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Solving wave problems in infinite domain by using variable local DtN operators

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This paper presents an iterative method for solving two-dimensional wave problems in infinite domains. The method yields a solution that satisfies Sommerfeld's radiation condition, as required for the correct solution of infinite domains excited only locally. This problem occurs in the solution of the wave equation in infinite domains when using an asymptotic local DtN (Dirichlet-to-Neumann) map in computational procedures applied to a finite domain. We are demonstrating that the amplitudes of the reflected fictive harmonics depend upon the wave number, the location of the fictive boundary, as well as on the DtN operator used in the computations. A constant value of the operator cannot sufficiently eliminate the amplitudes of all reflected waves, while the results are poor especially for higher harmonics. Thus, we are proposing an iterative method, which varies the tangential dependence of the operator in each computational step.

Keywords: wave motion, infinite domains, fictive boundary, radiation condition, DtN operators

1. Introduction

In solving wave problems in infinite domains the main problem is to satisfy the Sommerfeld's radiation condition – the boundary condition at infinity. The radiation condition is satisfied automatically as a part of the fundamental solution in the *boundary element method*. Unfortunately, the fundamental solution is not always available. Although the boundary element method is regarded as the most powerful procedure for modelling the unbounded medium, it requires a strong analytical and numerical background.

In several wave motion problems in infinite domains it is more convenient, or even necessary, to solve the problem only in a *finite computational domain* analytically, or numerically by finite difference, finite element or finite volume methods. To obtain the finite computational domain (Ω_f) as an interior boundary value problem, the infinite domain must be truncated by introducing a fictive finite boundary (β), see Fig. 1. As a consequence spurious reflections of waves from β are obtained. In order to diminish these reflections various authors have devised improved boundary conditions on β.

An idea is to use the *Dirichlet-to-Neumann (DtN)* map on the artificial fictive boundary approximated by a series of Hankel functions. Bayliss and Turkel [1] used the asymptotic expansion of displacements valid for the field far from fictive boundary, which yielded similar to approximate local boundary conditions. Keller and Givoli [2] and Givoli and Keller [3] obtained exact non-reflecting boundary conditions on β, which totally eliminate all reflections. Porat and Givoli [4] obtained solutions of the Helmholtz equation in elliptic coordinates by involving the so-called Mathieu functions. This approach is applicable when choose elliptic artificial boundaries. Givoli and Patlashenko [5] developed a systematic way to derive optimal local Non-Reflecting Boundary Condition (NRBC) of given order. The optimal NRBC may be of low order but still yield some high-order modes in the solution. Some authors developed special finite elements for exterior problems of wave propagation – [6] and [7]. Givoli [8] presented the state-of-the-art review of the standard DtN methods. He concentrates on two major recent advances: (a) the extension of the DtN finite element method to non-linear elliptic and hyperbolic problems; (b) procedures for localizing the non-local DtN map leading to a family of finite element schemes with local artificial boundary conditions. Pinsky and Thompson [9] used approximate local boundary conditions from [1] in the finite element formulation in order to solve two-dimensional time-dependent structural acoustic problem. Thompson and Pinsky [10] did the same for three-dimensional problems.

In all these *standard DtN methods* the Dirichlet to Neumann operator is introduced into the finite elements on the artificial finite boundary. The consequence is that difficulties with continuity between elements on the artificial boundary may occur when high-order local operators are used. In these cases Galerkin's discontinuous finite element formulation is used in order to get good results. Hohage et al. [11] presented a new efficient algorithm for the solution of direct time-harmonic scattering problems based on the Laplace transform. The starting point in the method is an alternative characterization of outgoing waves called pole condition, which is equivalent to Sommerfeld's radiation condition for problems with radially symmetric potentials. Furthered representations of the formula based on the pole condition are presented in [12].

Aiello et al. [13] presented a new iterative procedure for solving electrostatic problems in infinite domains. In their method they used Green's function to obtain various Dirichlet and Neumann boundary conditions. Unfortunately, the form of Green's function can be very complicated for certain mechanical problems or may not exist at all in an analytical form.

The aim of this paper is to solve the problem of wave motion in infinite domains when a solution is obtained with the DtN map on the inserted fictive boundary. In Sec. 2 the idea for an improved solution, achieved by an iterative variation of DtN operator, is presented. Some numerical examples with comparison in results obtained by the standard DtN method are given in Sec. 3.

