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A multiscale and Trefftz computational method for medium-frequency vibrations of assemblies of heterogeneous plates

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A new approach called the "Variational Theory of Complex Rays" has been developed in order to calculate the vibrations of slightly damped elastic plates in the medium-frequency range. The solution of a small system of equations, which does not result from a fine spatial discretization of the structure, leads to the evaluation of effective quantities (deformation energy, vibration amplitude, ...). Here we extend this approach, which was already validated for assemblies of homogeneous substructures, to the case of heterogeneous substructures.

Keywords: vibrations, medium-frequency range, complex rays, heterogeneous structures

1. Introduction

The modeling and analysis of the vibrational response of elastic structures are undoubtedly among the key issues which appear in the design of satellites or car chassis. Today, at least concerning modeling and analysis, there remain no major difficulties in the low-frequency range, even for complex structures [1]. Concerning high frequencies, computational tools quite distinct from those used for low frequencies do exist, particularly the SEA method [2–5].

Conversely, the modeling and analysis of medium-frequency vibrations, which constitute the topic of this paper, continue to raise certain problems. Difficulties are experienced in extending the SEA method, which is appropriate mostly for high frequencies. In particular, the scale on which space is described is too coarse for medium-frequency analysis since the spatial aspect disappears almost entirely. The difficulty in attempting to extend the low-frequency methods to this case is that the length of variation of the phenomena being studied is very small compared to the characteristic dimension of the structure. Therefore, the finite element calculation involved would require an unreasonable number of degrees of freedom. Apart from these serious numerical difficulties, the quantities involved in the calculation remain associated with small variation lengths which are not very significant. These are not "effective" quantities, i.e. they are not representative of the vibratory response of the structure on the time and space scales considered. Therefore, they lead to results which are very sensitive to imprecisions in the data. Nevertheless, various improved finite element approaches have been studied, in particular in [6–24].

There are few works dedicated specifically to the calculation of medium-frequency vibrations. For rods and beams, the problem was solved, in particular, in [25–27]. For more complex structures such as plates or shells, boundary element methods were proposed in [28] and methods based on the use of specific reduced bases can be found in [29–32]. Most of these works are closely related to finite element methods and the quantities calculated are still not "effective". Therefore, in our opinion, they are not "true" medium-frequency methods. On the contrary, the theory initiated by Belov

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and Ryback [33] is built upon the concepts of "effective energy density" and "effective power flow". However, despite the improvements proposed [34], this theory still encounters theoretical obstacles.

The approach followed here is the Variational Theory of Complex Rays (VTCR), a predictive tool designed specifically to deal with medium-frequency problems, which was introduced by Ladevèze [35]. This approach is a "true" medium-frequency method in the sense that the calculations are performed on "effective" quantities. Previous works already validated this strategy for two-dimensional and for three-dimensional assemblies of elastic homogeneous plates with low damping [36–38]. For example, an assembly of 54 plates is used to model the front part of the chassis of a car, made of steel sheet (Fig. 1). The mechanical properties of the 54 plates are: $E = 210$ GPa; $\eta = 0.001$; $\nu = 0.3$; $\rho = 7\,800$ kg/m^3; $h = 0.8$ mm. The harmonic excitation is a distributed force F_d on the front side frame: $F_d = 1$ N/m; $f = 650$ Hz; $\omega = 2\pi f$ rad/s. Symmetry conditions are prescribed on the appropriate boundaries; the rear boundaries are fixed. w designates the normal deflections.

