  and prescribed boundary values  $u = g(s)$  we approximate the potential function  $u$  using Ritz trial functions by

$$v(x, y) = g(x, y) + \sum_1^n c_h q_h(x, y) \quad (A1)$$

where  $g(x, y)$  is a function that satisfies the prescribed boundary values on  $K$  and  $q_h(x, y)$  are functions which are zero on  $K$ . Hence  $v(x, y)$  satisfies the boundary conditions for arbitrary coefficients  $c$ . Minimization of the DIRICHLET integral of  $v$  yields the parameters  $c_h$ . If we denote by  $f(x, y)$  the error

$$f(x, y) = v(x, y) - u(x, y) \quad (A2)$$



then we have  $f = 0$  on the boundary and obtain

$$\iint \text{grad}^2 v \, dx \, dy = \iint \text{grad}^2 u \, dx \, dy + 2 \iint \text{grad} u \cdot \text{grad} f \, dx \, dy + \iint \text{grad}^2 f \, dx \, dy = \iint \text{grad}^2 u \, dx \, dy + \iint \text{grad}^2 f \, dx \, dy, \quad (\text{A3})$$

using the condition

$$\iint \text{grad} u \cdot \text{grad} f \, dx \, dy = \int_K f \frac{\partial u}{\partial \nu} \, ds - \iint f \nabla^2 u \, dx \, dy = 0; \quad (\text{A4})$$

note that  $\nu$  is the outward normal to the boundary.

This leads us to a slightly modified version of the minimum potential energy problem. Recognizing that the DIRICHLET integrals of  $v$ , i.e.  $\iint \text{grad}^2 v \, dx \, dy$ , and of the error  $f$ , i.e.  $\iint \text{grad}^2 f \, dx \, dy$ , differ by a constant value which is the DIRICHLET integral of the true solution, the Ritz method minimizes both the integrals  $\iint \text{grad}^2 v \, dx \, dy$  and  $F = \iint \text{grad}^2 f \, dx \, dy$ . Among the family of trial functions in (A1), the Ritz analysis method thus yields the one that minimizes the integral of the error.

Departing from this formulation, we arrive immediately at an analogue to the Ritz method. Instead of using trial functions that satisfy the boundary conditions but violate the differential equation, we approximate the solution by selecting functions that violate the boundary conditions but satisfy the differential equation. We can view this procedure as a generalisation of series expansions using particular solutions and the new approach in fact contains the conventional series expansions as a special case. We select trial functions

$$w(x, y) = \sum_1^n c_h p_h(x, y) \quad (\text{A5})$$

where  $p_h(x, y)$  is an harmonic potential function and determine the coefficients  $c$  that minimize the integral of the error

$$F = \iint \text{grad}^2 f \, dx \, dy = \iint \text{grad}^2(w - u) \, dx \, dy, \quad (\text{A6})$$

where  $u$  is the actual solution to the problem. Hence

$$\frac{\partial F}{\partial c_h} = 2 \iint \text{grad}(w - u) \text{grad} \frac{\partial w}{\partial c_h} \, dx \, dy = 0, \quad (\text{A7})$$

with

$$\frac{\partial w}{\partial c_h} = p_h(x, y). \quad (\text{A8})$$

With  $\nu$  being the outward normal, we integrate using GAUSS' theorem and obtain

$$\iint \text{grad}(w - u) \text{grad} p_h \, dx \, dy = \int_K (w - u) \frac{\partial p_h}{\partial \nu} \, ds = 0, \quad (\text{A9})$$

or

$$\sum_1^n c_\rho \int p_\rho \frac{\partial p_h}{\partial \nu} \, ds = \int g(s) \frac{\partial p_h}{\partial \nu} \, ds.$$

We thus generate  $n$  linear equations for  $n$  coefficients  $c$ , since  $u = g(s)$  is prescribed on the boundary.