2. Theoretical background

The method yields an iterative solution for solving wave problems in infinite domains. The infinite domain, which represents the exterior boundary value problem, is first truncated by introducing a fictive finite boundary (β). The actual system is mathematically divided into two subsystems (Fig. 1):

- an infinite domain outside of the fictive boundary (exterior boundary value problem),
- a finite computational domain (Ω_f) (interior boundary value problem), bounded with obstacle (actual boundary Γ) and fictive boundary β.

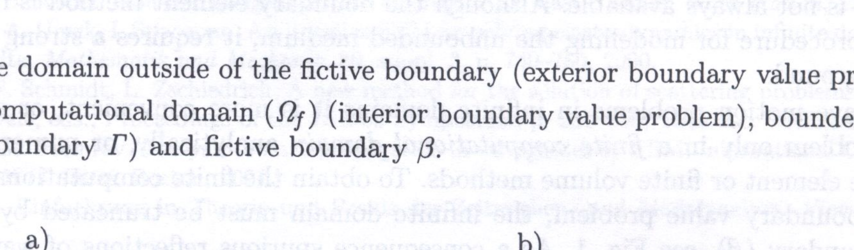


Fig. 1. a) Exterior boundary value problem; b) Interior boundary value problem

2.1. Infinite domain outside of the artificial boundary

The radiation condition in a space outside of the fictive boundary can be represented with DtN (Dirichlet-to-Neumann) operator:

$$\frac{\partial u}{\partial n} = \mathrm{DtN} \cdot \mathrm{u}\,; \quad \text{on } \beta. \tag{1}$$

In Eq. (1), $u(x)$ is the unknown displacement field and n is the outward normal on the fictive boundary. An exact operator (DtN) is the normal derivative of the displacements on the fictive boundary providing that the displacements exactly satisfy radiation conditions. The operator yields to the given displacements the belonging stresses, thus Diriclet-to-Neumann boundary conditions. When an exact operator is used the radiation conditions are exactly satisfied. In general, the exact formulation of the operator cannot be represented in an explicit analytical form, but only in an integral form and is usually not simple enough to be used in the finite element formulation on the fictive boundary. Thus, the use of a local operator usually approximates a non-local one.

To obtain local operators, some asymptotic expressions for Hankel functions must be introduced with respect to the independent value of the product (kR). Thus the accuracy of operators depends on the location of the fictive boundary (R) and on the considered wave number (k). Asymptotic local operators obtained by Bayliss and Turkel [1] are expressed in the following forms, starting from the crudest approximation to more exact ones:

$$S_{3/4} = -i \cdot k, \tag{2}$$

$$S_1 = S_{3/4} - \frac{1}{2R}, \tag{3}$$

$$S_2 = S_1 + \frac{1 - i\,k\,R}{8R\,(1 + k^2R^2)} + \left(\frac{1 - i\,k\,R}{2R\,(1 + k^2R^2)}\right) D, \tag{4}$$

$$\begin{aligned} S_3 = S_1 + \frac{2}{R}\left(\frac{46 - 73\,ikR + 16k^2R^2 - 16\,ik^3R^3}{529 + 1568\,k^2R^2 + 256\,k^4R^4}\right) \\ + \left(\frac{69 - 98\,ikR + 48\,k^2R^2 - 322\,ik^3R^3}{529 + 1568\,k^2R^2 + 256\,k^4R^4}\right)\cdot\left(\frac{6}{R}\right)\cdot D, \qquad D = \frac{\partial^2}{\partial\varphi^2}. \end{aligned} \tag{5}$$

The simplest operator $S_{3/4}$ is called Sommerfeld's operator. The operator S_1 is the axial symmetric operator, which does not depend on the tangential coordinate (φ), while the non-symmetric operators S_2 and S_3 do.