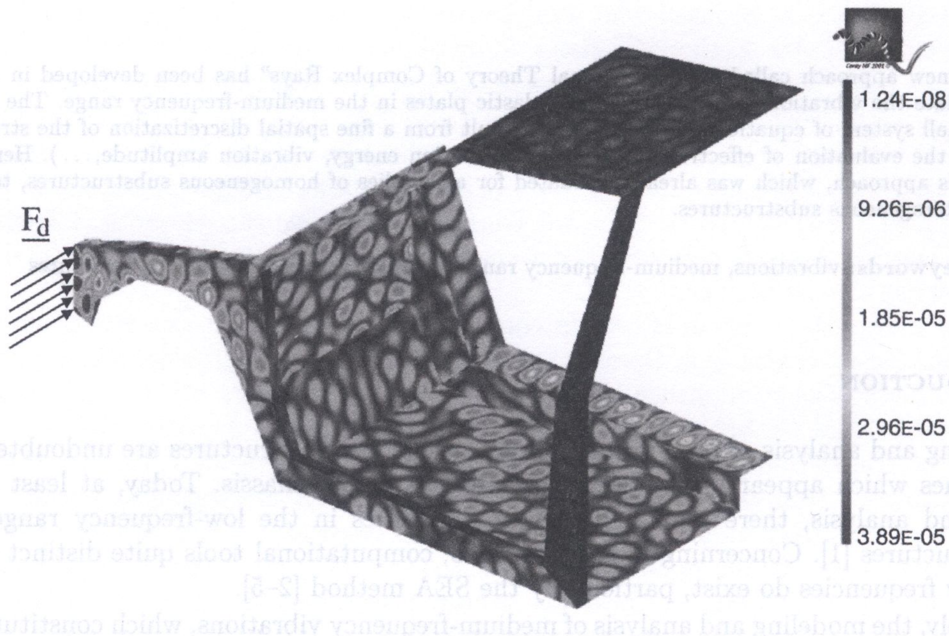

Fig. 1. VTCR solution: $|w|$ (m)

However, most industrial structures present some kind of structural heterogeneity, whether intentional (such as portholes or equipment connections) or unintentional (such as cracks). The vibratory response in the medium-frequency range is very sensitive to these structural discontinuities. While the VTCR was originally based on the assumption that the structure is an assembly of homogeneous substructures, this paper goes one step further and proposes an extension of the VTCR which enables the designer to take heterogeneity into account. The objective is to deal with three-dimensional assemblies of homogeneous and heterogeneous substructures.

2. Basic aspects of the VTCR

In the general case of a structure made of several substructures, the first step of the VTCR consists of associating each substructure with a superelement described by degrees of freedom which correspond to local basic modes, defined on two scales, which satisfy the dynamic equations (local equilibrium and constitutive relation) exactly. These modes are called complex rays. Besides having a strong mechanical meaning, they can be divided into families related to interior, edge or corner zones. In the vicinity of a point, the solution is assumed to be properly described locally as the superimposition of an infinite number of such local vibration modes. The unknowns are large-wavelength quantities, each representing the amplitude of the basic mode with which it is associated.

Next, an appropriate variational formulation enables us to verify, on average, the boundary and transmission conditions. Our choice of reasoning on a superelement level requires the formulation to allow approximations which are *a priori* independent within substructures. These approximations do not necessarily need to verify the transmission conditions *a priori*. In practice, the variational formulation associates each superelement with, on the one hand, an elementary matrix which represents the interaction of the fields with one another and, on the other hand, a right-hand side which represents the interaction between the fields and the boundary conditions. The transmission conditions at the interface are taken into account automatically at the assembly stage.

The last characteristic of the VTCR is that, from the calculated discretized amplitudes, it retains only effective quantities related to the elastic energy, the kinetic energy, the dissipation work, etc...

3. Extension of the VTCR to heterogeneous structures

3.1. The reference problem

Here, we will focus on the construction of the superelement associated with an isolated heterogenous substructure (Fig. 2). Let us consider the steady-state vibrations of a thin, isotropic and elastic Kirchhoff–Love plate with local heterogeneity. Let S be the plate and ∂S its boundary loaded harmonically at a fixed angular frequency ω. Classically, all quantities are defined in the complex domain: an amplitude $\underline{Q}(\underline{X})$ is associated with $\underline{Q}(\underline{X}).\exp(i\omega t)$. The boundary conditions are the deflection w^d on part $\partial_{wd}S$ of ∂S; the slope w_n^d on $\partial_{wnd}S$; the bending moment M^d on $\partial_{Md}S$ and the Kirchhoff shear K^d on $\partial_{Kd}S$. Each type of heterogeneity results in specific limitations at its boundary. In order to test the strategy, let us consider the case of a plate with an arbitrarily-shaped hole. Its boundary Γ_0 is assumed to be free.