We know from classical series expansion that the DIRICHLET integral of  $w$  is smaller than that of the true solution. A simple calculation confirms this statement. If  $f$  denotes the error, we have

$$\iint \text{grad}^2 u \, dx \, dy = \iint \text{grad}^2 w \, dx \, dy - 2 \iint \text{grad} w \cdot \text{grad} f \, dx \, dy + \iint \text{grad}^2 f \, dx \, dy.$$

Since

$$\iint \text{grad} w \text{grad} f \, dx \, dy = \int_K f \frac{\partial w}{\partial \nu} \, ds = \sum c_\rho \int (w - u) \frac{\partial p_\rho}{\partial \nu} \, ds \tag{A10}$$

and the coefficients  $c_\rho$  are determined by invoking (A9) to be zero, the central integral on the right hand side disappears. Hence

$$\iint \text{grad}^2 w \, dx \, dy = \iint \text{grad}^2 u \, dx \, dy - \iint \text{grad}^2 f \, dx \, dy \leq \iint \text{grad}^2 u \, dx \, dy, \tag{A11}$$

which proves the above statement.

For practical purposes, the new approach is superior to the Ritz method since we don't have to find trial functions that satisfy the boundary conditions which is not always easy. Moreover, error assessment is simplified: the largest error lies on the boundary. If we assume that a solution exists, we can easily prove convergence since both the field variables and their derivatives converge inside the domain  $G$ .

Both the Ritz and the new methods can be generalized, if we relax the continuity requirement inside the domain. Since this generalization is only of practical significance for simple cases, let us discuss a simple example for illustration.

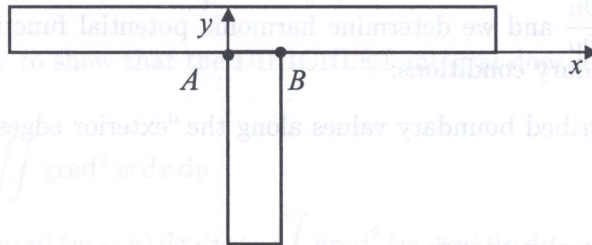


Fig. A1

Let us consider a domain that consists of two rectangles which share the common side  $AB$ , e.g. a T-section as shown in Fig. A1. The boundary of the domain and the side  $AB$  are termed the "exterior edge" and the "interior edge" respectively. The origin of the coordinate system is at point  $A$ , and point  $B$  is at  $x = b$  and  $y = 0$ .

**Ritz Method**

Since the values of  $u$  are unknown along side  $AB$ , and hence we are unable to solve the boundary value problem by dealing with the two rectangles individually, we use trial functions  $h_0, h_1$  etc. to write

$$v(x) = h_0 + \sum_1^n c_\rho h_\rho(x), \tag{A12}$$



in order to approximate  $u$ . Next we determine potential functions  $q_0, q_1, q_2$  etc. which satisfy the following boundary conditions:

$$\begin{aligned} q_0 &\text{ satisfies the prescribed boundary values along the "exterior edges" and} \\ q_0 &= 0 \text{ along } AB; \\ q_\rho &= 0 \text{ along the "exterior edges",} \\ q_\rho &= h_\rho(x) \text{ along } AB. \end{aligned} \tag{A13}$$

We obtain the functions  $q_\rho(x, y)$  solving the conventional boundary value problem for the two individual rectangles. Along line  $AB$ , the functions  $q_\rho(x, y)$  are continuous but not their derivatives in the normal direction. We now use the trial functions

$$v(x, y) = q_0(x, y) + \sum_1^n c_\rho q_\rho(x, y) \tag{A14}$$

and determine the coefficients  $c$  such that they minimize the DIRICHLET integral over the entire domain. Clearly, we obtain an upper bound to the true solution.