2.2. Finite computational domain

In the presented method an iterative procedure for solving the wave equation in the finite domain is proposed. The finite computational domain Ω_f is subjected in each iteration to actual (the same) boundary conditions on Γ and to various ones on the fictive boundary (Dirichlet or Neumann). The modified wave equation with starting boundary conditions is in the first computational step of the iteration in the form:

$$\nabla^2 u + k^2 u = 0\,; \qquad u = u\,(ka) \text{ on } \Gamma\,, \qquad u = FDBC \text{ on } \beta \tag{6}$$

$FDBC$ represents fictive Dirichlet boundary conditions on the fictive boundary (β). These conditions may be completely arbitrary. The simplest are the zero displacements ($FDBC = 0$), but the method needs less computing steps when choosing $FDBC$, which are similar to radiation conditions. Normal derivatives on β as a result of $FDBC$ must first be computed. While the interior boundary value problem is discussed, a general solution is composed of the two terms in the form:

$$\frac{\partial u}{\partial n} = \sum_{m=0}^{N-1} \Big[A_{1m} \cdot \Big(-\frac{m}{R} \cdot H_m^{(1)}(kR) + k \cdot H_{m-1}^{(1)}(kR) \Big) + A_{2m} \cdot \Big(-\frac{m}{R} \cdot H_m^{(2)}(kR) + k \cdot H_{m-1}^{(2)}(kR) \Big) \Big]. \tag{7}$$

In the solution N is the total number of the considered harmonics and $H_m^{(1)}(kr)$ and $H_m^{(2)}(kr)$ are Hankel functions of the first and the second kind and order m according to the independent value kR. The constants A_{1m} and A_{2m} are the amplitudes of the waves propagating inwards and outwards of the considered domain, respectively. They depend on the prescribed boundary conditions, on the selected asymptotic local operator (S_N) and also on the starting $FDBC$. As the asymptotic local operators are not exact, spurious reflections from the fictive boundary are obtained. They are physically represented with the amplitudes A_{1m}.

In the second computational step the displacements on β are obtained due to fictive Neumann boundary conditions ($FNBC$) on the fictive boundary. This step yields the second "point". For the choice of $FNBC$ applies the equivalent comment as for the $FDBC$ mentioned above. Of course, the finite computational domain and the exciting conditions are the same in all computational steps.

Thus, in the first two steps of the iteration procedure we have for every point on the fictive boundary a pair of values consisting of obtained displacements and derivatives. For the sake of the explanation of the method, we shall call these pairs "*the points*" *(P1 and P2)*. In the graph, with the displacements on the abscissa ($P1$) and the derivatives on the ordinate axis ($P2$), we shall present them symbolically as a single point (Fig. 2). It is worth noting that the "point" is complex.

In the third step of the iteration procedure the two above computed "points" yields a line $y^{(i)}$ (Fig. 2), while the DtN operator, which is usually in the asymptotic local form, is symbolically presented by another line (S_N). The solution of both lines is symbolically presented by the point $A^{(i)}$. This completes the first iteration, which yields new $FDBC$ and $FNBC$ to start the next one.