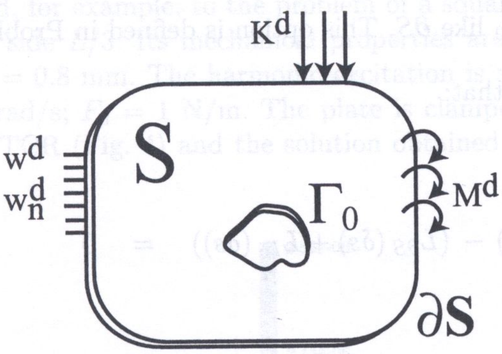

Fig. 2. The reference problem

Let us introduce for S the space $\mathcal{S}_{ad}$ of the displacement-stress pairs $s = (w, \mathbf{M})$ such that:

$$s \in \mathcal{S}_{ad} \quad \Leftrightarrow \quad \begin{cases} s \in \mathcal{U} \times \mathcal{S} \quad \text{(set of finite-energy fields)}, \\ \Delta\Delta w - k^4 w = 0 \quad \text{on } S, \\ \mathbf{M}_l = \dfrac{2h^3}{3}(1 + i\eta)\, \mathbf{K}_{PS}\, \mathcal{X}(w) \end{cases} \tag{1}$$

with

$$k^4 = \frac{3\rho\omega^2(1-\nu^2)}{(1+i\eta)Eh^2}$$

The reference problem is defined in Problem 1.

Problem 1. Find s such that:

$$\begin{array}{rclll} s & \in & \mathcal{S}_{ad} & & \\ w & = & w^d & \text{on} & \partial_{wd}S \\ w_{,n} & = & w_n^d & \text{on} & \partial_{wnd}S \\ M_n & = & M^d & \text{on} & \partial_{Md}S \\ K_n & = & K^d & \text{on} & \partial_{Kd}S \\ M_n & = & M^d = 0 & \text{on} & \Gamma_0 \\ K_n & = & K^d = 0 & \text{on} & \Gamma_0 \end{array} \tag{2}$$

with

$$M_n = \underline{n}\mathbf{M}\underline{n}$$

and

$$K_n = \underline{n}\,\underline{div}[\mathbf{M}] + (\underline{t}\mathbf{M}\underline{n})_{,t}.$$

3.2. Direct application of the VTCR

3.2.1. The variational formulation associated with the VTCR

The first option is to treat Γ_0 like ∂S. This option is defined in Problem 2.

Problem 2. Find s such that:

$$\begin{array}{l} s \in \mathcal{S}_{ad} \\ \left(A_{\partial S}(\delta s, s) + A_{\Gamma_0}(\delta s, s)\right) - \left(L_{\partial S}(\delta s) + L_{\Gamma_0}(\delta s)\right) \quad = \quad 0 \\ \forall\, \delta s \in \mathcal{S}_{ad} \end{array} \tag{3}$$

with:

$$A_{\partial S}(\delta s, s) - L_{\partial S}(\delta s) = \mathrm{Re}\Big\{ -i\omega\Big(-\int_{\partial_{wd}S} \delta K_n\,(w-w^d)^*\,\mathrm{d}L + \int_{\partial_{wnd}S} \delta\underline{n}\mathbf{M}\underline{n}\,(w_{,n}-w^d_{,n})^*\,\mathrm{d}L + \int_{\partial_{Md}S} (\underline{n}\mathbf{M}\underline{n} - M^d)\,\delta w_n^*\,\mathrm{d}L - \int_{\partial_{Kd}S} (K_n - K^d)\,\delta w^*\,\mathrm{d}L\Big)\Big\}$$

and

$$A_{\Gamma_0}(\delta s, s) - L_{\Gamma_0}(\delta s) = \mathrm{Re}\left\{-i\omega\left(\int_{\Gamma_0} \underline{n}\mathbf{M}\underline{n}\,\delta w_n^*\,\mathrm{d}L - \int_{\Gamma_0} K_n\,\delta w^*\,\mathrm{d}L\right)\right\}.$$

Note that the terms which appear in this formulation represent powers.