### New Method

Since the values of  $\frac{\partial u}{\partial \nu}$  are unknown along side  $AB$  and hence we are unable to solve the boundary value problem by dealing with the two rectangles independently, we use trial functions  $k_1, k_2$  etc. to write

$$\frac{\partial w}{\partial y} = \sum c_\rho k_\rho(x), \tag{A15}$$

as an approximation to  $\frac{\partial u}{\partial \nu}$  and we determine harmonic potential functions  $p_0, p_1, p_2$  etc. which satisfy the following boundary conditions:

$$\begin{aligned} p_0 &\text{ satisfies the prescribed boundary values along the "exterior edges",} \\ \frac{\partial p_0}{\partial y} &= 0 \text{ along } AB; \\ p_\rho &= 0 \text{ along the "exterior edges",} \\ \frac{\partial p_\rho}{\partial y} &= k_\rho(x) \text{ along } AB. \end{aligned} \tag{A16}$$

Again, we obtain the functions  $p_\rho(x, y)$  by solving the conventional boundary value problem for the two individual rectangles. Along line  $AB$ , the functions  $p_\rho(x, y)$  are discontinuous but their derivatives in the normal direction are continuous. We now use the trial functions

$$w(x, y) = p_0(x, y) + \sum_1^n c_\rho p_\rho(x, y) \tag{A17}$$

and determine the coefficients  $c$  such that they minimize the integral of the error

$$F = \iint \text{grad}^2(w - u) \, dx \, dy,$$

where  $u$  is the true solution. This results in the condition

$$\iint \text{grad}(w - u) \cdot \text{grad} p_h(x, y) \, dx \, dy = 0. \tag{A18}$$



We use integration by parts to rearrange terms. Since both  $u$  and  $w$  are harmonic potential functions, the area integrals are zero and we are left with the integral along the boundaries of the two rectangles

$$\int (w - u) \frac{\partial p_h}{\partial \nu} ds = 0. \tag{A19}$$

Since  $u = w$  along the exterior edge, the only remaining contribution is along  $AB$ , one for each rectangle. Let us denote by  $p_h^{(1)}(x, y)$  and  $p_h^{(2)}(x, y)$  the values of  $p_h(x, y)$  for rectangles 1 and 2, respectively. Since

$$\frac{\partial p_h}{\partial \nu} = \frac{\partial p_h}{\partial y} = k_h \tag{A20^I}$$

and

$$\frac{\partial p_h}{\partial \nu} = -\frac{\partial p_h}{\partial y} = -k_h, \tag{A20^{II}}$$

for rectangles 1 and 2, respectively, we obtain

$$\int_A^B (w^{(1)} - u) k_h dx - \int_A^B (w^{(2)} - u) k_h dx = \int_A^B (w^{(1)} - w^{(2)}) k_h dx = 0. \tag{A21}$$

Since the derivative along the normal is continuous the unknown  $u$  cancels out and we have the equations

$$\int_A^B \left\{ p_0^{(1)} - p_0^{(2)} + \sum_1^n c_\rho (p_\rho^{(1)} - p_\rho^{(2)}) \right\} k_h dx = 0, \tag{A22}$$

to determine  $c_h$ . It is easy to show that the DIRICHLET integral does in fact yield a lower bound. We have

$$\begin{aligned} \iint \text{grad}^2 u dx dy &= \iint \text{grad}^2 w dx dy \\ &- 2 \iint \text{grad} w \cdot \text{grad} (w - u) dx dy + \iint \text{grad}^2 (w - u) dx dy \end{aligned}$$

and the second integral on the right hand side becomes

$$\iint \text{grad} (w - u) \cdot \text{grad} p_0 dx dy + \sum c_\rho \iint \text{grad} (w - u) \cdot \text{grad} p_\rho dx dy.$$

The summation in the preceding expression is zero according to Eq. (A18). The first term is equal to the boundary integral

$$\int (w - u) \frac{\partial p_0}{\partial \nu} ds,$$

which disappears since  $w = u$  along the exterior edge and  $\frac{\partial p_0}{\partial \nu} = 0$  along  $AB$ .