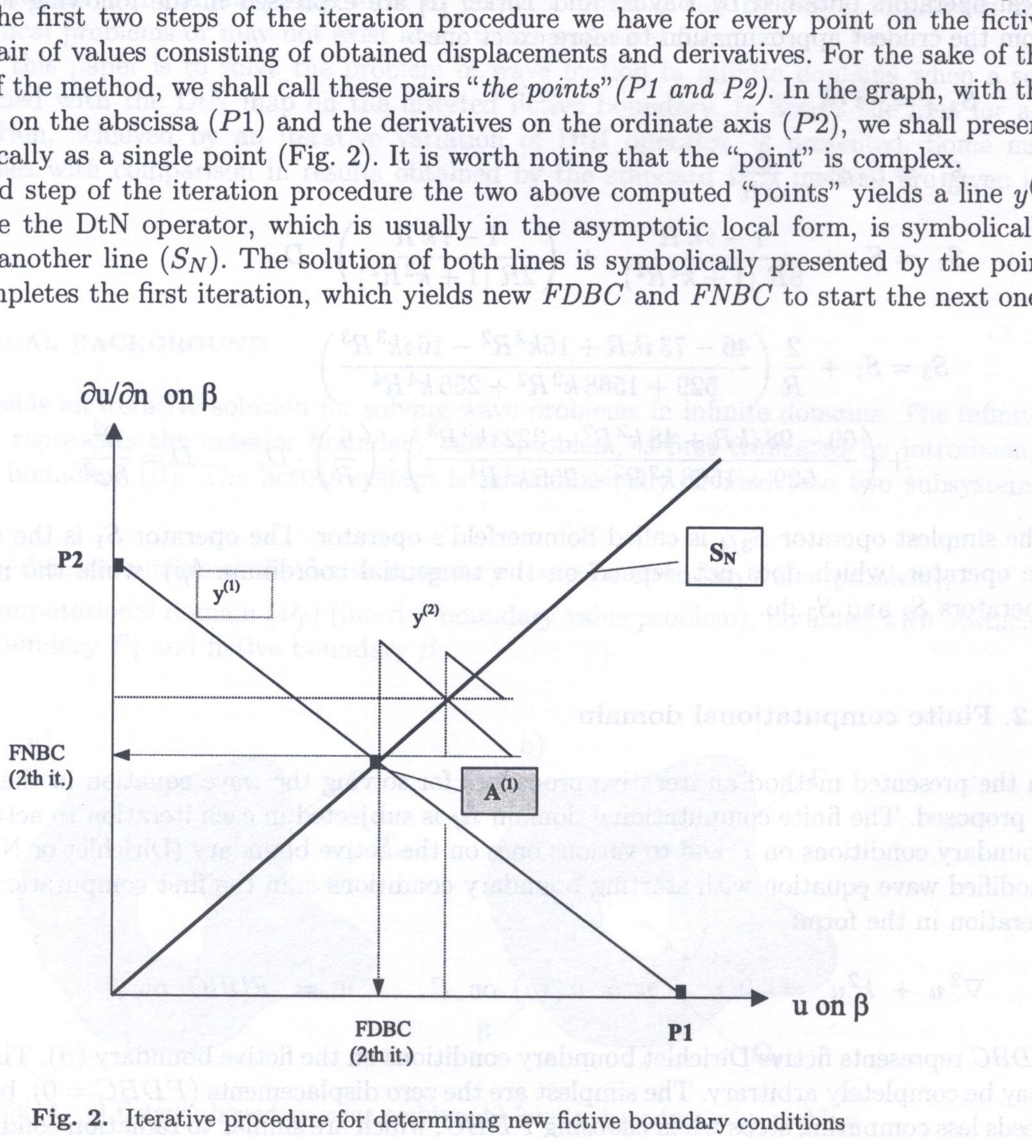


Fig. 2. Iterative procedure for determining new fictive boundary conditions

The lines $y^{(i)}$ and "points" $A^{(i)}, i = 1, 2, \ldots$, in Fig. 2 are only a symbolic graphic presentation of equations and their solutions in each iteration, because the arguments are complex functions, or in case of FEM complex vectors. Thus the graph suggests the idea of the method only symbolically. The operator S_N is complex and is simply the proportionality factor between the normal derivative and the displacement. It can be considered formally as a tangent function of a complex angle.

The examples presented later suggest that the method has several advantages:

- As the method is iterative, the iterations render the possibility to change the operator in each step of the iteration in order to eliminate various harmonics comprised in the waves reflected from the fictive boundary. Thus, although the iterations require more computations, they improve the results at the same time. However, we can achieve considerable accuracy of the results already in a couple of iterations providing we start with values on the fictive boundary, which at least poorly approximate the outgoing waves.
- The method does not include the DtN operator into the dynamic stiffness matrix, unlike the standard DtN methods. The advantage is that the problem of the continuity conditions of higher harmonics, which may occur between the finite elements, is not present at all. Else, there is no need for special finite elements on the fictive boundary as opposed to the standard DtN methods.
- Finally, the method can simply be employed in the standard computer programs for standard finite element method. It requires only some additional simple mathematical manipulations, which are in principle simple to integrate it into the computer program.

3. Numerical example

Consider a simple out-of-plane wave motion in a full space with a circular hole, having a radius a (Fig. 3).

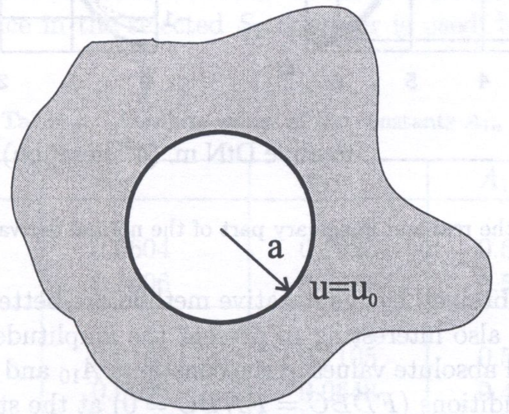


Fig. 3. Space with a hole

Wave number is k. Wave motion in a whole space ($r > a$) is described by a differential equation of Helmholtz type:

$$\nabla^2 u + k^2 u = 0 . \tag{8}$$

We are supposing that we have prescribed harmonic Dirichlet boundary conditions on the hole:

$$u(ka) = \sum u_0 \cdot \cos(m\varphi), \qquad m = 0, 1, \ldots, N-1 . \tag{9}$$