3.2.2. *Construction of admissible fields*

In practice, the first stage of the VTCR consists of building admissible fields. Let us define a subset of S_{ad} which can contain local interior vibration modes described using "complex rays of the n^{th} order". If the damping coefficient η ($\eta \ll 1$) is small, the basic mode can be written as:

$$\begin{cases} w(\underline{X},\underline{P}) = e^{i\sqrt{\omega}\underline{P}.\underline{X}}\, e^{\frac{\eta}{4}\sqrt{\omega}\underline{P}.\underline{X}}\,\mathbf{V}(\underline{X},\underline{P})\,[\underline{a}(\underline{P})] = \mathbf{W}(\underline{X},\underline{P})\,[\underline{a}(\underline{P})], \\ \mathbf{M}(\underline{X},\underline{P}) = (1+i\eta)\mathbf{K}_{PS}\mathfrak{X}(w(\underline{X},\underline{P})), \\ \underline{P} \in \mathfrak{C}. \end{cases}$$

Each mode is associated with a wave vector $\underline{P}$. The admissibility relation requires that the locus of the end of $\underline{P}$ be a curve characterizing the material. $\mathbf{V}(\underline{X},\underline{P})\,[\underline{a}(\underline{P})]$ is a polynomial expression of degree n in $\underline{X}$. The small-length part appears explicitly. The unknown generalized amplitudes $[\underline{a}(\underline{P})]$ do not depend on $\underline{X}$ and, therefore, are large-length quantities. The edge and corner modes are built in the same way.

3.2.3. *The discretized form of the VTCR*

In order to derive approximations from the VTCR, one discretizes the curve $\mathfrak{C}$ using finite elements; for example, $\underline{P} \to a(\underline{P})$ is assumed to be constant in each element. In other terms, $\underline{a}^h$ is associated with $\mathfrak{C}^r$ (i.e. $\mathfrak{C}$ discretized with r elements). The subspace $\mathcal{S}^h_{ad}$ of the approximations deduced from $\mathcal{S}_{ad}$ yields a discretized variational formulation leading to a system of equations of dimension r in the complex domain:

$$\left[\mathbf{A}^h\right]\left[\underline{a}^h\right] = \left[\underline{L}^h_d\right]. \tag{4}$$

3.2.4. *Example*

Such a method can be applied, for example, to the problem of a square steel plate of side $L = 1$ m with a center square hole of side $L/3$. Its mechanical properties are: $E = 210$ GPa; $\eta = 0.001$; $\nu = 0.3$; $\rho = 7800$ kg/m^3; $h = 0.8$ mm. The harmonic excitation is a distributed force F_d on one side: $f = 400$ Hz; $\omega = 2\pi f$ rad/s; $F_d = 1$ N/m. The plate is clamped on the opposite side. The solution obtained with the VTCR (Fig. 4) and the solution obtained with finite elements (Fig. 3) are very similar.

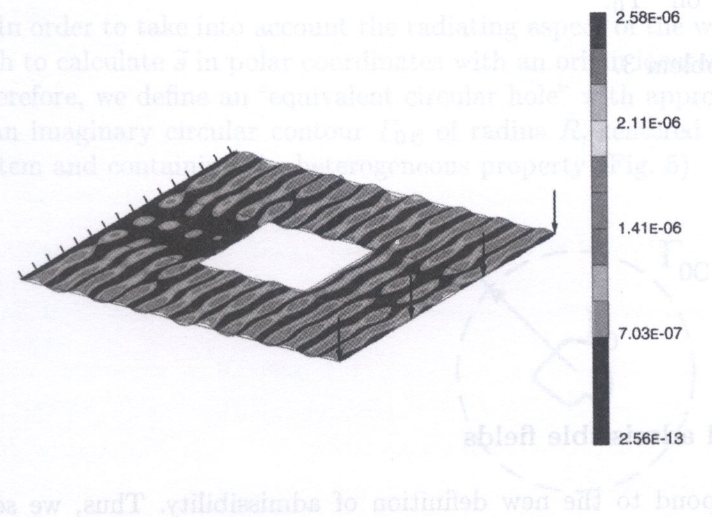

Fig. 3. Finite element solution: $|w|$ (m)