Thus we have in fact

$$\iint \text{grad}^2 w dx dy = \iint \text{grad}^2 u dx dy - \iint \text{grad}^2 (w - u) dx dy \leq \iint \text{grad}^2 u dx dy. \tag{A23}$$



### APPLICATION: TORSIONAL STIFFNESS OF PRISMATIC BEAMS

In order to illustrate the above we first discuss the simple example of the torsional stiffness of a square cross section of side length  $2a$ . The origin of the coordinate system lies at the centroid of the section. The shear stresses acting in the section plane can be derived from a stress function  $\varphi$  as

$$\tau_y = \frac{\partial \varphi}{\partial x}, \quad \tau_x = -\frac{\partial \varphi}{\partial y}. \quad (\text{A24})$$

Equilibrium along the boundary requires  $\varphi = \text{const}$  or, since the constant is irrelevant,

$$\varphi = 0 \quad (\text{A25})$$

for simplicity. Determination of  $\varphi$  can be cast into the following variational problem: find the function  $\varphi$  with boundary values  $\varphi = 0$  which minimizes the strain energy

$$A = \frac{1}{2G} \iint (\tau_x^2 + \tau_y^2) dx dy = \frac{1}{2G} \iint \text{grad}^2 \varphi dx dy, \quad (\text{A26})$$

for a given torque

$$M = \iint (x\tau_y - y\tau_x) dx dy = \iint \left( x \frac{\partial \varphi}{\partial x} + y \frac{\partial \varphi}{\partial y} \right) dx dy. \quad (\text{A27})$$

Once this variational problem is solved we obtain the twist  $\omega$  from the relation

$$2A = M\omega. \quad (\text{A28})$$

### Ritz Method

Considering symmetry and boundary conditions we select the trial functions

$$\varphi = (x^2 - a^2)(y^2 - a^2) \{ \alpha a^2 + \beta(x^2 + y^2) \}. \quad (\text{A29})$$

Torque and strain energy then become

$$M = \frac{16}{45} a^8 \{ 10\alpha + 4\beta \} \quad (\text{A30})$$

and

$$A = \frac{128a^{12}}{7 \times 5^2 \times 3^8 G} \{ 105\alpha^2 + 72\alpha\beta + 44\beta^2 \}, \quad (\text{A31})$$

respectively. The parameters  $\alpha$  and  $\beta$  that minimize the strain energy for a given moment are

$$\alpha = \frac{74\lambda}{831}, \quad \beta = \frac{15\lambda}{831}, \quad (\text{A32})$$

where

$$\lambda = \frac{45 \times 831 M}{16 \times 800 a^8}, \quad (\text{A33})$$

is a LAGRANGE multiplier. Eliminating  $\lambda$  yields

$$2A = \frac{6 \times 6648 M^2}{700 \times 128 G a^4}. \quad (\text{A34})$$

Clearly, this is an upper bound for  $A$ . Hence

$$2A^* = M\omega < 0.4452 \frac{M^2}{G a^4}, \quad (\text{A35})$$

where the \* - symbol denotes the true strain energy, or

$$\omega < 0.4452 \frac{M}{G a^4}. \quad (\text{A36})$$



**The author's approach**

The minimum potential energy problem is equivalent to solving the differential equation

$$\nabla^2 \varphi = \text{const},$$

with the boundary condition  $\varphi = 0$ . If we let

$$\varphi = C \{a^2 r^2 - u\} \tag{A37}$$

( $C$  is the LAGRANGE multiplier of the constrained variational principle) then we need to determine the potential function  $u$  that takes on the value  $a^2 r^2$  along the boundary. Using symmetry we select a trial function

$$\varphi = C \{a^2 r^2 + \alpha (x^4 - 6x^2 y^2 + y^4)\}. \tag{A38}$$

With  $x^4 - 6x^2 y^2 + y^4 = p$ , the coefficient  $\alpha$  is determined using the condition

$$\int \{a^2 r^2 + \alpha p\} \frac{\partial p}{\partial \nu} ds = 0 \tag{A39}$$

and we obtain

$$\alpha = 7/36. \tag{A40}$$

Hence

$$M = \frac{8 \times 76}{135} C a^6 \tag{A41}$$

and

$$2A = \frac{8 \times 456}{3 \times 135} \frac{C^2 a^8}{G}, \tag{A42}$$

or after eliminating  $C$

$$2A = \frac{135}{304} \frac{M^2}{G a^4}. \tag{A43}$$

This is a lower bound for  $A$ , thus

$$2A^* = M \cdot \omega > 0.4441 \frac{M^2}{G a^4}. \tag{A44}$$

If we take, say,

$$\omega = 0.4446 \frac{M}{G a^4}, \tag{A45}$$

as the approximate value, then the resulting error is less than 0.14%.