3.1. Approximate solution obtained by iterative DtN method

The interior boundary value problem solution is in form of Eq. (7) but with respect to the prescribed actual boundary conditions (Eq. (9)):

$$\frac{\partial u}{\partial n} = \sum_{m=0}^{N-1} \left[A_{1m} \cdot \left(-\frac{m}{R} \cdot H_m^{(1)}(kR) + k \cdot H_{m-1}^{(1)}(kR) \right) + A_{2m} \cdot \left(-\frac{m}{R} \cdot H_m^{(2)}(kR) + k \cdot H_{m-1}^{(2)}(kR) \right) \right] \cdot \cos(m\varphi). \tag{10}$$

We will discuss two different numerical examples:

a) with two considered harmonics ($N = 2$),
b) and with first four considered harmonics ($N = 4$).

a) N = 2; Discuss the numerical example with $k = 1.0$ and $a = 1.0$. The fictive boundary is located very close to the origin ($R = 1.5$). Figure 4 shows the solutions for the real and the imaginary part of the normal derivative obtained with the iterative method in the fourth iteration and the with the standard DtN method. The same non-symmetric operator S_2 with $D = -\mathrm{Cos}(1\phi)$ is used by the both methods.

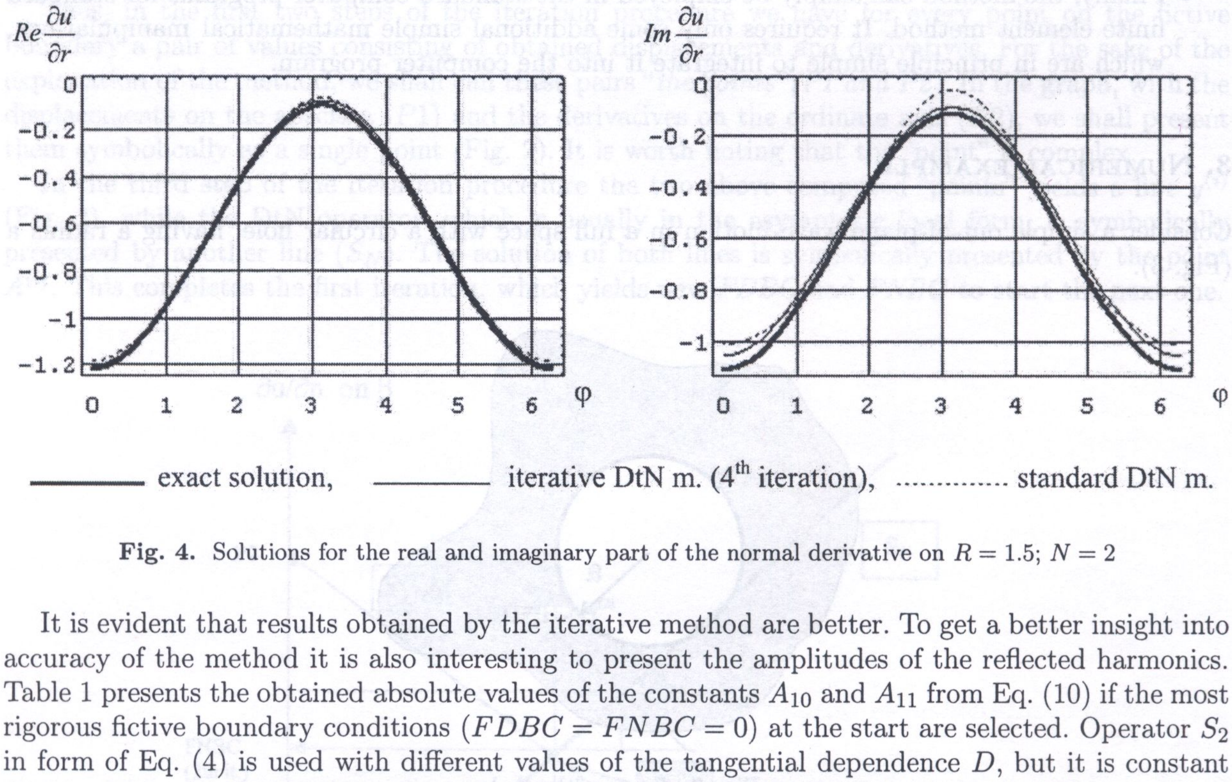


Fig. 4. Solutions for the real and imaginary part of the normal derivative on $R = 1.5$; $N = 2$