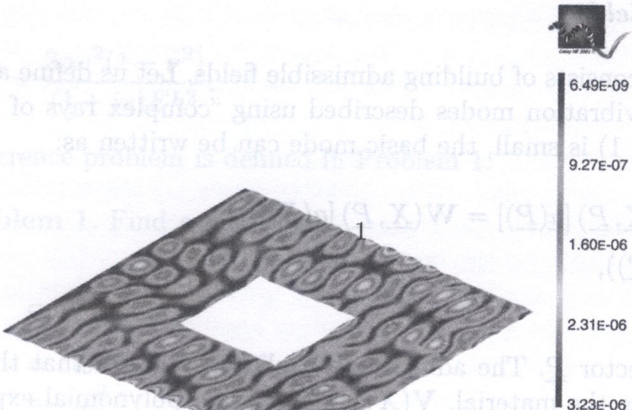

Fig. 4. VTCR solution: $|w|$ (m)

However, the influence of heterogeneity is taken into account through the terms integrated on Γ_0, whereas we would like the treatment of local heterogeneity to be distinguished from that of the master structure. Indeed, from a mechanical point of view, the hole creates only a perturbation of the vibratory response. One should note that this interpretation is not valid for all types of heterogeneity. For example, a long stiffener would behave differently. The case of stiffeners will be developed in a subsequent paper.

3.3. Adaptation of the VTCR to the treatment of local heterogeneity

The adaptation consists of taking the perturbation due to local heterogeneity into account as soon as the admissible fields of $\mathcal{S}_{ad}$ have been built. The purpose is to develop *a priori*, and only once, families of admissible fields which satisfy the boundary conditions resulting from each type of heterogeneity. Therefore, let us define the set $\mathcal{S}^c_{ad}$ (i.e. corrected $\mathcal{S}_{ad}$) of the pairs s such that:

$$s \in \mathcal{S}^c_{ad} \quad \Leftrightarrow \quad \begin{cases} s \in \mathcal{S}_{ad}, \\ M_n = M^d = 0 \quad \text{on} \quad \Gamma_0, \\ K_n = K^d = 0 \quad \text{on} \quad \Gamma_0. \end{cases} \tag{5}$$

Then, the problem is defined in Problem 3.

Problem 3. Find s such that:

$$\begin{aligned} & s \in \mathcal{S}^c_{ad}, \\ & A_{\partial S}(\delta s, s) - L_{\partial S}(\delta s) = 0, \\ & \forall\, \delta s \in \mathcal{S}^c_{ad}. \end{aligned} \tag{6}$$

3.4. Construction of the corrected admissible fields

We want to build fields which correspond to the new definition of admissibility. Thus, we seek $s^c \in \mathcal{S}^c_{ad}$ among the pairs $\bar{s}$ admissible in the sense of $\mathcal{S}_{ad}$ which already exist, to which we apply a correction $\widetilde{s}$ so that s^c verifies the boundary conditions resulting from heterogeneity. Of course, this correction must belong to $\mathcal{S}_{ad}$ also. One can view each s^c as the superimposition of an incident field $\overline{s}$ and a diffracted field $\widetilde{s}$ due to heterogeneity:

$$(w^c, \mathbf{M}^c) = (\overline{w}, \overline{\mathbf{M}}) + (\widetilde{w}, \widetilde{\mathbf{M}}) = \overline{\mathbf{W}}(\underline{X}, \underline{P})\,[\overline{a}(\underline{P})] + \widetilde{\mathbf{W}}(\underline{X}, \underline{P})\,[\overline{a}(\underline{P})]. \tag{7}$$