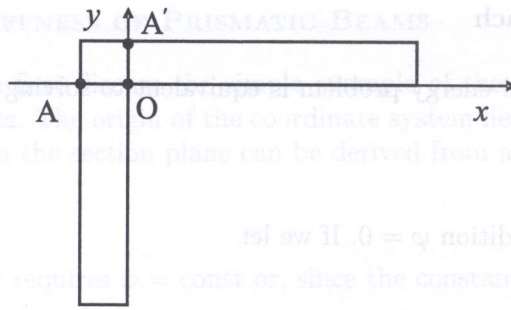


Fig. A2

### The angle

Using the approach described above, we estimate the torsional stiffness of the angle shown in Fig. A2 with a leg aspect ratio of 1:5. The coordinate axes are along the interior edges. For simplicity, we ignore any fillets.

For a given torque, we first determine the strain energy as if the angle consisted of two individual rectangles meeting along  $OA$ . The resulting stress function is continuous and differentiable in both rectangles, but violates the boundary conditions along  $OA$  by taking on zero value along that line. The stress function is thus an admissible function according to the Ritz method and yields an upper bound to the true strain energy. The result is

$$\omega < 0.388 \frac{M}{Gd^4}. \quad (\text{A46})$$

Let us now assume that the domain is composed of two rectangles and a square connected along  $OA$  and  $OA'$ . Determining the stress functions for these sub-domains so that the normal slopes at the interfaces are zero yields a lower bound for  $A$ . For each rectangle, the stress function is equal to that for a rectangle of twice the length. For the square, it is equal to that for a square of twice the side length. The result is

$$\omega > 0.332 \frac{M}{Gd^4}. \quad (\text{A47})$$

Taking as the approximate value the average of the upper and lower bounds we obtain

$$\omega = 0.360 \frac{M}{Gd^4} \quad (\text{A48})$$

and the error is less than 8%. In light of the uncertainty associated with the shear modulus  $G$ , this result is satisfactory. Little effort is required to come up with a better approximation.

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## 1. INTRODUCTION

The finite element method is widely used and accepted as a numerical tool to 'solve' certain classes of problems for which an analytical solution is not available. The solutions thus obtained are, as a matter of rule, approximate and it is therefore important to assess the accuracy of the approximation.

Whatever the approach, the lack of an exact solution implies that only an approximation of the error may be obtained. Usually two different behaviours may be required from these approximations: either they must present asymptotic exactness or they must bound the exact error. From a practical point of view it is generally more important to guarantee that, for a given solution, the true error is smaller than the prescribed tolerance. This may be ensured only if an upper bound of the error is computed, from which an upper and a lower bound of the exact solution may be obtained.

The concept of dual analysis, developed by Fraeijs de Veubeke [7], may be used to obtain a bound for the global error of the solution.

In practice, it is generally more relevant to know the error of the solution at specific points or sub-regions, rather than having a global average error. It is therefore important to formulate techniques which provide this type of information.

Greenberg [8] and Washizu [20] proved that, given two pairs of dual solutions, one pair for the real problem and the other for a virtual action (equivalent unit load for displacements and equivalent unit relative displacement for stresses), dual of the desired local effect, bounds of the exact value of a displacement or a stress at a point can be obtained.

In the context of the finite element method, Yang, Kelly and Isles [22] used Washizu's approach to obtain upper bound estimates of the pointwise error in the finite element method and Albanese [1] used Greenberg's approach for electromagnetic problems.

When the goal is to obtain a more accurate approximation of the local value, but not bound it for this value, it is sufficient to have one solution for the real problem and another for the virtual action. This approach was followed by Babuska [4], who also used the associated element error indicators to drive an adaptive refinement process directed at reducing the local error.