It is evident that results obtained by the iterative method are better. To get a better insight into accuracy of the method it is also interesting to present the amplitudes of the reflected harmonics. Table 1 presents the obtained absolute values of the constants A_{10} and A_{11} from Eq. (10) if the most rigorous fictive boundary conditions ($FDBC = FNBC = 0$) at the start are selected. Operator S_2 in form of Eq. (4) is used with different values of the tangential dependence D, but it is constant during iterations. Some solutions with another $FDBC$ and $FNBC$ can be found in [14].

It is evident that by using the symmetric form of the operator ($D = 0$) we have problems with the second (non-symmetric) harmonic. By introducing the tangential dependence with $D = -\mathrm{Cos}(1\phi)$ we can improve the accuracy of the second harmonic, but then the reflections of the first (symmetric) harmonic are bigger. We explained in Sec. 2 that it is in the iterative method possible to change the operator during the iterations. The results of the case with the variable D are presented in Table 2. The tangential dependence is changed in the following form:

- $D = 0$ in each odd iteration – symmetric form,
- $D = -\mathrm{Cos}(1\phi)$ in each even iteration – non-symmetric form.

Table 1. Absolute values of the constants A_{10} and A_{11}

	A_{10}		A_{11}	
	$D = 0$	$D = -\text{Cos}(1\phi)$	$D = 0$	$D = -\text{Cos}(1\phi)$
2. iteration	4.516E−3	0.0526	0.1160	0.0119
3. iteration	4.614E−3	0.0579	0.1102	0.0110
4. iteration	1.989E−3	0.0471	0.1141	0.0174

Table 2. Absolute values of the constants A_{10} and A_{11}

	A_{10}	A_{11}
2. iteration	4.516E−3	0.1160
3. iteration	0.0435	0.0926
4. iteration	0.0443	0.0805
5. iteration	0.0356	0.0423

The results in the second iteration are of course the same as by previous example $D = 0 =$ const. The amplitude of the reflected second harmonic (A_{10}) is in the second iteration bigger as a consequence of non-symmetric form of D in the previous iteration. But in general, in comparing with Table 1, the obtained results are improved by changing D during the iterations.

b) N = 4; Discuss now the problem with first four considered harmonics. We explained that we usually have problems to obtain accurate solution for higher harmonics if the fictive boundary is located close to the origin. In the previous subsection we showed that a true way to an accurate solution is in a suitable selection of the asymptotic local DtN operator, which is used to satisfy a radiation condition in the truncated area.

For the same numerical problem as by $N = 2$ we keep the same distance of the fictive boundary ($R = 1.5$). Table 3 presents the obtained absolute values of the constants A_{1n} from Eq. (10) if a different tangential dependence in the selected S_2 operator is used, but it is still constant during the iterations.

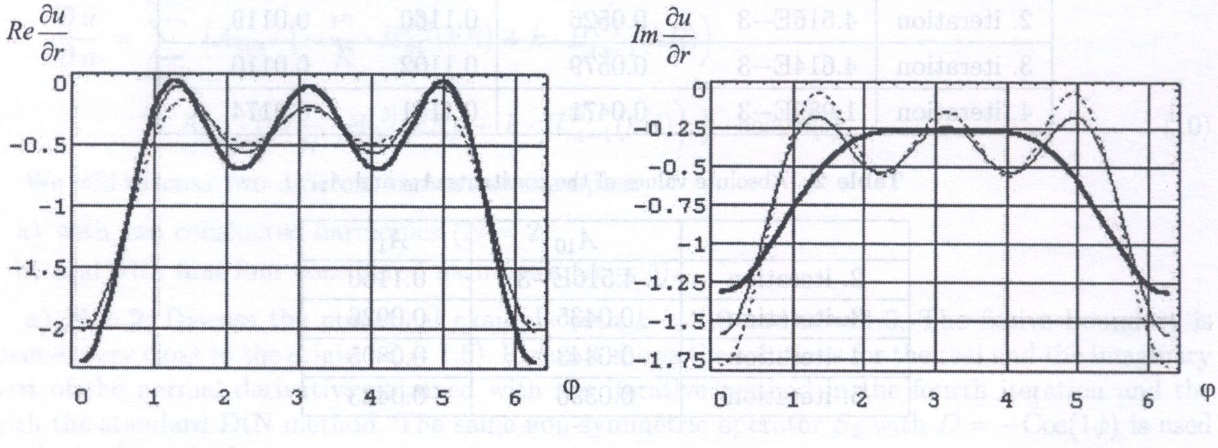
Table 3. Absolute values of the constants A_{1n}