The fields s^c of $\mathcal{S}^c_{ad}$ are deduced from those of $\mathcal{S}_{ad}$:

$$s^c \in \mathcal{S}^c_{ad} \;\Leftrightarrow\; \begin{cases} w^c(\underline{X}, \underline{P}) &= (\overline{\mathbf{W}}(\underline{X}, \underline{P}) + [\mathbf{H}(\underline{X})]\,[\mathbf{C}_\Gamma(\overline{\mathbf{W}}(\underline{X}, \underline{P}))])[\overline{a}(\underline{P})], \\ \mathbf{M}^c(\underline{X}, \underline{P}) &= (1 + i\eta)\,\mathbf{K}_{PS}\,\mathcal{X}(w^c), \end{cases} \tag{8}$$

with:

$$\widetilde{\mathbf{W}}(\underline{X}, \underline{P}) = [\mathbf{H}(\underline{X})], \tag{9}$$

$$[\widetilde{a}(\underline{P})] = [\mathbf{C}_\Gamma(\overline{\mathbf{W}}(\underline{X}, \underline{P}))]\;[\overline{a}(\underline{P})]. \tag{10}$$

Thus, the correction associated with each $\overline{s}$ is built individually as a linear combination of functions H, which are additional basic modes. These modes are weighted by coefficients C_Γ adjusted to the incident field. In the final calculation, each correction function enriches the basic modes and the associated generalized amplitude becomes a new unknown connected to the amplitudes associated with $\overline{s}$.

3.5. Example of the construction of corrected admissible fields

In the case of the reference problem, corrections $\widetilde{s}(\underline{X}, \underline{P})$ associated with an incident field of unit amplitude $\overline{s}(\underline{X}, \underline{P})$ are defined in Problems 4 and 5.

Problem 4. Find $\widetilde{s}$ such that:

$$\begin{aligned} &\widetilde{s} \in \mathcal{S}_{ad}, \\ &\widetilde{M}_n = -\overline{M}_n|_{\Gamma_0}, \\ &\widetilde{K}_n = -\overline{K}_n|_{\Gamma_0}. \end{aligned} \tag{11}$$

In order to take into account the radiating aspect of the wave diffracted due to heterogeneity, we wish to calculate $\widetilde{s}$ in polar coordinates with an origin located at the "center of gravity" of the hole. Therefore, we define an "equivalent circular hole" with appropriate boundary conditions. This hole is an imaginary circular contour $\Gamma_{0\,\mathrm{C}}$ of radius R, centered at the origin of the polar coordinates system and containing the heterogeneous property (Fig. 5).

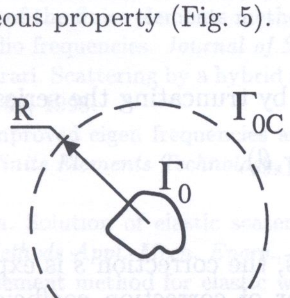

Fig. 5. Definition of the imaginary circular contour

Then, $\widetilde{w}$ is sought in the classical form:

$$\widetilde{w}(r,\theta) = \sum_{\substack{m\in\mathbf{Z}\\ f=1,2}} C_{fm}(\overline{w})\, H_{fm}(r,\theta), \tag{12}$$

with

$$H_{fm} = H_m^{(f)}(\alpha_f kr)\, e^{im\theta} \quad \text{and} \quad \alpha_1 = i\,,\ \alpha_2 = 1, \tag{13}$$

where $H_m^{(1)}$ and $H_m^{(2)}$ denote the Hankel functions of the first and second kind respectively. The solution is a combination of an outbound propagating wave and a boundary wave which decreases rapidly. Both waves are emitted by Γ_0 and neither reflects at infinity. The coefficients C_{fm} are determined through the boundary conditions on Γ_0, which are relocated on $\Gamma_{0\,\mathrm{e}}$. These relocated boundary conditions are the restrictions to the contour $\Gamma_{0\,\mathrm{e}}$ of the solution to Problem 4. In fact, outside $\Gamma_{0\,\mathrm{e}}$, the solution to Problem 4 is the same as the solution to Problem 5.

Problem 5. Find $\widetilde{s}$ such that:

$$\begin{aligned} \widetilde{s} &\in \mathcal{S}_{ad}, \\ \widetilde{w} &= \widetilde{w}^d|_{\Gamma_{0\,\mathrm{e}}}, \\ \widetilde{w}_{,n} &= \widetilde{w}^d_n|_{\Gamma_{0\,\mathrm{e}}}. \end{aligned} \tag{14}$$

Problem 4 is solved by a boundary element method, e.g. [39], or an extension of the VTCR to unbounded media which is currently under development. Then, $\widetilde{w}^d|_{\Gamma_\mathrm{e}}$ and $\widetilde{w}^d_n|_{\Gamma_\mathrm{e}}$ are expanded into Fourier series:

$$\begin{aligned} \widetilde{w}^d(R,\theta) &= \textstyle\sum_{m\in\mathbf{Z}} \widetilde{w}^{dm}(R)\, e^{im\theta}, \\ \widetilde{w}^d_{,n}(R,\theta) &= \textstyle\sum_{m\in\mathbf{Z}} \widetilde{w}^{dm}_{,n}(R)\, e^{im\theta}, \end{aligned} \tag{15}$$

with

$$\widetilde{w}^{dm}(R) = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta}\,\widetilde{w}^d(R,\theta)\,\mathrm{d}\theta\,, \qquad \widetilde{w}^{dm}_{,n}(R) = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta}\,\widetilde{w}^d_{,n}(R,\theta)\,\mathrm{d}\theta\,. \tag{16}$$

The coefficients C_{fm} are calculated by inverting linear systems of dimension 2:

$$\begin{aligned} C_{1m}H_m^{(1)}(ikr) &+ C_{2m}H_m^{(2)}(kr) = \widetilde{w}^{dm}(R), \\ C_{1m}H_m^{(1)\prime}(ikr) &+ C_{2m}H_m^{(2)\prime}(kr) = \widetilde{w}^{dm}_{,n}(R), \end{aligned} \tag{17}$$

or

$$[\mathcal{H}_m][\underline{C_m}(\overline{w})] = [\underline{F_{dm}}(\overline{w})]. \tag{18}$$

One obtains an approximate solution by truncating the series, i.e.:

$$\widetilde{w}(r,\theta) \simeq \sum_{\substack{m\in\mathbf{Z}\,,\,|m|\le M\\ f=1,2}} C_{fm}(\overline{w})\, H_{fm}(r,\theta), \tag{19}$$

with $|m| \le M$. For any incident pair $\overline{s}$, the correction $\widetilde{s}$ is expressed on the same basis of $(2M+1)$ local fields H_{fm}. Let $\underline{C}$ be the vector of correction coefficients associated with the vector $\underline{H}$ of functions H_{fm}. Then, $\widetilde{w}(r,\theta)$ can be expressed as:

$$\widetilde{w}(r,\theta) \simeq [\underline{C}(\overline{w})]^{\mathrm{T}}[\underline{H}(r,\theta)], \tag{20}$$

and $\underline{C}$ is the solution to the following linear system, which results from the assembly of systems of the same type as 18:

$$[\mathcal{H}][\underline{C}(\overline{w})] = [\underline{F_d}(\overline{w})]. \tag{21}$$

Then, all the $\underline{C}$s obtained while correcting each incident field are assembled to build the operator $\mathbf{C}_{\Gamma}$ defined in 3.4.

In short, the superelements corresponding to heterogeneous substructures include a family of additional basic modes weighted in order to verify conditions on the boundary of the discontinuity *a priori*. The weights for ranges of nondimensional parameters corresponding to each type of local heterogeneity are processed and stored in advance. This approach can be generalized to two or more zones containing local heterogeneities.

4. Conclusion

The "Variational Theory of Complex Rays", initially introduced to calculate the vibrations of slightly damped elastic plates in the medium-frequency range, is a very general approach with a strong mechanical basis. This tool is not only efficient for assemblies of homogeneous plates, but also flexible enough to deal with locally heterogeneous substructures economically.

In the near future, the capabilities of the VTCR will be extended in several directions e.g. from plates to shells and from local heterogeneity to more complex cases such as multi-stiffened substructures. In addition, an approach will be proposed which will be appropriate for dealing with wide-band frequency excitations.

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