	A_{10}	A_{11}	A_{12}	A_{13}
2. iteration				
$D = 0$	0.0604	0.1025	0.6826	2.3858
$D = -\text{Cos}(1\phi)$	0.1006	0.0817	0.5382	2.2317
3. iteration				
$D = 0$	0.0160	0.1105	0.5349	1.8099
$D = -\text{Cos}(1\phi)$	0.0605	0.0648	0.4007	1.6658
4. iteration				
$D = 0$	4.268E−3	0.1132	0.5416	1.8250
$D = -\text{Cos}(1\phi)$	0.0595	0.0617	0.4054	1.6742

The problem of higher harmonics ($m = 2$ and $m = 3$) is evident. Again a better solution can be obtained by iterative variation of the tangential dependence (D) in the selected operator in the following form (*Form 1*):

- $D = -\text{Cos}(1\phi)$ in each odd iteration – first non-symmetric mode,
- $D = -\text{Cos}(3\phi)$ in each even iteration – last non-symmetric mode.

Obtained results in the fourth iteration are graphically presented in Fig. 5. They are also compared with the results obtained by the standard DtN method in which a constant value $D = -\text{Cos}(1\phi)$ is used.



exact solution, iterative DtN m. (4[th] iteration), ... standard DtN m.

Fig. 5. Solutions for the real and imaginary part of the normal derivative on $R = 1.5$; $N = 4$

It is easy to see again that with the iterative variation of the tangential dependence (D) in the selected S_N operator better solutions can be obtained. The iterative method also permits other possibilities how to change D during iterations. Table 4 presents also an option with the simultaneous variation in the following form (*Form 2*):

- 1. iteration: $D = 0$
- 2. iteration: $D = -\text{Cos}(1\phi)$
- 3. iteration: $D = -\text{Cos}(2\phi)$
- 4. iteration: $D = -\text{Cos}(3\phi)$

Table 4. Absolute values of the constants A_{1n}

	A_{10}	A_{11}	A_{12}	A_{13}
2. iteration				
$D =$ (*Form 1*)	0.0970	0.0759	0.5533	2.3014
$D =$ (*Form 2*)	0.0545	0.0994	0.6975	2.4566
3. iteration				
$D =$ (*Form 1*)	0.0903	0.0943	0.4034	1.3937
$D =$ (*Form 2*)	0.0509	0.0610	0.3912	1.6800
4. iteration				
$D =$ (*Form 1*)	0.0592	0.1530	0.2408	1.2831
$D =$ (*Form 2*)	0.0429	0.0738	0.4265	1.7818

In comparing the obtained amplitudes with those from Table 3 it is evident that especially for higher harmonics better results can be obtained by iterative variation of the tangential dependence in the selected operator. In comparing only between *Form 1* and *Form 2* better results for higher harmonics ($m = 2$ and $m = 3$) are obtained with *Form 1*, but for a good solution for the first and the second harmonic it is more convenient to use *Form 2*. Of course there exist also other possibilities how to change D during iterations. The presented two possibilities are on our opinion the most interesting.

4. Concluding remarks

With the method presented, wave problems in infinite domains can be solved. The proposed iterative method is based on an iterative variation of fictive boundary conditions on the artificial boundary. It is very important that the method does not include the DtN operator into the dynamic stiffness matrix as opposed to the standard DtN methods.

We showed that by using a constant value of the operator an accurate solution for all harmonics cannot be obtained. Thus it is recommended to change the tangential dependence during iterations. The method permits many possibilities how to variety the operator, we presented only the most logical and interesting solutions. The most accurate was the variant with the iterative changing between the first and the last non-symmetric mode. Comparisons with the results, obtained with the standard DtN method, where it is not possible to change the operator, show many advantages of the proposed iterative method, especially if the artificial boundary is located close to the origin and if the problem of higher harmonics is discussed.

Acknowledgments

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